Conditional Heteroscedastic Models

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Today we are going to learn...

1. Volatility
2. The ARCH model
3. The GARCH models

Characteristics of volatility I

- A special feature of stock **volatility** is that it is not directly observable.
- The daily volatility is *not directly observable* from the return data because there is only one observation in a trading day. If intraday data of the stock, such as 10-minute returns, are available, then one can estimate the daily volatility.
- The high-frequency intraday returns contain only very limited information about the overnight volatility.
- The unobservability of volatility makes it difficult to evaluate the forecasting performance of conditional heteroscedastic model
- In options markets, if one accepts the idea that the prices are governed by an econometric model such as the Black-Scholes formula, then one can use the price to obtain the **implied volatility**.
- This implied volatility is derived under the assumption that the price of the underlying asset follows a geometric Brownian motion. It might be different from the actual volatility

Characteristics of volatility II

- Although volatility is not directly observable, it has some characteristics that are commonly seen in asset returns.
  - Volatility may be high for certain time periods and low for other periods (**volatility clusters**).
  - Volatility evolves over time in a continuous manner–that is, volatility jumps are rare.
  - Volatility does not diverge to infinity–that is, volatility varies within some fixed range. This means that volatility is often stationary.
  - Volatility seems to react differently to a big price increase or a big price drop (leverage effect).

The autoregressive conditional heteroscedastic (ARCH) model I

- Let $a_t = r_t - \mu_t$ be the residuals of the mean equation, the basic idea of ARCH models:
  - the shock a_t of an asset return is serially uncorrelated, but dependent,
  - the dependence of a_t can be described by a simple quadratic function of its lagged values.
- An ARCH(m) model assumes that

$$a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + ... + \alpha_m a_{t-m}^2$$

  where ϵ_t is a sequence of independent and identically distributed (iid) random variables with mean zero and variance 1, $\alpha_0 > 0$, and $a_i \geqslant 0$ for $i > 0$.
- Under the ARCH framework, large shocks tend to be followed by another large shock.

Estimating ARCH models

- Several likelihood functions are commonly used in ARCH estimation, depending on the distributional assumption of ϵ_t

$$
\begin{aligned}
f(a_1, ..., a_T|\alpha_0, ..., \alpha_m) &= f(a_1, ..., a_m|\alpha_0, ..., \alpha_m) \\
&\times f(a_{m+1}|F_m) \times ... \times f(a_{T-1}|F_{T-2}) \times f(a_T|F_{T-1})
\end{aligned}
$$

- Since the exact form of $f(a_1, ..., a_m|\alpha_0, ..., \alpha_m)$ is complicated, it is commonly dropped from the prior likelihood function. This results in using the **conditional-likelihood function**

$$
\begin{aligned}
f(a_{m+1}, ..., a_T|\alpha_0, ..., \alpha_m, a_1, ..., a_m) &= f(a_{m+1}|F_m) \\
&\times ... \times f(a_{T-1}|F_{T-2}) \times f(a_T|F_{T-1})
\end{aligned}
$$

Exampe

- **Tsy (2010) P. 127**

Properties of ARCH Models

- The unconditional mean of a t remains zero

$$E(a_t) = E(E(a_t|F_{t-1})) = E(\sigma_t E(\epsilon_t)) = 0$$

- The unconditional variance of a_t can be obtained as

$$Var(a_t) = E(a_t^2) = E(\sigma_t^2 E(\epsilon_t^2)) = E(\alpha_0 + \alpha_1 a_{t-1}^2)$$

  Because a_t is a stationary process, it is easily obtained as $Var(a_t) = \alpha_0/(1 - \alpha_1)$.

- The excess kurtosis of a_t is positive and the tail distribution of a_t is heavier than that of a normal distribution. In other words, the shock a_t of a conditional Gaussian ARCH(1) model is more likely than a Gaussian white noise series to produce "outliers."
- These properties continue to hold for general ARCH models, but the formulas become more complicated for higher order ARCH models.

Weaknesses of ARCH Models

- The model assumes that positive and negative shocks have the same effects on volatility because it depends on the square of the previous shocks. In practice, it is well known that the price of a financial asset responds differently to positive and negative shocks.
- The ARCH model is rather restrictive. In practice, it limits the ability of ARCH models with Gaussian innovations to capture excess kurtosis.
- The ARCH model does not provide any new insight for understanding the source of variations of a financial time series. It merely provides a mechanical way to describe the behavior of the conditional variance. It gives no indication about what causes such behavior to occur.
- ARCH models are likely to overpredict the volatility because they respond slowly to large isolated shocks to the return series.

Introduction to GARCH models

- Although the ARCH model is simple, it often requires many parameters to adequately describe the volatility process of an asset return.
- Bollerslev (1986) proposes a useful extension known as the generalized ARCH (GARCH) model.
- For a log return series r_t , let $a_t = r_t - \mu_t$ be the innovation at time t. Then, a_t follows a GARCH(m,s) model if

$$a_t = \sigma_t \epsilon_t, \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2$$

  where again ϵ_t is a sequence of iid random variables with mean 0 and variance 1.
- The conditions: $\alpha_0 > 0$, $\alpha_i \geqslant 0$, $\beta_j \geqslant 0$, and $\sum_{i=1}^{max(m,s)} (\alpha_i + \beta_j) < 1$.
- The GARCH model reduces to a pure ARCH(m) model if $s = 0$

Properties of GARCH models

- A large a_{t-1} tends to be followed by another large a_t , generating, again, the well-known behavior of volatility clustering in financial time series.
- Similar to ARCH models, the tail distribution of a GARCH(1,1) process is heavier than that of a normal distribution.
- The model provides a simple parametric function that can be used to describe the volatility evolution.

Estimating GARCH models

- A two-pass estimation method can be used to estimate GARCH models.
- First, ignoring any ARCH effects, one estimates the mean equation of a return series using the methods discussed in Chapter 2 (e.g., maximum-likelihood method). Denote the residual series by a_t
- Second, treating a_t^2 as an observed time series, one applies the maximum-likelihood method to estimate parameters. Denote the AR and MA coefficient estimates by $\hat{\phi}_i$ and $\hat{\theta}_i$. The GARCH estimates are obtained as $\hat{\beta}_i = \hat{\theta}_i$ and $\hat{\alpha}_i = \hat{\phi}_i - \hat{\theta}_i$.
- The above estimates are approximations to the true parameters and their statistical properties have not been rigorously investigated. However, limited experience shows that this sim- ple approach often provides good approximations, especially when the sample size is moderate or large.

The Integrated GARCH model

- If the AR polynomial of the GARCH representation has a unit root, then we have an **IGARCH model**. Thus, IGARCH models are unit-root GARCH models.

The GARCH-M model

- In finance, the return of a security may depend on its volatility. To model such a phenomenon, one may consider the GARCH-M model, where M stands for GARCH in the mean.

$$
\begin{aligned}
r_t &= \mu + c\sigma_t^2 + a_t, \\
a_t &= \sigma_t \epsilon_t, \\
\sigma_t^2 &= \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2
\end{aligned}
$$

  where μ and c are constants.

- The parameter c is called the **risk premium parameter**. A positive c indicates that the return is positively related to its volatility.
- The formulation implies that there are serial correlations in the return series r_t .

The exponential GARCH (EGARCH) model

- Nelson (1991) proposes the exponential GARCH (EGARCH) model. In particular, to allow for asymmetric effects between positive and negative asset returns, he considered the weighted innovation

$$g(\epsilon_t) = \theta\epsilon_t + \gamma(|\epsilon_t| - E(|\epsilon_t|))$$

where θ and γ are real constants.

The threshold GARCH (TGARCH) model

- Another volatility model commonly used to handle leverage effects is the threshold GARCH. A TGARCH(m,s) model assumes that

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{s}(\alpha_i + \gamma_i N_{t-1})a_{t-i}^2 + \sum_{j=1}^{m}\beta_j \sigma_{t-j}^2$$

where N_{t-i} is an indicator for negative a_{t-i}, that is

$$N_{t-i} = \begin{cases} 1, & a_{t-i} < 0 \\ 0, & elsewhere. \end{cases}$$

The stochastic volatility (SV) model

- An alternative approach to describe the volatility evolution of a financial time series is to introduce an innovation to the conditional variance equation of a_t

$$a_t = \sigma_t \epsilon_t,$$
$$(1 - \alpha_1 B - ... - \alpha_m B^m) \log(\sigma_t^2) = \alpha_0 + \nu_t$$

  where ϵ_t are iid $N(0, 1)$ and ν_t are iid $N(0, \sigma_\nu^2)$
- Adding the innovation ν_t substantially increases the flexibility of the model in describing the evolution of σ_t^2 , but it also increases the difficulty in parameter estimation.
- To estimate an SV model, we need a quasi-likelihood method via Kalman filtering or a Monte Carlo method.

The long-memory stochastic volatility model

- More recently, the SV model is further extended to allow for long memory in volatility, using the idea of **fractional difference**.
- A time series is a long-memory process if its autocorrelation function decays at a hyperbolic ($(1-B)^d$), instead of an exponential, rate as the lag increases.
- The extension to long-memory models in volatility study is motivated by the fact that the autocorrelation function of the squared or absolute-valued series of an asset return often decays slowly, even though the return series has no serial correlation

$$
\begin{aligned}
a_t &= \sigma_t \epsilon_t, \\
\sigma_t &= \sigma \exp(u_t/2) \\
(1-B)^d u_t &= \eta_t
\end{aligned}
$$

Suggested Reading

- Tsay (2010) **Chapter 3**