Time series essentials

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Today we are going to learn...

1. General Information
2. The lag operators
3. The difference operators
4. Linear models for stationary time series
5. Stationary
6. White noise

General information

- **Lecturer** : Feng Li, Email: `feng.li@cufe.edu.cn`
- **Language**: The course is taught in English. The course materials are in English.
- **Reception hours**: Questions concerned with this course can be asked after each lecture or via email.
- **Lecture notes and course materials**
  - Tsay, Ruey S. Analysis of financial time series. 3rd Edition. John Wiley & Sons, 2010.
  - Tsay, Ruey S. Multivariate Time Series Analysis: With R and Financial Applications. John Wiley & Sons, 2013.
  - Course materials and updated news are available at the course homepage (`http://feng.li/teaching/ts2017spring/`)
- **Working load**: Depending on your own situation and you ambition, you decide how much time you want to input. But there are compulsory assignments including presentation and discussions should be done in order to pass the exam.

Assignments and examinations

To pass the final exam you have to have at least 70p in the total score with the following tasks.

- **Lab assignments and presentation** (50%)
- **Projects** (50%)

Don’t worry and it’s fun!

The lag operators

- Suppose X_t is the GDP for past ten years $(t = 1, 2, ..., 10)$.
- The X_{t-1} is called the GDP with a lapse of time (i.e. **a lag**).
- In time series analysis, the **lag operator** or **backshift (**L **is the notation) operator** operates on an element of a time series to produce the previous element.
    - Given some time series $X_t = \{X_1, X_2, \dots\}$
    - then $LX_t = X_{t-1}$ for all $t > 1$
    - or equivalently $X_t = LX_{t+1}$ for all $t \geqslant 1$
    - and this also works $L^{-1}X_t = X_{t+1}$
    - and $L^k X_t = X_{t-k}$.
- How many lags can we have maximumly?

Why lags?

- Psychological reasons
  - Those who become instant millionaires by winning lotteries may not change the lifestyles intermediately.
  - People do not change their consumption habits immediately following a price decrease or an income increase.
- Technological reasons
  - We obtained the data from the stock market maybe always 5 seconds behind real time due to technological reasons.
  - The data obtain from authorities maybe always delayed due to confidential reasons.
- Institutional reasons
  - Employers often give their employees a choice among several health insurance plans, but once a choice is made, an employee may not switch to another plan for at least 1 year.
  - You are only allowed to take the re-exam next year if you fail this time.

The difference operator

- Assume your yearly salaries are X_t, how much do you earn compared to previous year?
- That should be $\Delta_t X_t = X_t - X_{t-1}$ which is called **the first difference** operator in time series analysis.
- It could be written in terms of lag operators $\Delta_t X_t = X_t - X_{t-1} = (1 - L)X_t$
- Similarly, the second difference operator works as follows:

$$\begin{aligned}
\Delta(\Delta X_t) &= \Delta X_t - \Delta X_{t-1} \\
\Delta^2 X_t &= (1 - L)\Delta X_t \\
\Delta^2 X_t &= (1 - L)(1 - L)X_t \\
\Delta^2 X_t &= (1 - L)^2 X_t \ .
\end{aligned}$$

Autocovariance and Autocorrelation Functions

- The covariance between y_t and its value at another time period, say, y_{t+k} is called the **autocovariance** at lag k,

$$\gamma_k = Cov(y_t, y_{t+k}) = E((y_t - \mu)(y_{t+k} - \mu))$$

- The collection of the values of γ_k, $k = 0, 1, 2, ...$ is called the **autocovariance function**.
- The autocovariance at lag $k = 0$ is just the variance of the time series;
- The **autocorrelation coefficient** at lag k is

$$\rho_k = \frac{Cov(y_t, y_{t+k})}{Var(y_t)} = \frac{\gamma_k}{\gamma_0}$$

- Note that by definition $\rho_0 = 1$.
- The collection of the values of ρ_k. $k = 0.1.2....$ is called the **autocorrelation function** (ACF).
- The ACF is independent of the scale of measurement of the time series.
- The autocorrelation function is symmetric around zero $\rho_k = \rho_{-k}$.

Sample autocorrelation function and partial autocorrelation I

- It is necessary to estimate the autocovariance and autocorrelation functions from a time series of finite length. The usual estimate of the autocovariance function is

$$c_k = \hat{\gamma}_k = \frac{1}{T} \sum_{t=1}^{T-k} (y_t - \bar{y})(y_{t+k} - \bar{y})$$

- The autocorrelation function is estimated by the **sample autocorrelation function**

$$r_k = \hat{\rho}_k = \frac{c_k}{c_0}$$

Sample autocorrelation function and partial autocorrelation II

- **The partial correlation** is the correlation between two variables after being adjusted for a common factor that may be affecting them.
- The partial correlation between X and Y after adjusting for Z is defined as

  $$\mathrm{Corr}(X - \hat{X}, Y - \hat{Y})$$

  where $\hat{X} = a_1 + b_1 Z$ and $\hat{Y} = a_2 + b_2 Z$
- The **partial autocorrelation function** between y_t and y_{t-k} is the autocorrelation between y_t and y_{t-k} after adjusting for y_{t-1}, y_{t-2},...,y_{t-k+1} and y_{t-k}.

Linear models for stationary time series

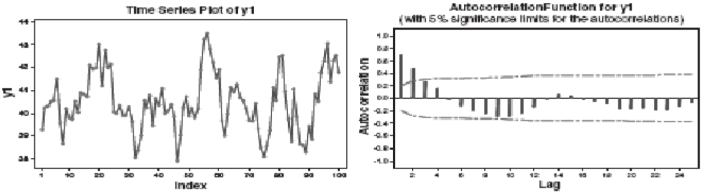
- Consider a linear operation from one time series x_t to another time series y_t

$$y_t = \sum_{i=-\infty}^{\infty} \psi_i x_{t-i}$$

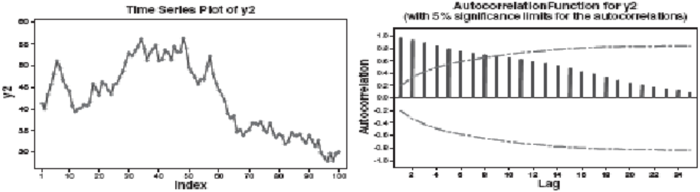
  which is called a **linear filter**.
- The linear filter should have the flowing properties
  - **Time-invariant**: ψ do not depend on time.
  - **Stable** if $\sum_{i=-\infty}^{\infty} |\psi_i| < \infty$

Stationary

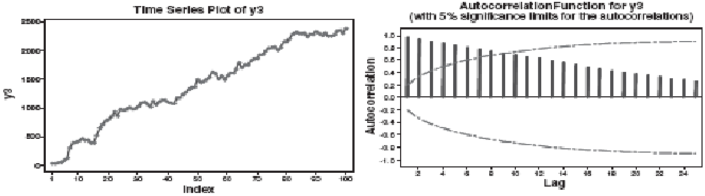
- A **stationary time series** exhibits similar "statistical behavior" in time and this is often characterized as a **constant** probability distribution (in terms of mean, variance, skewness, kurtosis, or even higher moments) in time.
- If we only consider the first two moments of the time series, we are talking about **weak stationarity** which is defined
  - The expected value of the time series does not depend on time.
  - The autocovariance function defined as $Cov(y_t, y_{t-k})$ for any lag k is only a function of k and not time t.
- If the time series is not stationary, it can be examined by observing **autocorrelation function (ACF)** and **particial autocorrelation function (PACF)**.



(a) $y_{1,t} = 10 + 0.75 y_{1,t-1} + \varepsilon_t$



(b) $y_{2,t} = 2 + 0.95 y_{2,t-1} + \varepsilon_t$



(c) $y_{3,t} = 20 + y_{3,t-1} + \varepsilon_t$

Example: Calculating ACF with R.

White noise

- If a time series consists of uncorrelated observations and has constant variance. we say that it is **white noise**.
- If in addition, the observations in this time series are normally distributed, the time series is **Gaussian white noise**.
- If a time series is white noise, the distribution of the sample autocorrelation coefficient at lag k in large samples is approximately normal with mean zero and variance $1/T$.

Stationary time series

- Many time series do not exhibit a stationary behavior.
- The stationarity is in fact a rarity in real life.
- However it provides a foundation to build upon since (as we will see later on) if the time series in not stationary, its first difference ($y_t - y_{t-1}$) will often be stationary.

Stationary time series

- For a time-invariant and stable linear filter and a stationary input time series x_t

$$y_t = \sum_{i=-\infty}^{\infty} \psi_i x_{t-i}$$

with $\mu_x = E(x_t)$ and $\gamma_x(k) = Cov(x_t, x_{t+k})$.

- The output time series y_t is also a stationary time series where

$$E(y_t) = \mu_y = \sum_{-\infty}^{\infty} \psi_i \mu_x$$

$$\gamma_y(k) = Cov(y_t, y_{t+k}) = \sum_{-\infty}^{\infty} \psi_i \psi_j \gamma_x(i - j + k)$$

Stationary time series

- The following stable linear process with white noise time series, ϵ_t,

$$y_t = \mu + \sum_{-\infty}^{\infty} \psi_i \epsilon_{t-i}$$

is also stationary where ϵ_t has $E(\epsilon_t) = 0$ and

$$\gamma_\epsilon(k) = Cov(\epsilon_t, \epsilon_{t+k}) = \begin{cases} \sigma^2 & k = 0 \\ 0 & otherwise \end{cases}$$

- The autocovariance function of y_t is

$$\begin{aligned} \gamma_y(k) = Cov(y_t, y_{t+k}) &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j \gamma_\epsilon(i - j + k) \\ &= \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \end{aligned}$$

Suggested Reading

- Tsay (2010) **Chapter 1**