Time Series Analysis

2011

Index

时间序列分析

第一讲（上）

授课教师

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授课教师

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- Email : george.tiao@gsb.uchicago.edu
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助教

- 黄达 huangda@gsm.pku.edu.cn

Course Grade

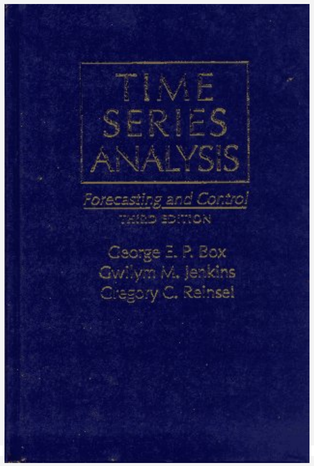
- Homework 20%
- Exam 40%
- Project 40%

  (Three in a Group)

Text Required

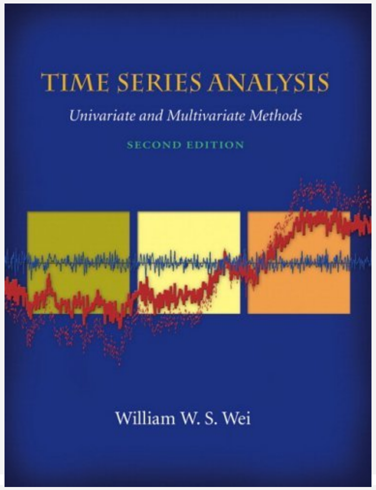
- Packet
- Additional notes distributed in class

Text Recommended



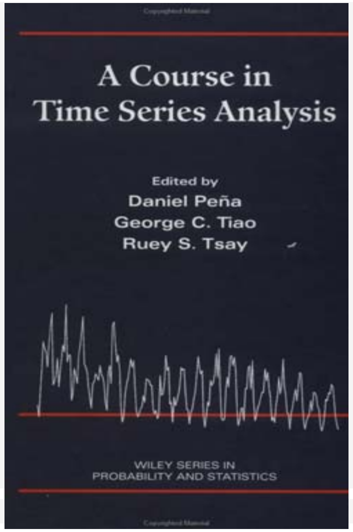
- Box, G., Jenkins, G. and Reinsel, R. (1994)
- *Time Series Analysis, Forecasting and Control (3rd Edition)*
- Prentice Hall
- 人民邮电出版社有影印版

Text Recommended



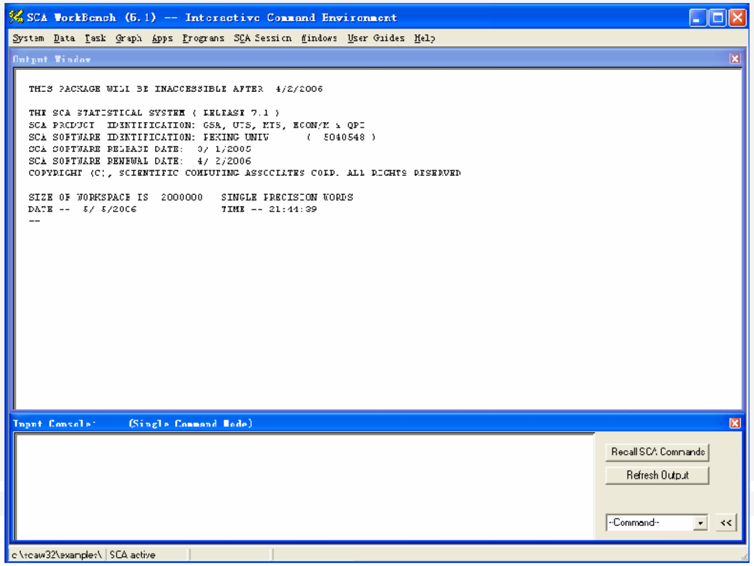
- Wei, W.S. (2006)
- *Time Series Analysis: Univariate and Multivariate Methods(2nd Edition)*
- Addison Wesley

Text Recommended



- Pena, D., Tiao, G. C., and Tsay, R.S. (2001)
- *A Course in Time Series Analysis (1st Edition)*
- John Wiley &Sons

Software Required : SCA



修课要求

- 预修课程
  - 数理统计
  - 回归分析
- 课后自习时间
  - 每周除上课外至少10小时

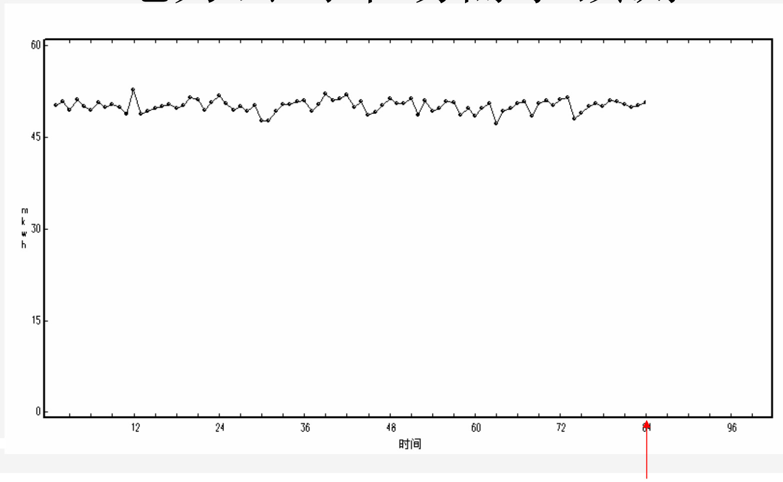
Nature of Course

- 主要内容 : 分析时间序列
  - 方法
  - 预测
- 主要目的 : 学会理性预测

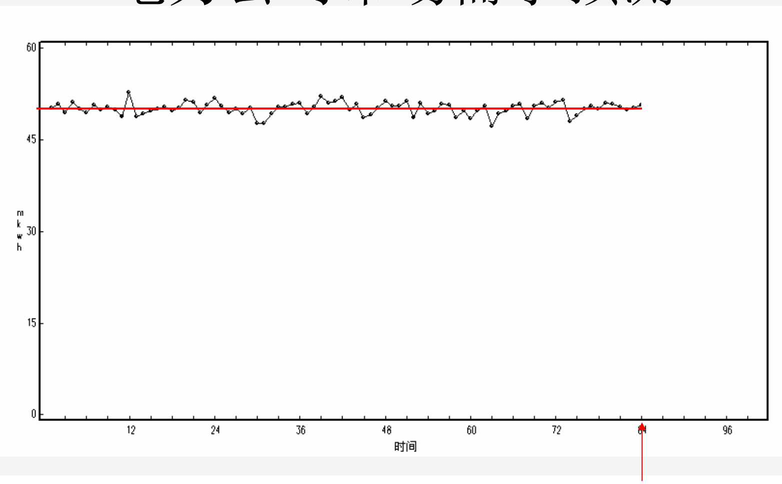
电力公司市场需求预测

- 预测时间长度
  - 短期 (周)
  - 中期 (季)
  - 长期 (年)
- 怎么预测？
  - 从过去的数据中获取信息！

电力公司市场需求预测



电力公司市场需求预测



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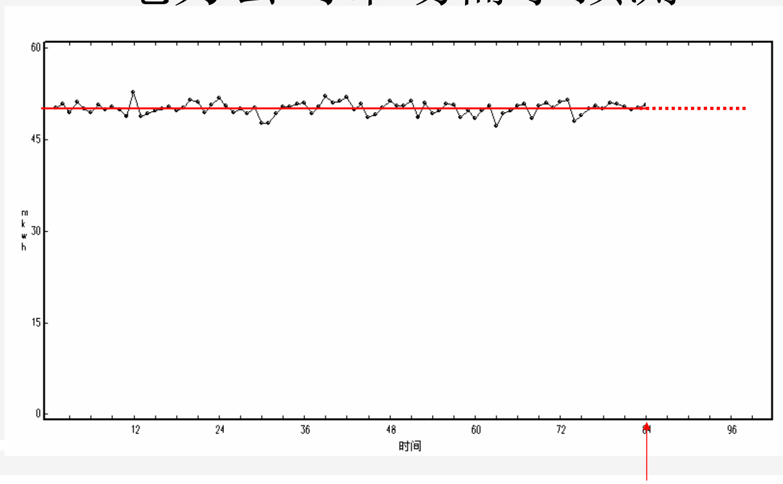

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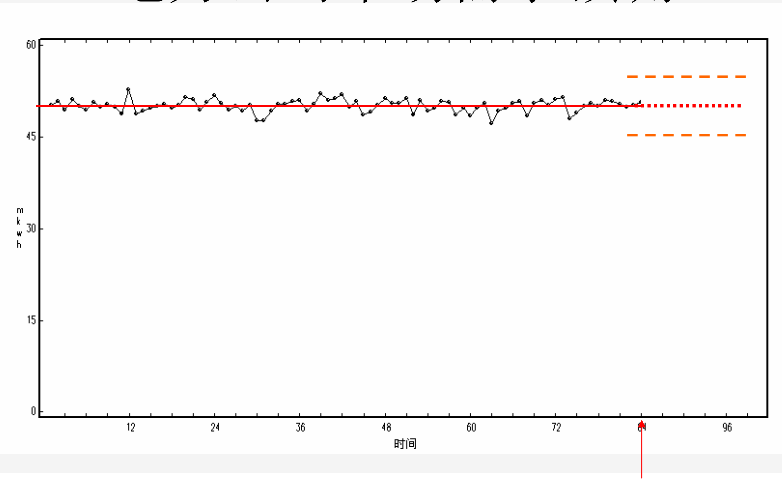
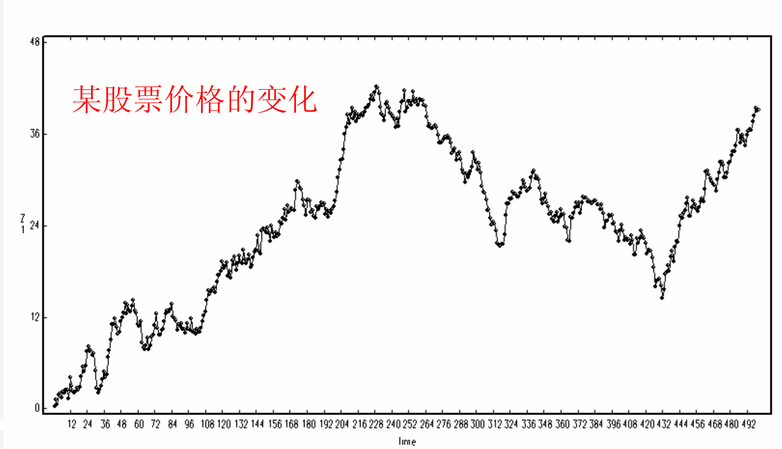
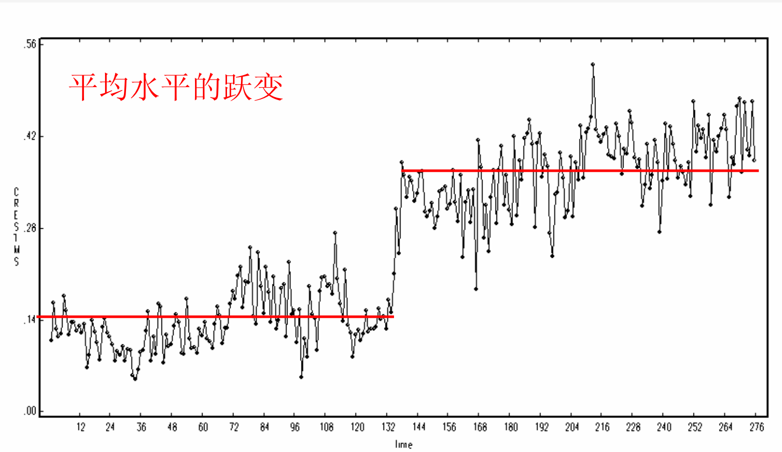


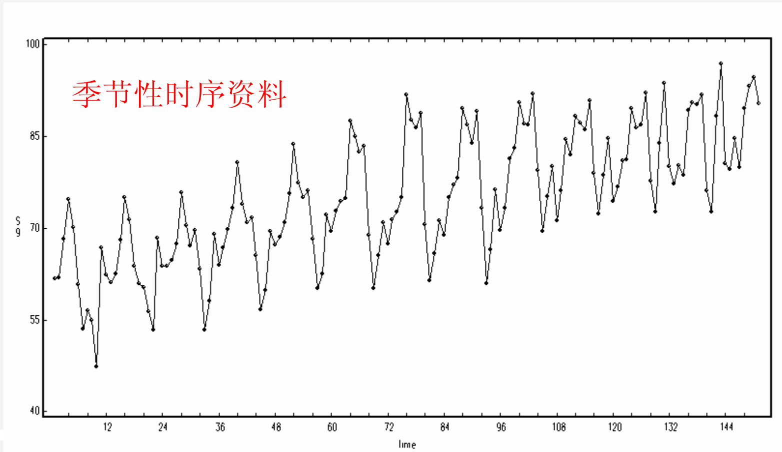
other patterns - 1



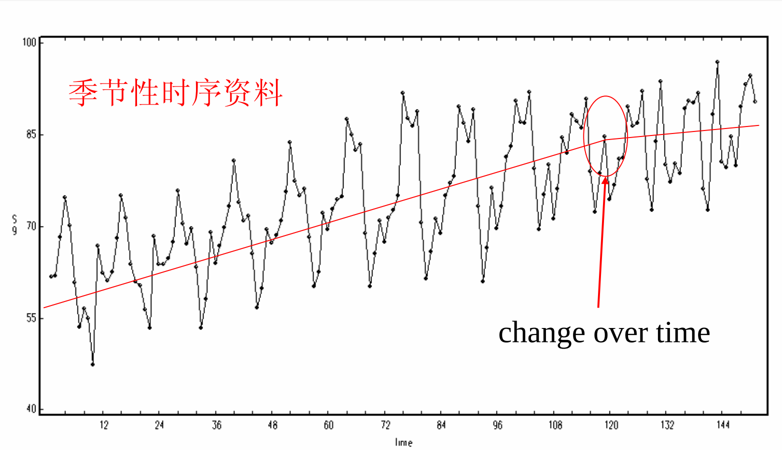
other patterns – 2



other patterns – 3



other patterns – 3



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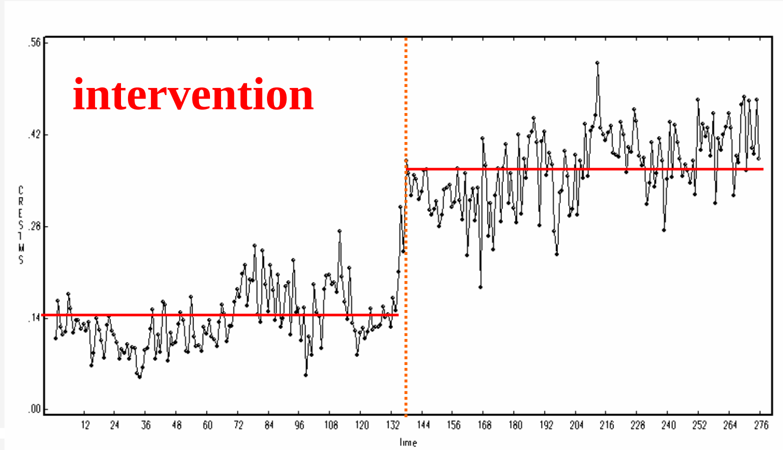
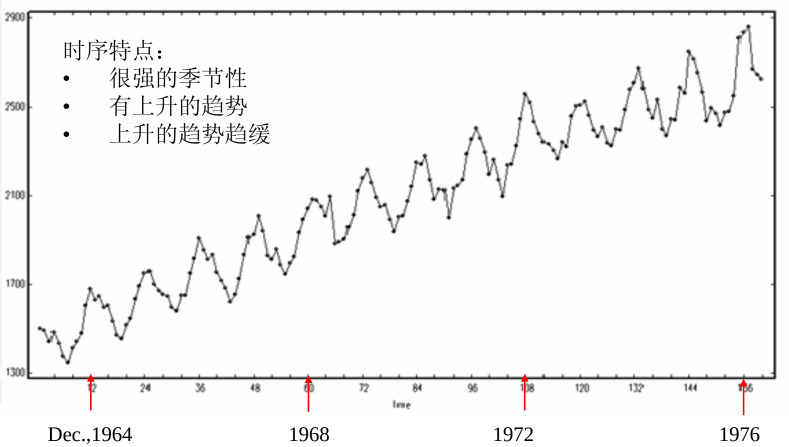


电力公司市场需求预测

Monthly Electricity Demand (logged), 1/64 to 4/77



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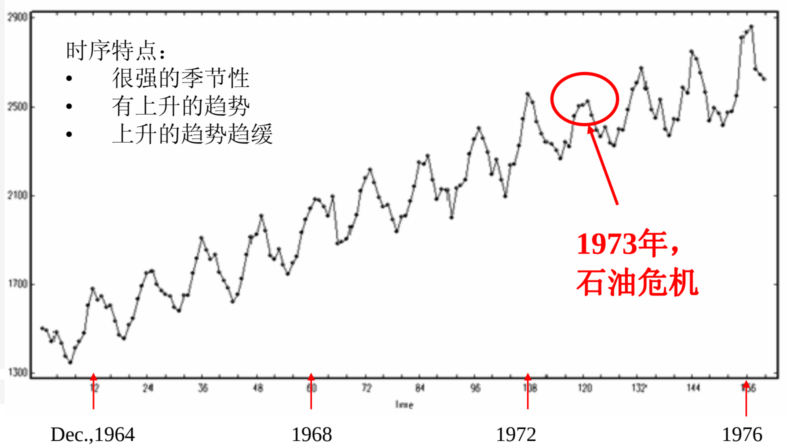
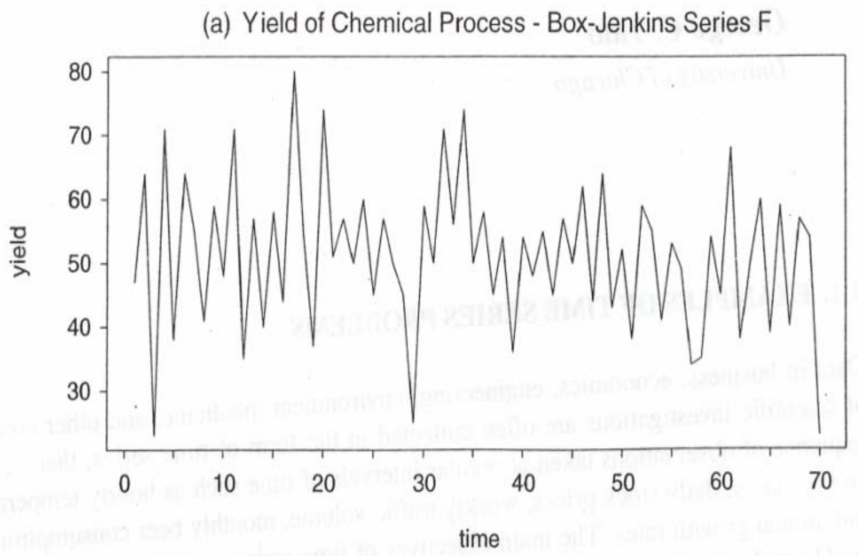


平稳时间序列 (1)



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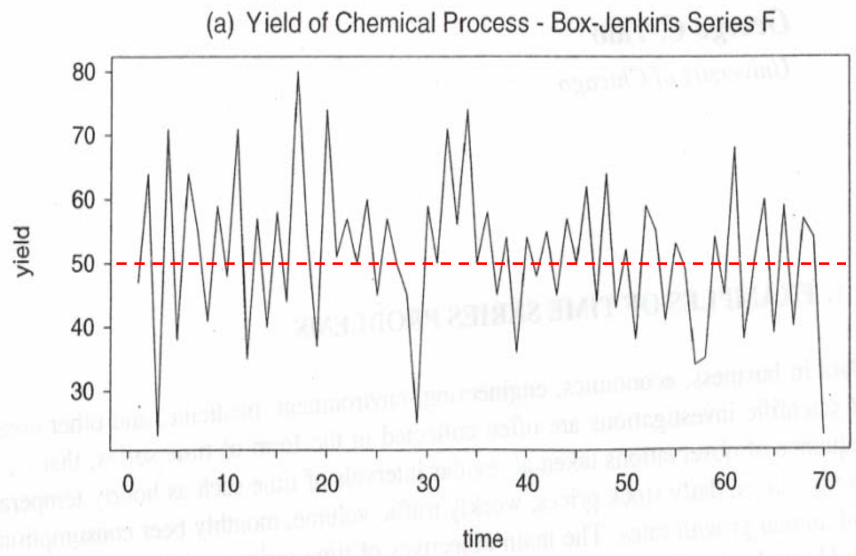

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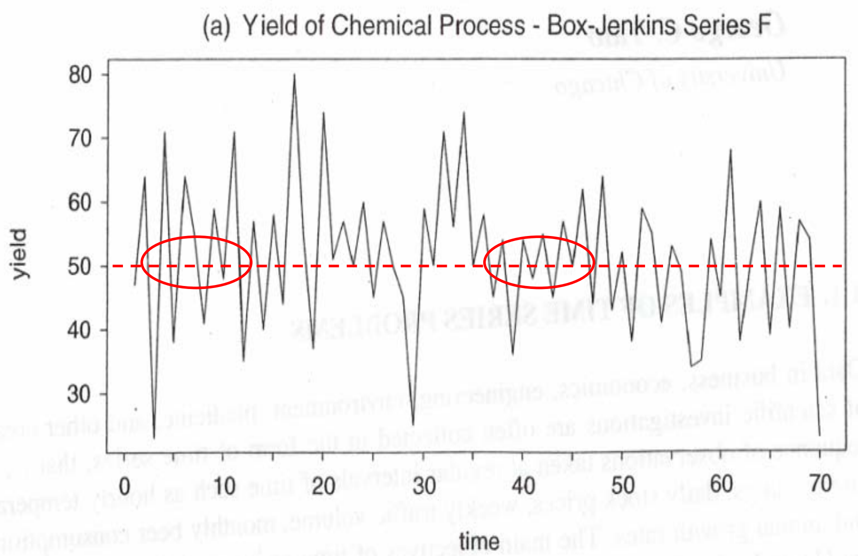
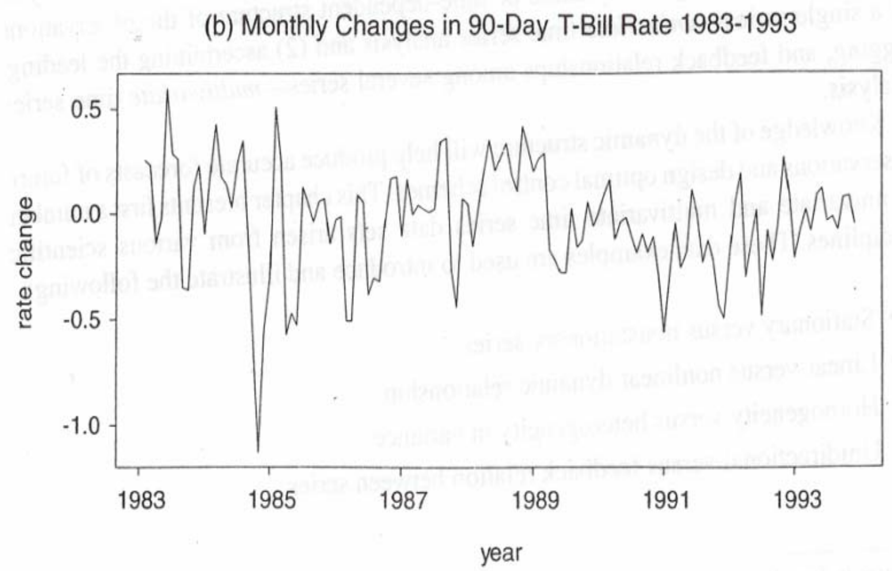
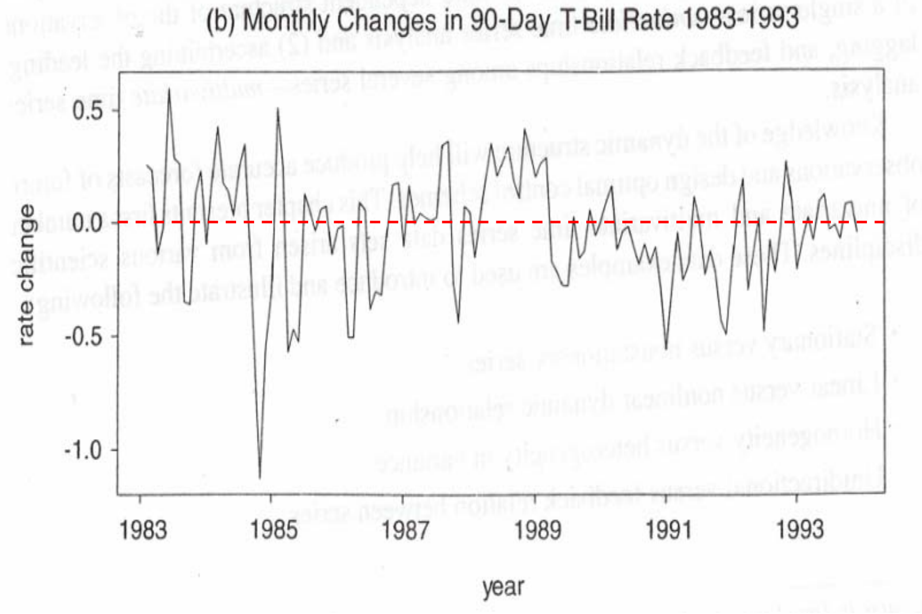


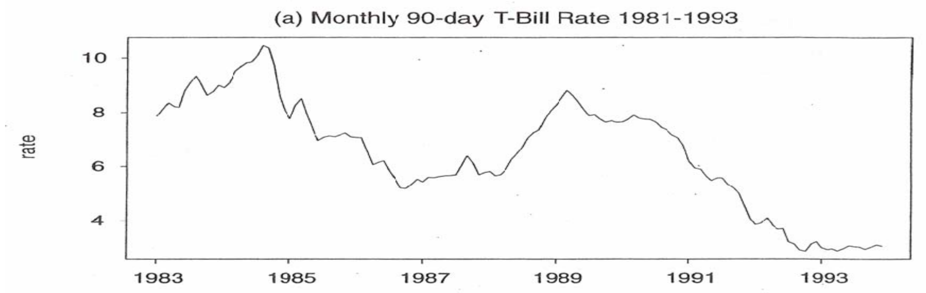
平稳时间序列 (2)



平稳时间序列 (2)



非平稳时间序列 (1)



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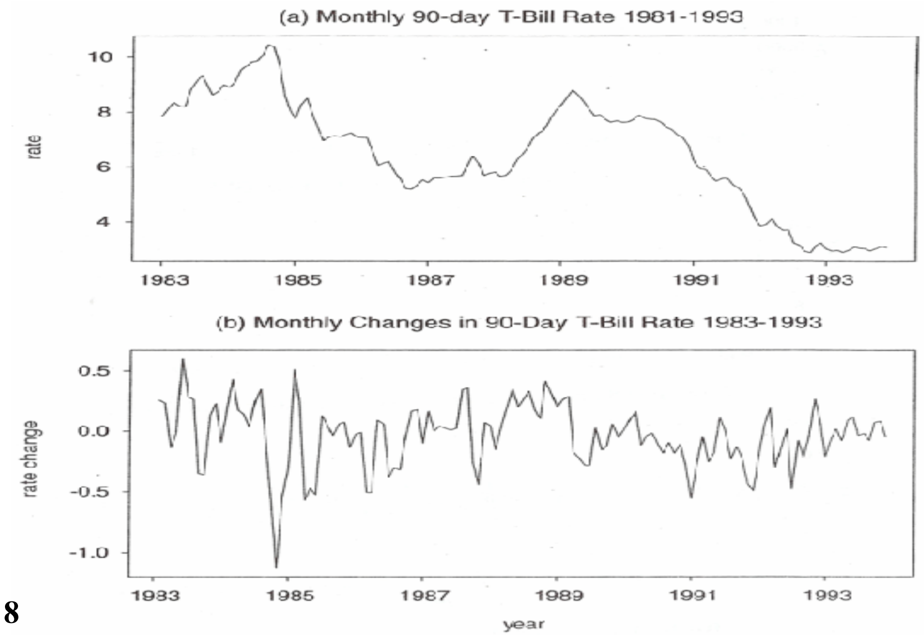

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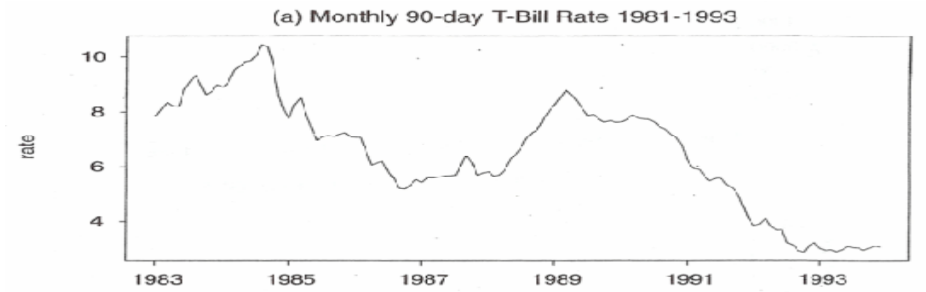

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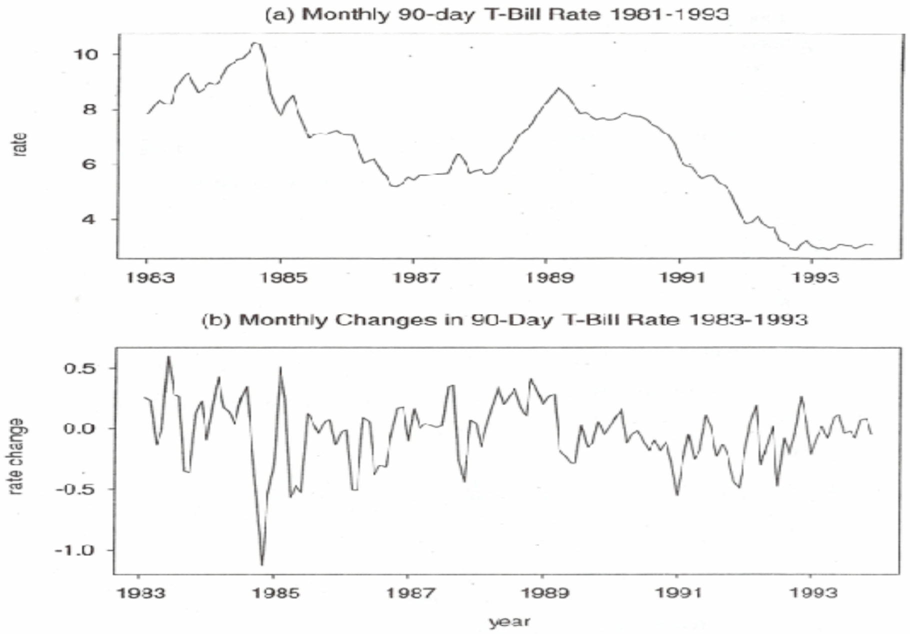

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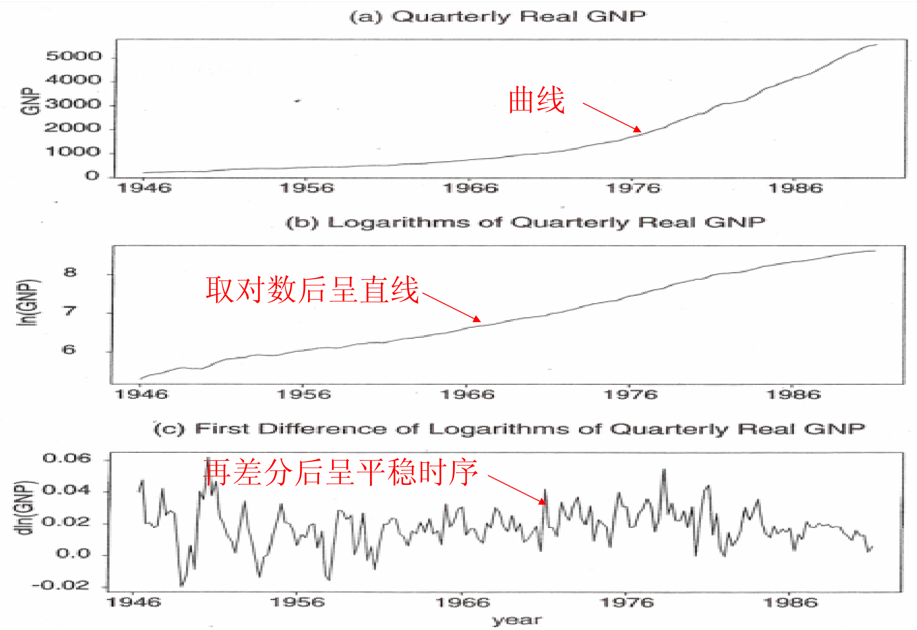


平稳时序？非平稳时序？

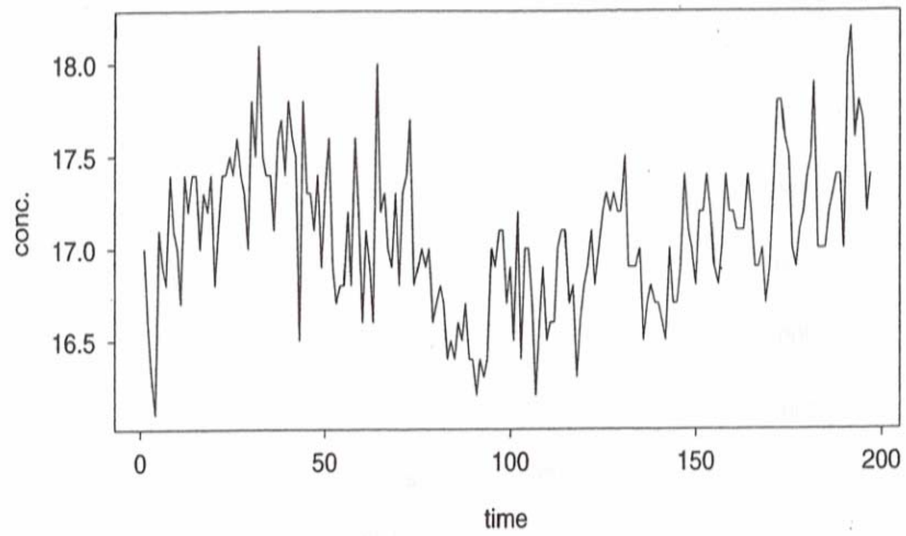


FIGURE 1.4 Concentration readings of a chemical process: Box–Jenkins series A.

平稳时序？非平稳时序？

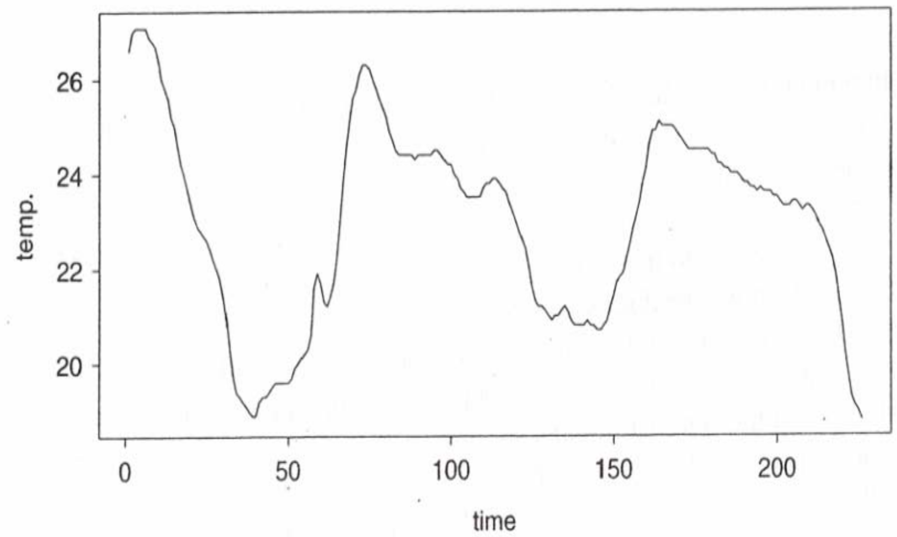


FIGURE 1.5 Temperature readings: Box–Jenkins series C.

季节性时间序列 (1)

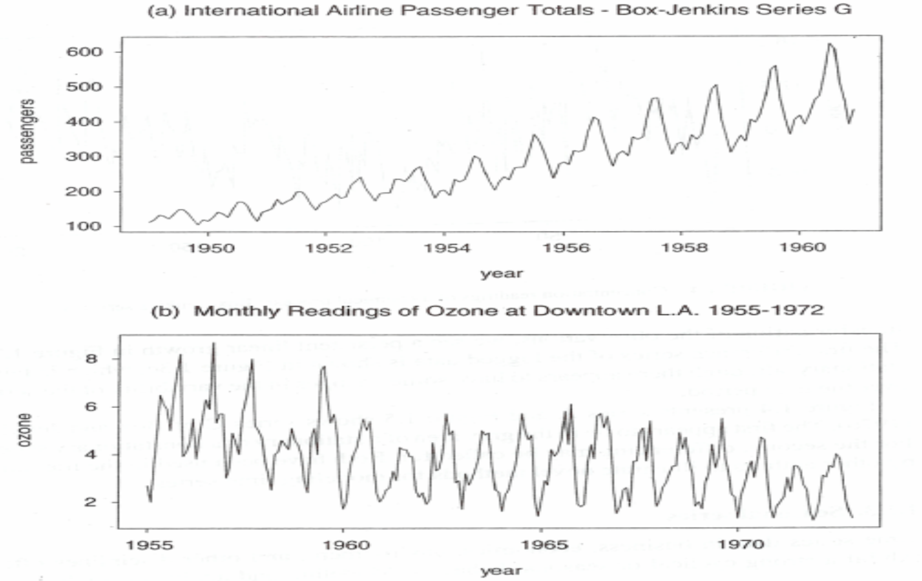


FIGURE 1.6 Two examples of seasonal series.

季节性时间序列 (1)

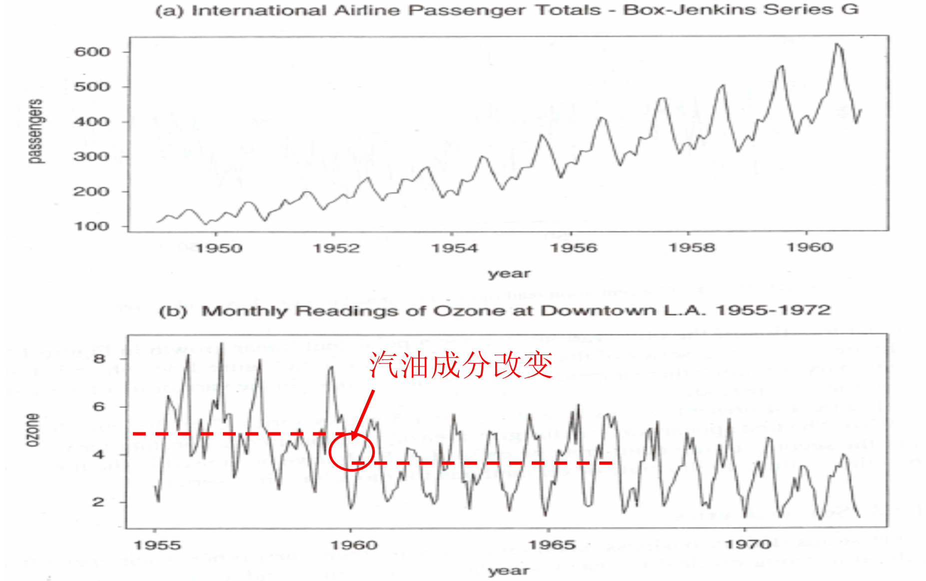


FIGURE 1.6 Two examples of seasonal series.

季节性时间序列 (1)

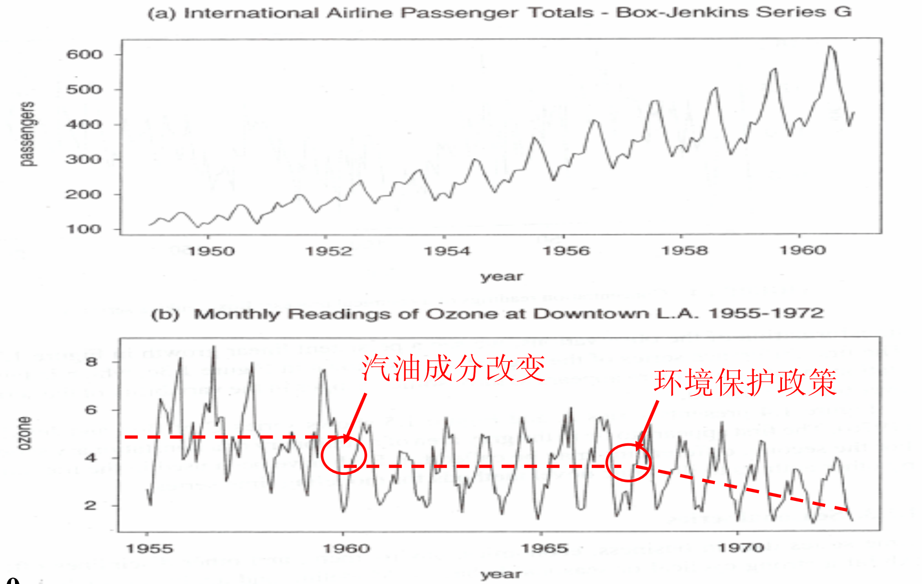


FIGURE 1.6 Two examples of seasonal series.

季节性时间序列 (2)

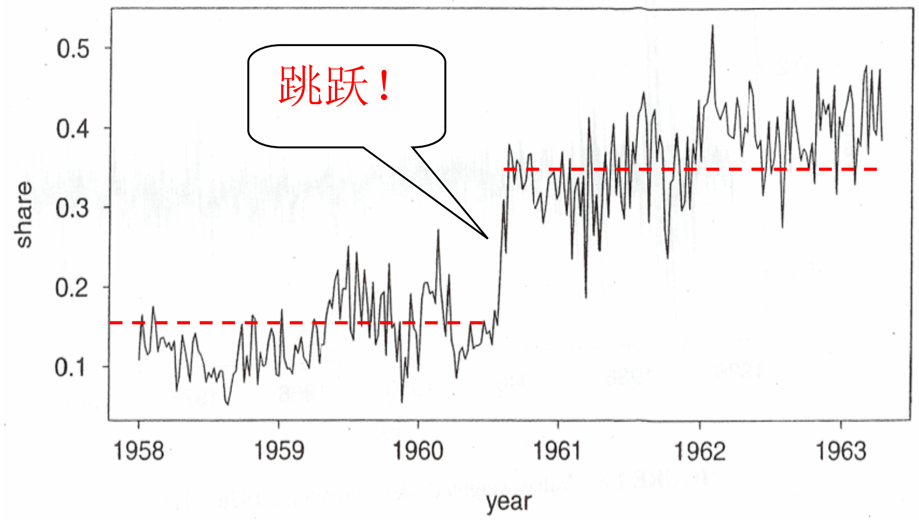


FIGURE 1.7 The Crest market share weekly data 1958–1963.

时间序列分析

第一讲(下)

Variance Change

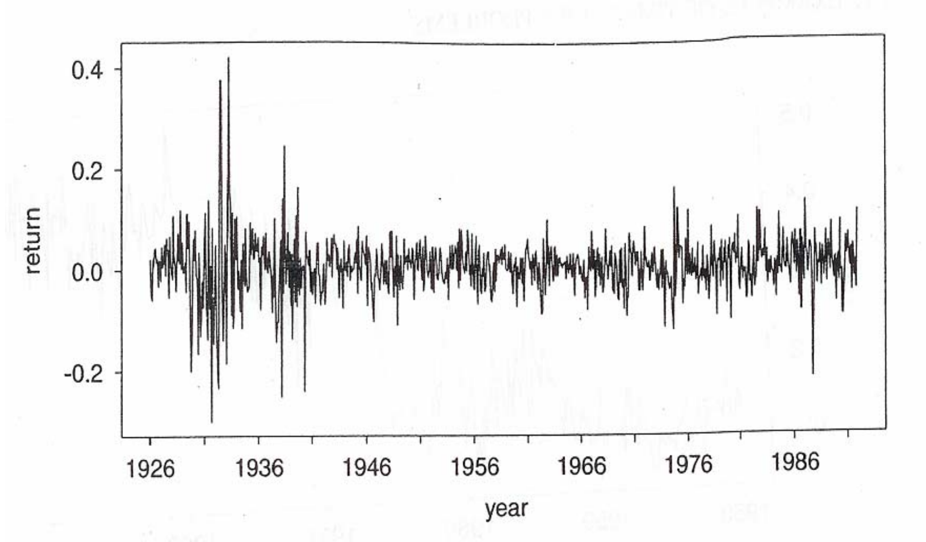
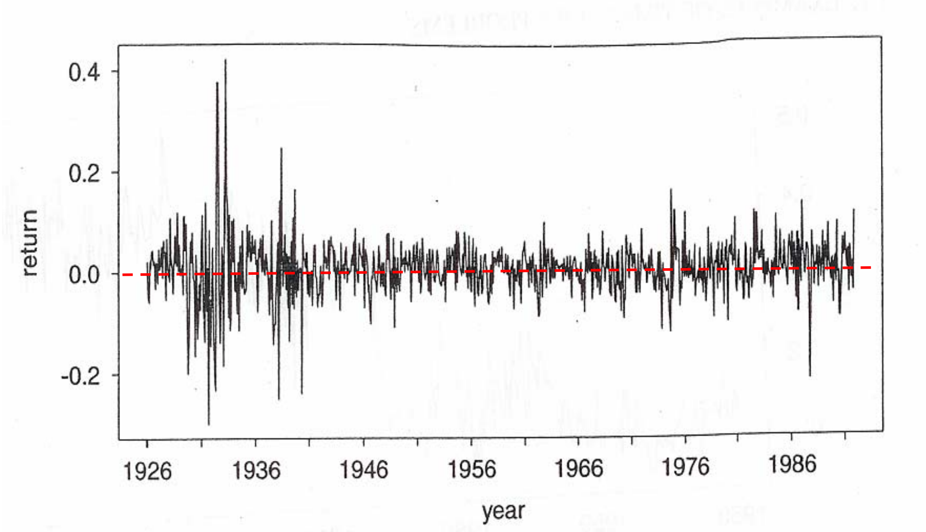


FIGURE 1.8 Value-weighted S&P 500 returns 1926–1991.

Variance Change



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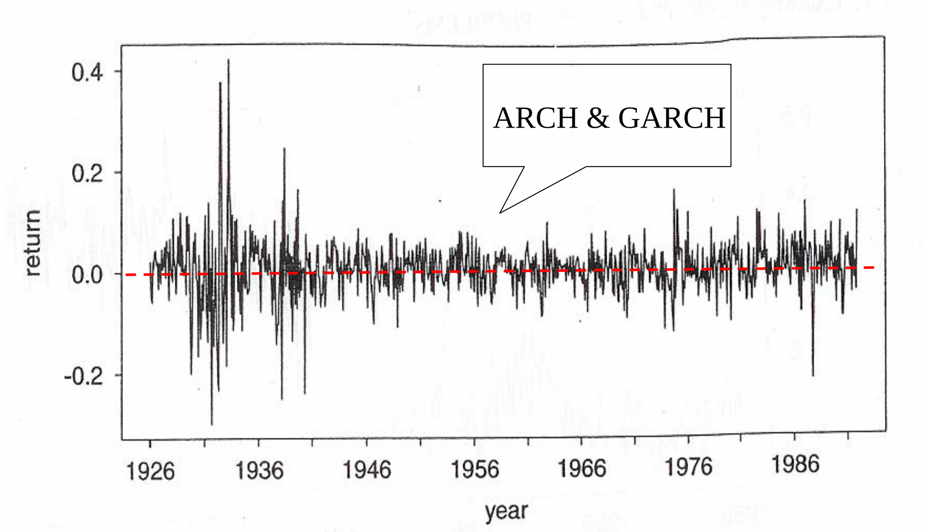
Variance Change



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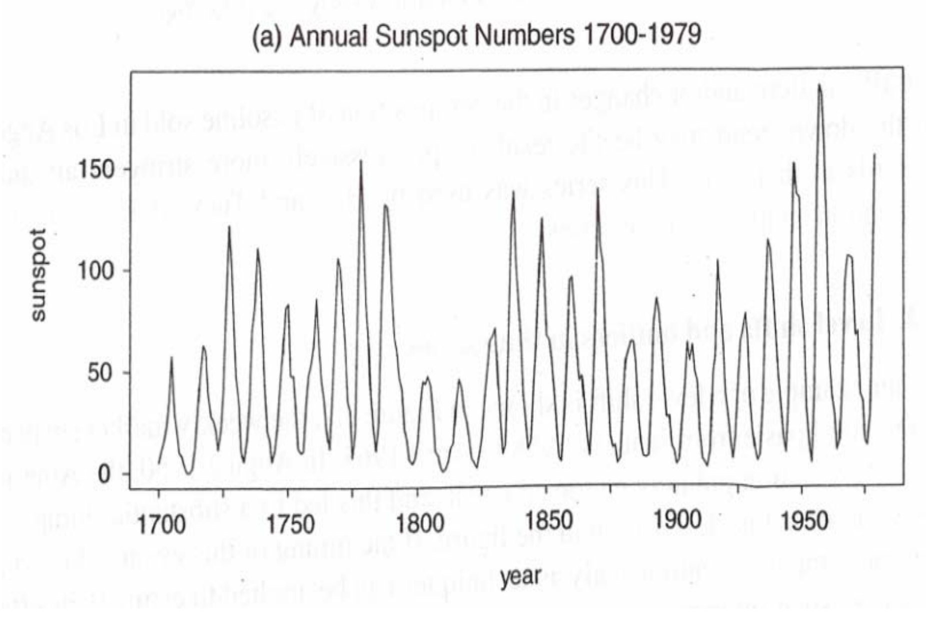
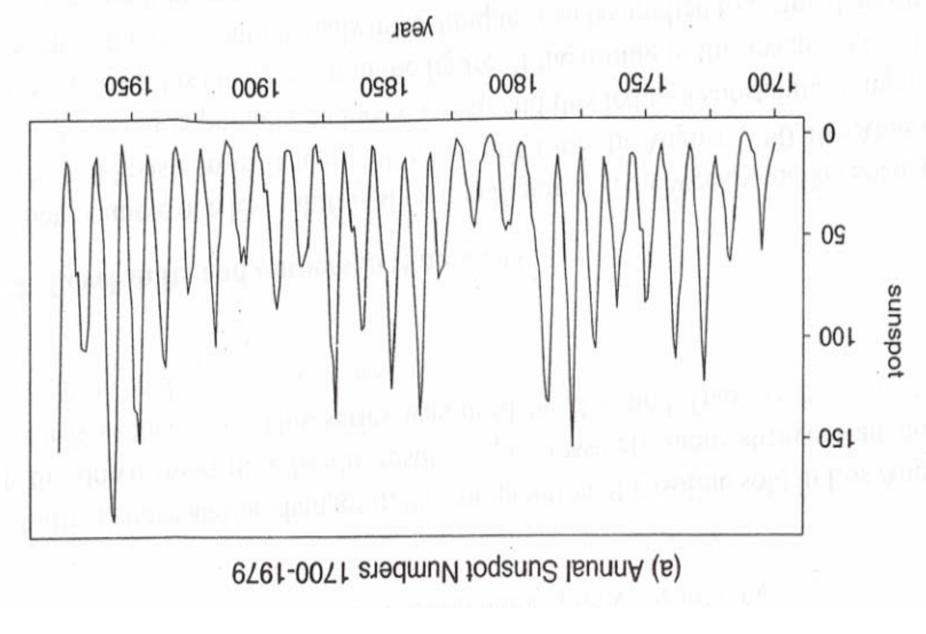
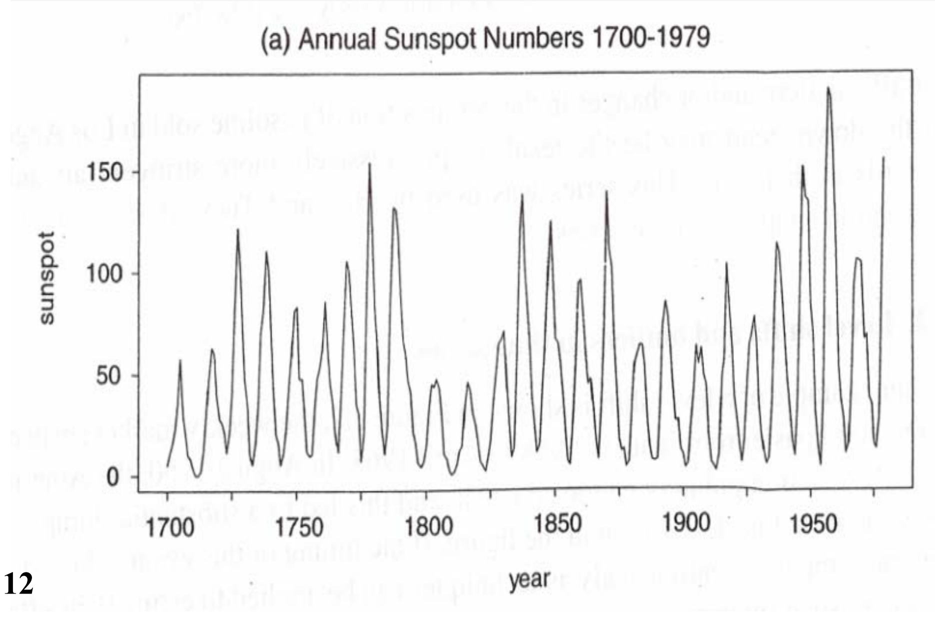
Asymmetric time series (1)



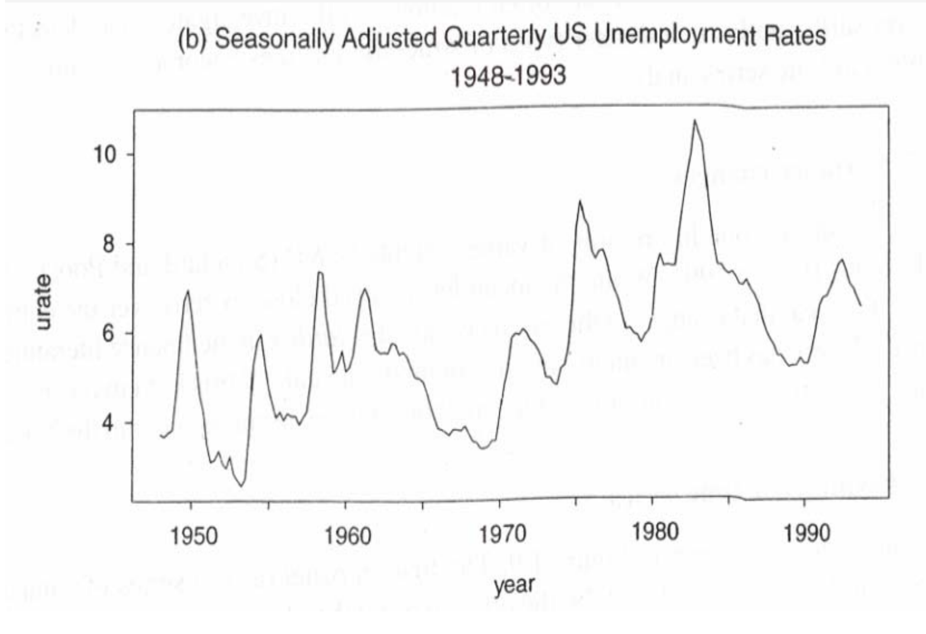
Asymmetric time series (1)



Asymmetric time series (1)



Asymmetric time series (2)



Unidirectional Relation between Series

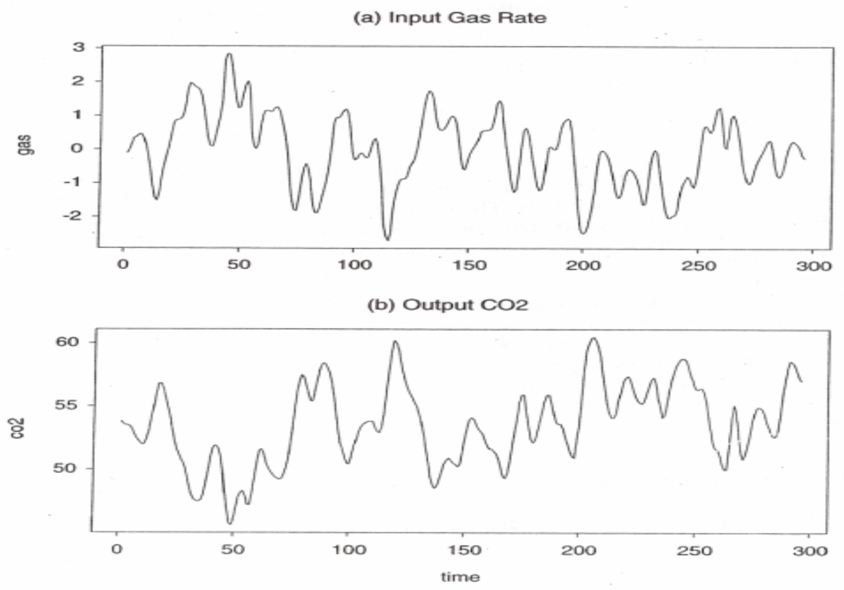


FIGURE 1.10 The gas furnace data: Box–Jenkins series J.

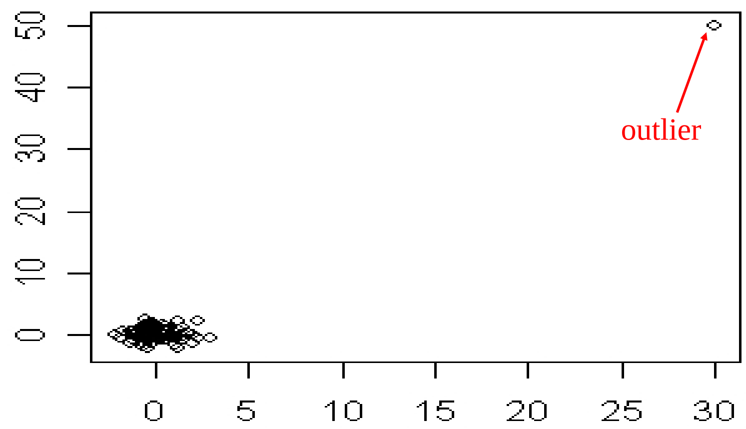
时间序列的各种类型

- Stationary
- Non-stationary
- Seasonality
- Level Shift
- Variance Changes
- Intervention
- Bivariate Leading-Lagging Relationship
- …
- GOAL: model these different patterns for prediction (forecasting)

Course Objects – Topics (1)

- Linear Dynamic Models
  - ARIMA Models
  - Forecasting
  - Model Building
  - Dynamic transfer function
  - Intervention + Outliers
- Unit Root Tests
- Signal Extraction, Unobserved components, Seasonal Adjustment

Outlier



Course Objects – Topics (1)

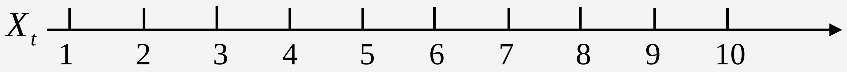
- Linear Dynamic Models
  - ARIMA Model
  - Forecasting
  - Model Building
  - Dynamic transfer function
  - Intervention + Outliers
- Unit Root Tests
- Signal Extraction, Unobserved components, Seasonal Adjustment

Course Objects – Topics (2)

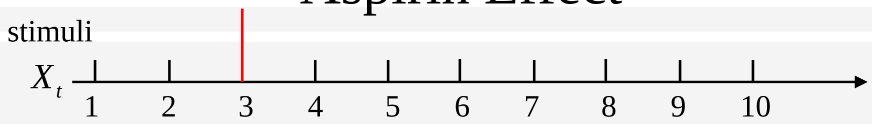
- Models with nonhomogeneous variances
  - ARCH & GARCH Models, others
- Adaptive modeling
  - Nonlinear Models
- Introduction to multiple time series

14 HALF DAYS!

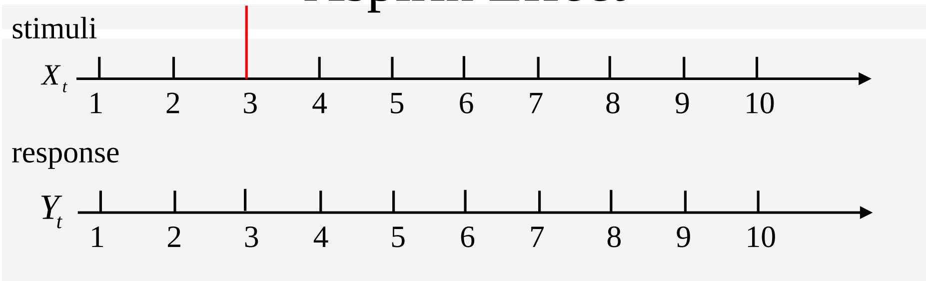
Aspirin Effect



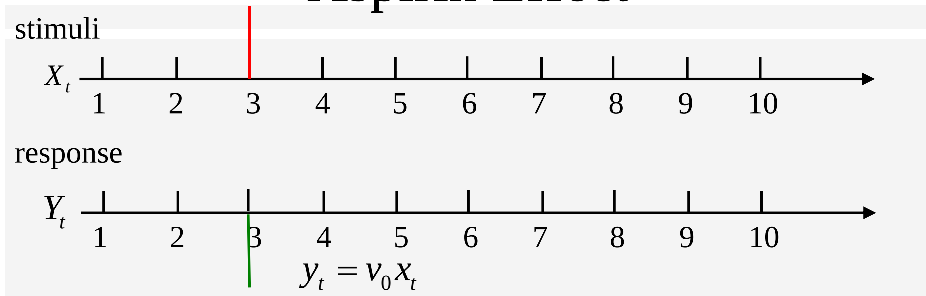
Aspirin Effect



Aspirin Effect



Aspirin Effect



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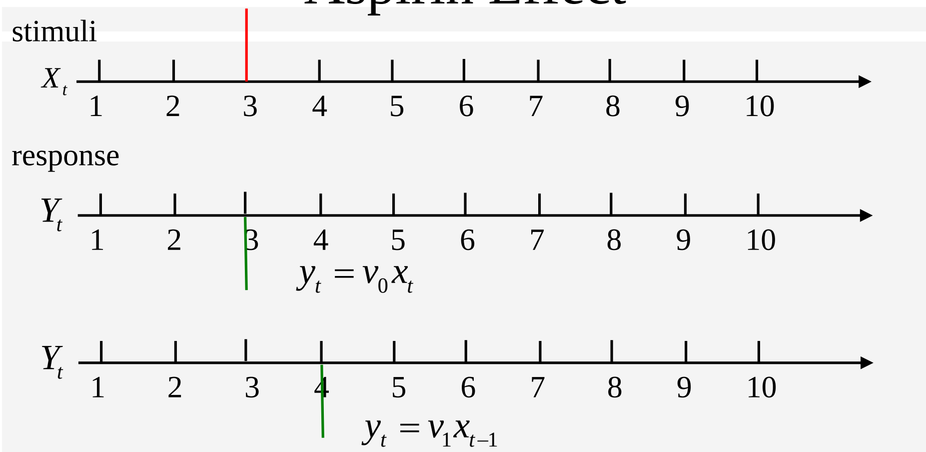

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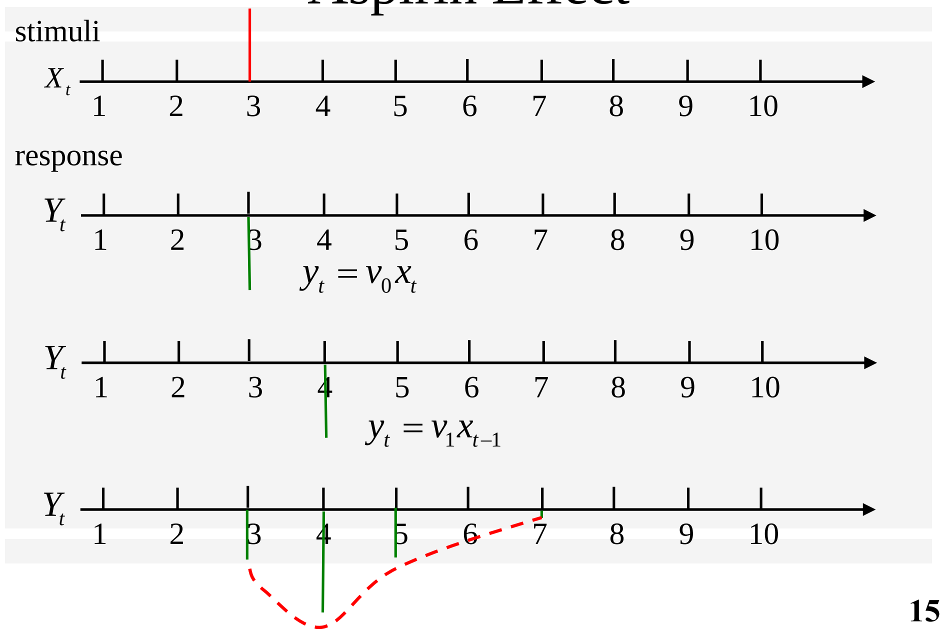

Aspirin Effect

$$Y_t = \nu_0 X_t + \nu_1 X_{t-1} + \nu_2 X_{t-2} + \nu_3 X_{t-3}$$

Aspirin Effect

$$Y_t = \nu_0 X_t + \nu_1 X_{t-1} + \nu_2 X_{t-2} + \nu_3 X_{t-3}$$

X_t : 第t期时受到的刺激(stimuli)

Y_t : 当期和过去所受刺激的影响之和

Aspirin Effect

$$Y_t = \nu_0 X_t + \nu_1 X_{t-1} + \nu_2 X_{t-2} + \nu_3 X_{t-3}$$

X_t : 第t期受到的刺激(stimuli)

Y_t : 当期和过去所受刺激的影响之和

$$\Downarrow$$

$$Y_t = \nu_0 X_t + \nu_1 X_{t-1} + \nu_2 X_{t-2} + \cdots + \nu_j X_{t-j} + \cdots$$

ν_j : j期以前的刺激对现在Y_t的影响 或

现在的刺激对j期以后Y_t的影响

Linear Dynamic Model - 1

$$Y_t = \nu(B) X_t$$

- B在计量经济学中常写作L
- $BX_t = X_{t-1}$
- $B^2 X_t = BX_{t-1} = X_{t-2}$
- $\cdots$

Linear Dynamic Model - 2

$$Y_t = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots)X_t = \nu(B)X_t$$

Linear Dynamic Model - 2

$$Y_t = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots)X_t = \nu(B)X_t$$

- ν_j
  - effect of X_{t-j} on Y_t
  - j-th impulse response

Linear Dynamic Model - 2

$$Y_t = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots)X_t = \nu(B)X_t$$

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  - j-th impulse response

- $\nu(B)$
  - $\nu(B) = \nu_0 + \nu_1 B + \nu_2 B^2 + \cdots$
  - transfer function

Linear Dynamic Model - 2

Linear Dynamic Model

$$Y_t = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots)X_t = \nu(B)X_t$$

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Linear Dynamic Model - 2

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- ν_j
  - effect of X_{t-j} on Y_t
  - j-th impulse response
- $\nu(B)$
  - $\nu(B) = \nu_0 + \nu_1 B + \nu_2 B^2 + \cdots$
  - transfer function

计量经济学中称为 Distributed Lags Model

Linear Dynamic Model - 2

Linear Dynamic Model

$$Y_t = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots)X_t = \nu(B)X_t$$

- ν_j Memory Function
  - effect of X_{t-j} on Y_t
  - j-th impulse response
- $\nu(B)$
  - $\nu(B) = \nu_0 + \nu_1 B + \nu_2 B^2 + \cdots$
  - transfer function

记忆的长短

- 记当期：$Y_t = \nu_0 X_t$ 参数：ν_0
- 记1期：$Y_t = \nu_0 X_t + \nu_1 X_{t-1}$ 参数：(ν_0, ν_1)
- 记2期：$Y_t = \nu_0 X_t + \nu_1 X_{t-1} + \nu_2 X_{t-2}$ 参数：(ν_0, ν_1, ν_2)
- $\cdots$

参数越来越多,估计越来越困难

记忆的长短

$$\begin{aligned} Y_t &= X_t + .5X_{t-1} + .25X_{t-2} + .125X_{t-3} + .0625X_{t-4} + \cdots \\ &\Downarrow \quad (\text{令 } \phi = 0.5) \\ Y_t &= X_t + \phi X_{t-1} + \phi^2 X_{t-2} + \phi^3 X_{t-3} + \phi^4 X_{t-4} + \cdots \end{aligned}$$

Model Coefficients $\longrightarrow$ ARIMA Models

Linear Dynamic Model - 2

Linear Dynamic Model

$$Y_t = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots) X_t = \nu(B) X_t$$

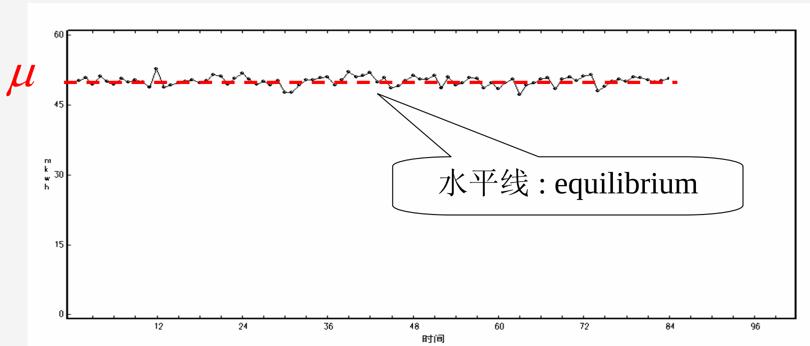
- ν_j
  - effect of X_{t-j} on Y_t
  - j-th impulse response
- $\nu(B)$
  - $\nu(B) = \nu_0 + \nu_1 B + \nu_2 B^2 + \cdots$
  - transfer function

记忆类型的观念架构

参见Packet第111页

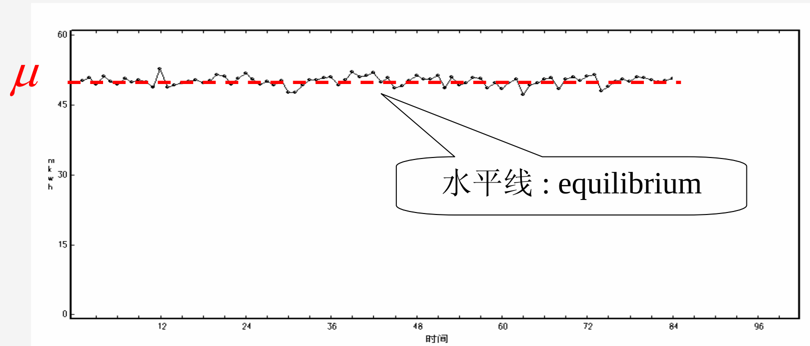
记忆类型的观念架构

参见Packet第111页



记忆类型的观念架构

参见Packet第111页



$$\begin{aligned} Z_t &= \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\ a_t &\sim N(0, \sigma_a^2) \end{aligned}$$

记忆类型的观念架构

$$
\begin{aligned}
Z_t &= \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
a_t &\sim N(0, \sigma_a^2)
\end{aligned}
$$

Possible conceptual formulation as to how different patterns arise

$$
Z_t = \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots
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记忆类型的观念架构

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Z_t &= \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
a_t &\sim N(0, \sigma_a^2)
\end{aligned}
$$

Possible conceptual formulation as to how different patterns arise

$$
\begin{aligned}
Z_t &= \mu + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&= \mu + a_t + \psi_1 B a_t + \psi_2 B^2 a_t + \psi_3 B^3 a_t + \cdots \\
&= \mu + (1 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) a_t
\end{aligned}
$$

No Memory

$$
Z_t = \mu + a_t
$$

A Simple Example

$$
Z_t = \mu + a_t + \psi_1 a_{t-1}
$$

A Simple Example

$$Z_t = \mu + a_t + \psi_1 a_{t-1}$$

- MA(1) model

A Simple Example

$$Z_t = \mu + a_t + \psi_1 a_{t-1}$$

- MA(1) model
- a short memory model
  since $\psi_2 = \psi_3 = \psi_4 = \cdots = 0$ and $\psi_1 \neq 0$

A Simple Example

$$Z_t = \mu + a_t + \psi_1 a_{t-1}$$

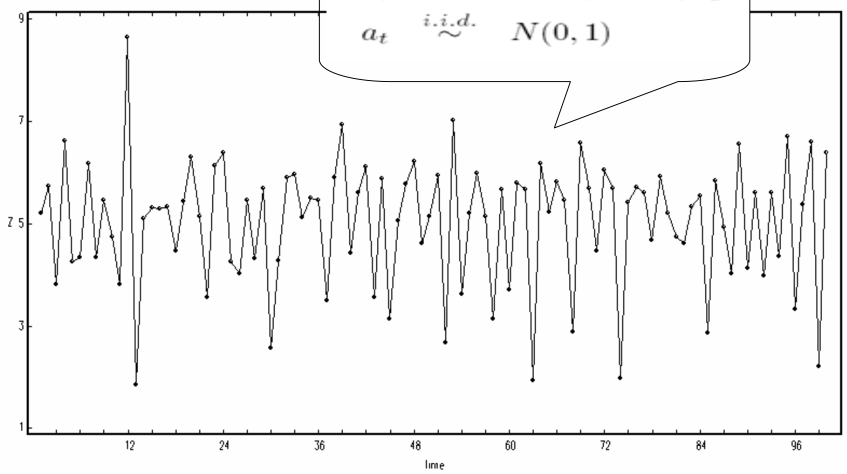
- MA(1) model
- a short memory model
  since $\psi_2 = \psi_3 = \psi_4 = \cdots = 0$ and $\psi_1 \neq 0$
- $Z_t = \mu + a_t - \theta a_{t-1} \Rightarrow \theta = -\psi_1$

A Generated MA(1) Example

$$\begin{aligned} Z_t &= 5 + a_t - .7a_{t-1} \\ a_t &\overset{i.i.d.}{\sim} N(0,1) \end{aligned}$$

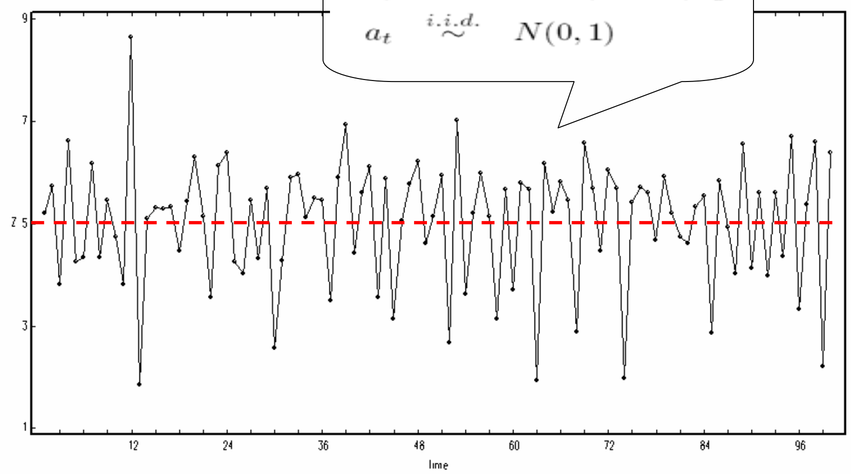
A Generated MA(1) Example

$$Z_t = 5 + a_t - .7a_{t-1}$$
$$a_t \overset{i.i.d.}{\sim} N(0,1)$$



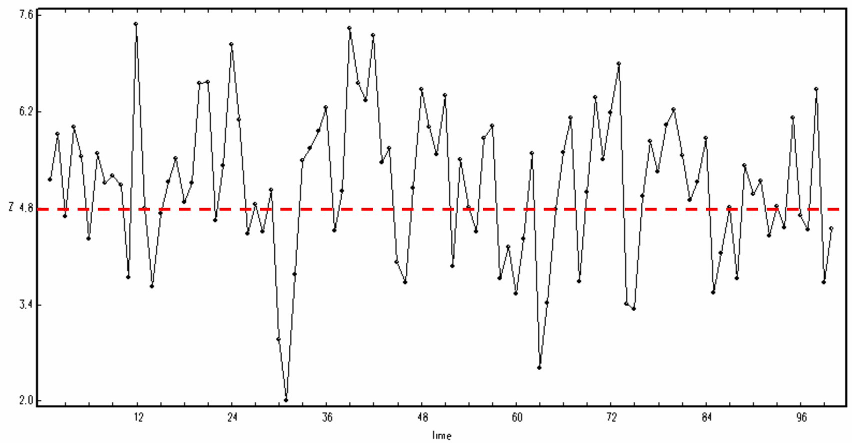
A Generated MA(1) Example

$$Z_t = 5 + a_t - .7a_{t-1}$$
$$a_t \overset{i.i.d.}{\sim} N(0,1)$$

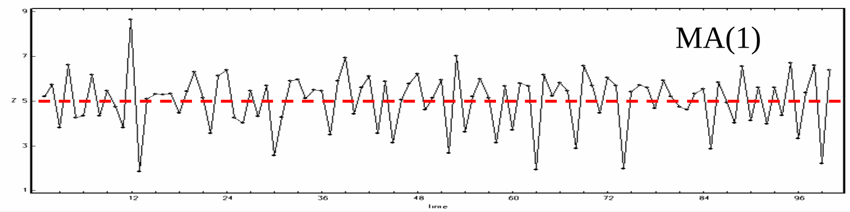


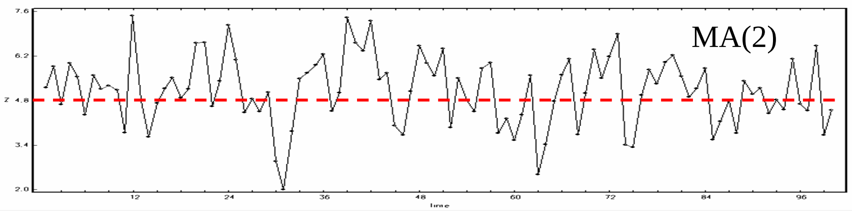
A Generated MA(2) Example

$$Z_t = 5 + a_t + .3a_{t-1} - .4a_{t-2}$$
$$a_t \overset{i.i.d.}{\sim} N(0,1)$$



MA(1) v.s. MA(2)





总结

时间序列分析

第二讲(上)

Notes : from page 109 to page 232

Notes : from page 109 to page 232

Examples: from page 233 to page 373

Notes : from page 109 to page 232

Examples: from page 233 to page 373

Text Recommended：

- ✓ Box, G., Jenkins, G. and Reinsel, R. (1994)
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Notes : from page 109 to page 232

Examples: from page 233 to page 373

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Notes : from page 109 to page 232

Examples: from page 233 to page 373

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  Addison Wesley
- ✓ Pena, D, Tiao, G. C., and Tsay, R.S. (2001)
  A Course in Time Series Analysis （1st Edition)
  John Wiley &Sons

Today Topics

Conceptual Formulation
Stochastic Processes – Stationary P113~114
Stationary Gaussian
ARMA
MA(1)
MA(2)
MA(q)
ACF & SACF ← Var
AR(1) ← Cov. Structure
AR(2)

Stochastic Process

Stochastic Process:

A sequence of Random Valuable with specified prob. Structure

See page 113 of notes

Stochastic Process

Stochastic Process:

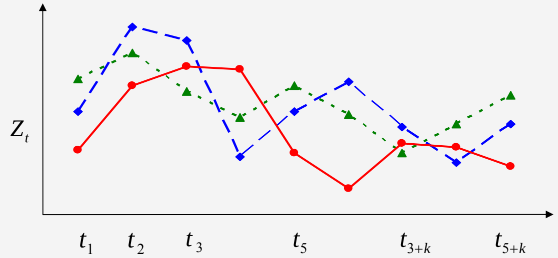
A sequence of Random Valuable
with specified prob. Structure

$P\,(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_k}) \equiv$

Stationary *process*

See page 113 of notes

平稳时间序列



Stochastic Process

Stochastic Process:

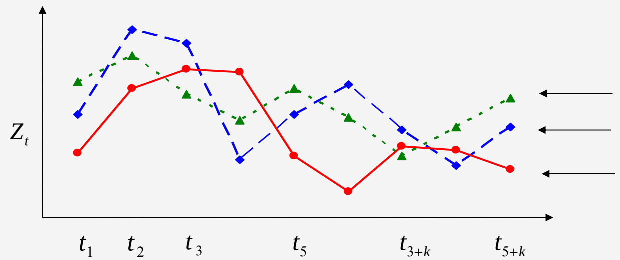
A sequence of Random Valuable
with specified prob. Structure

Stationary Process:

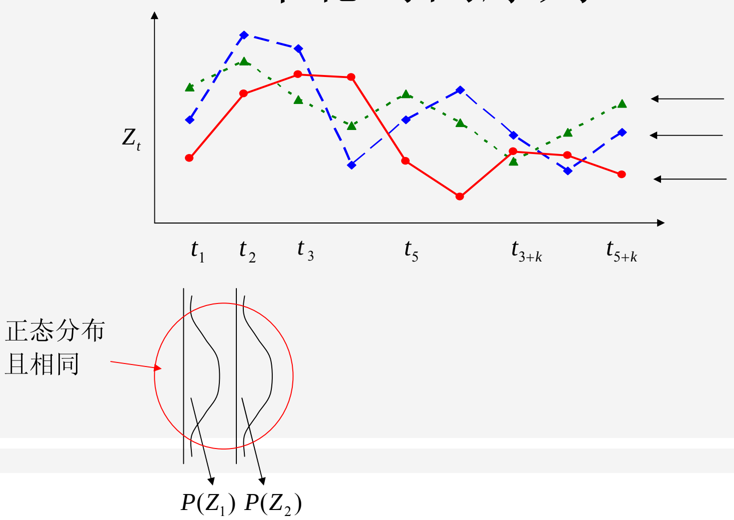
$$P(Z_{t_1}, Z_{t_2}, \cdots, Z_{t_\ell}) \equiv P(Z_{t_1+k}, Z_{t_2+k}, \cdots, Z_{t_\ell+k}) \quad \forall t, k$$

See page 114 of notes

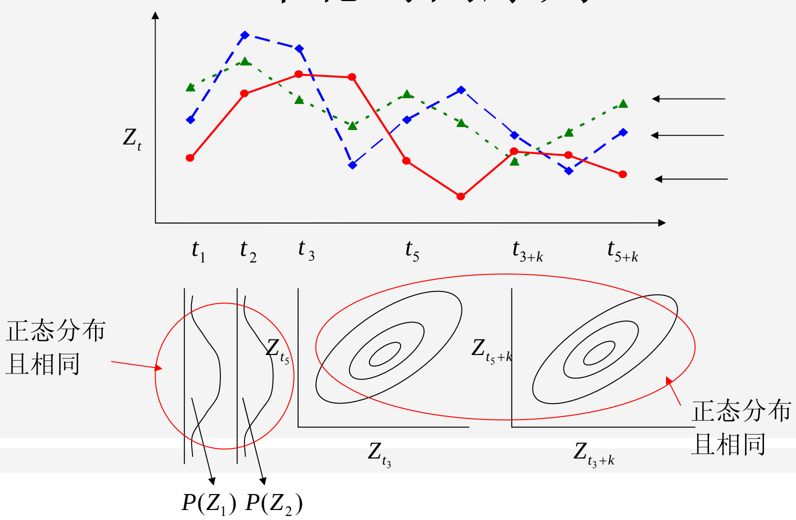
平稳时间序列



平稳时间序列



平稳时间序列



协方差系数和相关系数

协方差系数： $\gamma_0 = \text{cov}(Z_t, Z_t)$

$\gamma_k = \text{cov}(Z_t, Z_{t-k}) \quad k \geq 0$

Depend on $|K|$:K , -K

协方差系数和相关系数

协方差系数： $\gamma_0 = \text{cov}(Z_t, Z_t)$

$\gamma_k = \text{cov}(Z_t, Z_{t-k}) \quad k \geq 0$

Depend on $|K|$:K , -K

相关系数矩阵：

$$Cov\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \ddots & \ddots & \vdots \\ \gamma_2 & \gamma_1 & \gamma_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \gamma_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \gamma_1 \\ \gamma_{n-1} & \cdots & \cdots & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}$$

协方差系数和相关系数

协方差系数：$\gamma_0 = \text{cov}(Z_t, Z_t)$

$\gamma_k = \text{cov}(Z_t, Z_{t-k}) \qquad k \geq 0$

Depend on $|K|$:K , -K

相关系数矩阵：

$$\gamma_k = \gamma_{-k} \qquad Cov\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \ddots & \ddots & \vdots \\ \gamma_2 & \gamma_1 & \gamma_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \gamma_2 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \gamma_1 \\ \gamma_{n-1} & \cdots & \cdots & \gamma_2 & \gamma_1 & \gamma_0 \end{pmatrix}$$

协方差系数和相关系数

协方差系数：$\gamma_0 = \text{cov}(Z_t, Z_t)$

$\gamma_k = \text{cov}(Z_t, Z_{t-k}) \qquad k \geq 0$

Depend on $|K|$:K , -K

相关系数矩阵：

$$\gamma_k = \gamma_{-k} \qquad Cov\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{pmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_k & \gamma_{k-1} & \gamma_{k-2} & \cdots & \gamma_0 \end{bmatrix}$$

相关系数：$\rho_k = \dfrac{\gamma_k}{\gamma_0}$

Conceptual Formulation

Conceptual Formulation ARIMA

$$Z_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

$$a_t \quad iid \quad N(0, \sigma_a^2)$$

Conceptual Formulation

Conceptual Formulation ARIMA

$$Z_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

$$a_t \quad iid \quad N(0, \sigma_a^2)$$

Stationary if $\sum \psi_j^2 < \infty$

Memory long $\rightarrow$ many ψ's

Conceptual Formulation

Conceptual Formulation ARIMA

$$Z_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

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Memory long $\rightarrow$ many ψ's

Simplify by making ψ's functionally related

$$\psi_1 = \phi \qquad \psi_2 = \phi^2 \qquad \psi_3 = \phi^3 \qquad \psi_4 = \phi^4$$

Conceptual Formulation

Conceptual Formulation ARIMA

$$Z_t - \mu = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

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Simplify by making ψ's functionally related

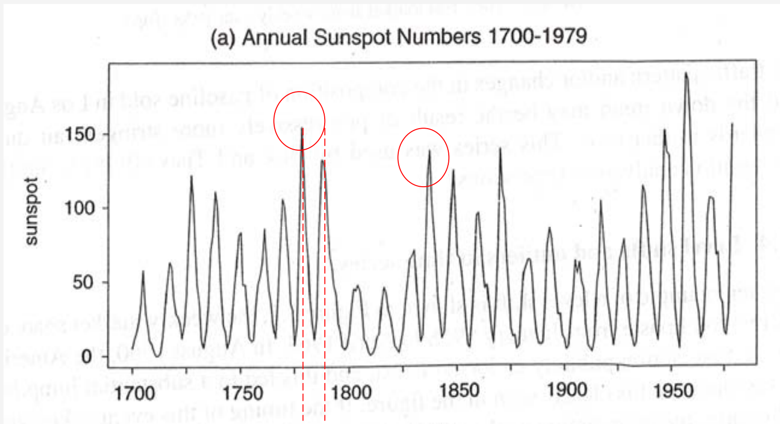
$$\psi_1 = \phi \qquad \psi_2 = \phi^2 \qquad \psi_3 = \phi^3 \qquad \psi_4 = \phi^4$$

$\longrightarrow$ ARIMA 由此而来

ARMA

Yule (1927)

ARMA



about 11years

ARMA

Yule (1927)

$$Z_t = \alpha_1 Z_{t-1} + \alpha_2 Z_{t-2} + e_t$$

ARMA

Yule (1927)

$$Z_t = \alpha_1 Z_{t-1} + \alpha_2 Z_{t-2} + e_t$$

Slutsky (1927)

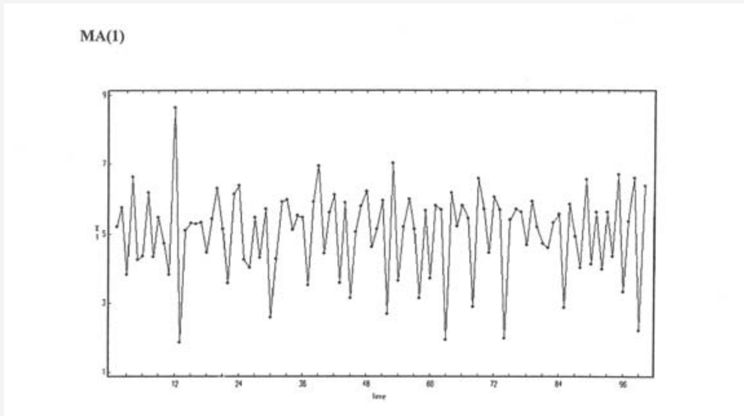
extend to MA models

ARMA

MA(1)

$$Z_t = \mu + a_t - \theta a_{t-1}$$

ARMA



see page 233~236 of notes

ARMA

MA(1)

$Z_t = \mu + a_t - \theta a_{t-1}$

$Z_t = a_t - \theta a_{t-1} \qquad Z_{t-1} = a_{t-1} - \theta a_{t-2} \qquad Z_{t-2} = a_{t-2} - \theta a_{t-3}$

ARMA

MA(1)

$Z_t = \mu + a_t - \theta a_{t-1}$

$Z_t = a_t - \theta a_{t-1} \qquad Z_{t-1} = a_{t-1} - \theta a_{t-2} \qquad Z_{t-2} = a_{t-2} - \theta a_{t-3}$

$\gamma_0 = (1+\theta^2)\sigma_a^2 \qquad \gamma_1 = Cov(Z_t, Z_{t-1}) = -\theta\sigma_a^2$

$\gamma_2 = 0 \qquad \gamma_3 = 0 \qquad \gamma_k = 0 \qquad k \geq 2$

ARMA

MA(1)

$Z_t = \mu + a_t - \theta a_{t-1}$

$Z_t = a_t - \theta a_{t-1} \qquad Z_{t-1} = a_{t-1} - \theta a_{t-2} \qquad Z_{t-2} = a_{t-2} - \theta a_{t-3}$

$\gamma_0 = (1+\theta^2)\sigma_a^2 \qquad \gamma_1 = Cov(Z_t, Z_{t-1}) = -\theta\sigma_a^2$

$\gamma_2 = 0 \qquad \gamma_3 = 0 \qquad \gamma_k = 0 \qquad k \geq 2$

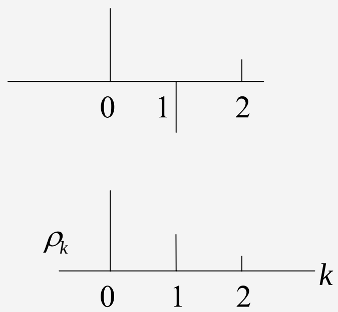
$$Cov = \sigma_a^2 \begin{bmatrix} 1+\theta^2 & -\theta & \cdots & 0 \\ -\theta & 1+\theta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & -\theta \\ 0 & \cdots & -\theta & 1+\theta^2 \end{bmatrix}$$

see page 115~116 of notes

ARMA

ACF auto correlation function: MA(1)

$$\rho_k = \begin{cases} \dfrac{-\theta}{1+\theta^2} & k = 1 \\ 0 & k \geq 2 \end{cases}$$



ARMA

$$Cov = \sigma_a^2 \begin{bmatrix} 1+\theta^2 & -\theta & \cdots & 0 \\ -\theta & 1+\theta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & -\theta \\ 0 & \cdots & -\theta & 1+\theta^2 \end{bmatrix}$$

$$\Downarrow$$

$$= \sigma_a^2\theta^2 \begin{bmatrix} \frac{1}{\theta^2}+1 & -\frac{1}{\theta} & \cdots & 0 \\ -\frac{1}{\theta} & \frac{1}{\theta^2}+1 & \cdots & 0 \\ \vdots & \vdots & \ddots & -\frac{1}{\theta} \\ 0 & \cdots & -\frac{1}{\theta} & \frac{1}{\theta^2}+1 \end{bmatrix}$$

$$\eta = \frac{1}{\theta} \Downarrow$$

$$= \sigma_\eta^2 \begin{bmatrix} \eta^2+1 & -\eta & \cdots & 0 \\ -\eta & \eta^2+1 & \cdots & 0 \\ \vdots & \vdots & \ddots & -\eta \\ 0 & \cdots & -\eta & \eta^2+1 \end{bmatrix}$$

$$\Downarrow$$

ARMA

$$Z_t = b_t - \eta b_{t-1}$$

$$b_t \quad iid \quad N(0,\sigma_\eta^2)$$

$$\sigma_\eta^2 = \theta^2\sigma_a^2$$

ARMA

$$Z_t = b_t - \eta b_{t-1} \qquad Z_t = a_t - \theta a_{t-1}$$

$$b_t \quad iid \quad N(0,\sigma_\eta^2) \qquad a_t \quad iid \quad N(0,\sigma_a^2)$$

$$\sigma_\eta^2 = \theta^2\sigma_a^2$$

two equivalent representations

Invertibility question

$$Z_t = a_t - \theta a_{t-1}$$
$$= (1-\theta B)a_t \quad \Leftarrow \quad a_{t-1} = \theta B a_t$$

ARMA

$$Z_t = a_t - \theta a_{t-1}$$

$$Z_t = (1-\theta B)a_t$$
$$|\theta| < 1$$

$$\begin{cases} a_t = Z_t + \theta a_{t-1} \\ a_{t-1} = Z_{t-1} + \theta a_{t-2} \end{cases}$$

$$\begin{aligned} a_t &= Z_t + \theta(Z_{t-1} + \theta a_{t-2}) \\ &= Z_t + \theta Z_{t-1} + \theta^2 a_{t-2} \\ &= Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \theta^3 a_{t-3} \end{aligned}$$

ARMA

$$Z_t = a_t - \theta a_{t-1}$$

$$Z_t = (1 - \theta B) a_t$$

$$|\theta| < 1$$

$$\begin{cases} a_t = Z_t + \theta a_{t-1} \\ a_{t-1} = Z_{t-1} + \theta a_{t-2} \end{cases}$$

$$\begin{aligned} a_t &= Z_t + \theta(Z_{t-1} + \theta a_{t-2}) \\ &= Z_t + \theta Z_{t-1} + \theta^2 a_{t-2} \\ &= Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \theta^3 a_{t-3} \end{aligned}$$

$$(1 - \theta B)^{-1} Z_t = a_t$$

If $|\theta| < 1$

$$(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \cdots) Z_t = a_t$$

$$a_t = Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \theta^3 Z_{t-3} + \cdots$$

$$\Downarrow$$

$$Z_t = -\theta Z_{t-1} - \theta^2 Z_{t-2} - \theta^3 Z_{t-3} - \cdots + a_t$$

ARMA

$$Z_t = a_t - \theta a_{t-1}$$

$$Z_t = (1 - \theta B) a_t$$

$$|\theta| < 1$$

$$\begin{cases} a_t = Z_t + \theta a_{t-1} \\ a_{t-1} = Z_{t-1} + \theta a_{t-2} \end{cases}$$

$$\begin{aligned} a_t &= Z_t + \theta(Z_{t-1} + \theta a_{t-2}) \\ &= Z_t + \theta Z_{t-1} + \theta^2 a_{t-2} \\ &= Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \theta^3 a_{t-3} \end{aligned}$$

$$(1 - \theta B)^{-1} Z_t = a_t$$

If $|\theta| < 1$

$$(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \cdots) Z_t = a_t$$

$$a_t = Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \theta^3 Z_{t-3} + \cdots$$

invertible form

$$\Downarrow$$

$$Z_t = -\theta Z_{t-1} - \theta^2 Z_{t-2} - \theta^3 Z_{t-3} - \cdots + a_t$$

时间序列分析

第二讲(下)

ARMA

$$Z_t = a_t - \theta a_{t-1}$$

$$Z_t = (1-\theta B)a_t$$

$$|\theta| < 1$$

$$\begin{cases} a_t = Z_t + \theta a_{t-1} \\ a_{t-1} = Z_{t-1} + \theta a_{t-2} \end{cases}$$

$$\begin{aligned} a_t &= Z_t + \theta(Z_{t-1} + \theta a_{t-2}) \\ &= Z_t + \theta Z_{t-1} + \theta^2 a_{t-2} \\ &= Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \theta^3 a_{t-3} \end{aligned}$$

$$(1-\theta B)^{-1} Z_t = a_t$$

If $|\theta| < 1$

$$(1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \cdots) Z_t = a_t$$

$$a_t = Z_t + \theta Z_{t-1} + \theta^2 Z_{t-2} + \theta^3 Z_{t-3} + \cdots$$

invertible form

$$\Downarrow$$

$$Z_t = -\theta Z_{t-1} - \theta^2 Z_{t-2} - \theta^3 Z_{t-3} - \cdots + a_t$$

ARMA

Autocovariance Generating Function

ARMA

Autocovariance Generating Function

$$\gamma(B) = (1-\theta B)(1-\theta F)\sigma_a^2$$

For example: MA(1) $\theta \Leftarrow Z_t = (1-\theta B)a_t$

$$a_t \quad iid \quad N(0, \sigma_a^2)$$

$$F = Forward \quad Shift \quad Operator$$

$$= B^{-1}$$

ARMA

Autocovariance Generating Function

$$\gamma(B) = (1-\theta B)(1-\theta F)\sigma_a^2$$

For example: MA(1) $\quad Z_t = (1-\theta B)a_t$

$$a_t \quad iid \quad N(0, \sigma_a^2)$$

$$F = Forward \quad Shift \quad Operator$$

$$= B^{-1}$$

$$\gamma(B) = \sigma_a^2 \left[1 - \theta(B+F) + \theta^2\right]$$

$$= \sigma_a^2 \left[(1+\theta^2) - \theta(B+F)\right]$$

$$= \gamma_0 + \gamma_1(B+F)$$

ARMA

In general

$$\gamma(B) = \gamma_0 + \gamma_1(B+F) + \gamma_2(B^2+F^2) + \gamma_3(B^3+F^3) + \cdots$$

ARMA

In general

$$\gamma(B) = \gamma_0 + \gamma_1(B+F) + \gamma_2(B^2+F^2) + \gamma_3(B^3+F^3) + \cdots$$

$$\frac{1}{\gamma_0}\gamma(B) = 1 + \rho_1(B+F) + \rho_2(B^2+F^2) + \rho_3(B^3+F^3) + \cdots$$

(autocorrelation generating function)

ARMA

In general

$$\gamma(B) = \gamma_0 + \gamma_1(B+F) + \gamma_2(B^2+F^2) + \gamma_3(B^3+F^3) + \cdots$$

$$\frac{1}{\gamma_0}\gamma(B) = 1 + \rho_1(B+F) + \rho_2(B^2+F^2) + \rho_3(B^3+F^3) + \cdots$$

(autocorrelation generating function)

$$B = e^{-i\pi f}$$

$$\gamma_0 + \gamma_1(B+F) = [\gamma_0 + 2\gamma_1 \cos\ 2\pi f\ ]$$

ARMA

In general

$$\gamma(B) = \gamma_0 + \gamma_1(B+F) + \gamma_2(B^2+F^2) + \gamma_3(B^3+F^3) + \cdots$$

$$\frac{1}{\gamma_0}\gamma(B) = 1 + \rho_1(B+F) + \rho_2(B^2+F^2) + \rho_3(B^3+F^3) + \cdots$$

(autocorrelation generating function)

$$B = e^{-i\pi f}$$

$$\gamma_0 + \gamma_1(B+F) = [\gamma_0 + 2\gamma_1 \cos 2\pi f]$$

$2\gamma(B) = P(f)$ (spectrum density function)

See page 118 of notes

ARMA

Stationary Gaussian Process

$\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \cdots$

$\gamma_0 \quad \rho_1 \quad \rho_2 \quad \rho_3 \quad \cdots$

Learn it yourself

ARMA

Stationary Gaussian Process

$\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \cdots$

$\gamma_0 \quad \rho_1 \quad \rho_2 \quad \rho_3 \quad \cdots$

MA(1)

$$Z_t = (1-\theta B)a_t$$

系数 (θ, σ_a^2)

Learn it yourself

ARMA

Stationary Gaussian Process

$\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \cdots$

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the series parameters & lag moments 关系何在？

Learn it yourself

ARMA

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$\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \cdots$

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MA(1)

$$Z_t = (1 - \theta B) a_t$$

系数 (θ, σ_a^2)

the series parameters & lag moments 关系何在?

$$\boxed{\begin{aligned} \gamma_0 &= (1 + \theta^2)\sigma_a^2 \\ \gamma_1 &= -\theta\sigma_a^2 \end{aligned}}$$

$$\gamma_k = 0 \quad k \geq 2$$

Learn it yourself

ARMA

Stationary Gaussian Process

$\gamma_0 \quad \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \cdots$

$\gamma_0 \quad \rho_1 \quad \rho_2 \quad \rho_3 \quad \cdots$

MA(1)

$$Z_t = (1 - \theta B) a_t$$

系数 (θ, σ_a^2)

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$$\boxed{\begin{aligned} \gamma_0 &= (1 + \theta^2)\sigma_a^2 \\ \gamma_1 &= -\theta\sigma_a^2 \end{aligned}}$$

$$\gamma_k = 0 \quad k \geq 2$$

If know (θ, σ_a^2) , get all moments

If know $\gamma_0 \ \gamma_1 \ \gamma_2 \ \cdots$, how to get (θ, σ_a^2) ?

Learn it yourself

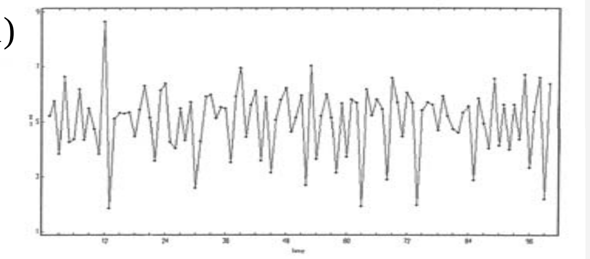
ARMA

MA(2)

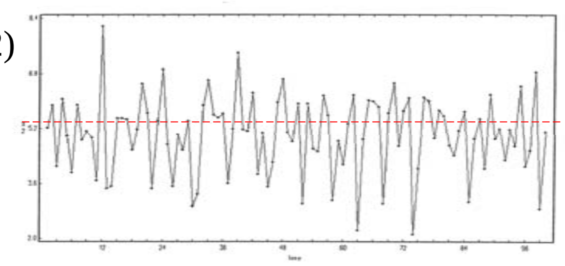
$$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

ARMA

MA(1)



MA(2)



ARMA

MA(2)

$$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

$$Z_{t-1} = a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3}$$

$$Z_{t-2} = a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4}$$

ARMA

MA(2)

$$Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

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$$Z_{t-2} = a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4}$$

$$\rho_1 \overset{?}{\neq} 0 \qquad \boxed{\rho_2 \neq 0} \qquad \rho_k = 0 \quad k \geq 3$$

ARMA

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$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \neq 0$$

ARMA

MA(2)

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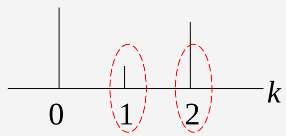
$$\gamma(B) = (1 - \theta_1 B - \theta_2 B^2)(1 - \theta_1 F - \theta_2 F^2)\sigma_a^2$$



ARMA

Inverted form

$$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$$

$$(1-\theta_1 B - \theta_2 B^2)^{-1} Z_t = a_t$$

ARMA

Inverted form

$$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$$

$$(1-\theta_1 B - \theta_2 B^2)^{-1} Z_t = a_t$$

$$(1-\alpha_1 B)(1-\alpha_2 B)^{-1} Z_t = a_t$$

ARMA

Inverted form

$$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$$

$$(1-\theta_1 B - \theta_2 B^2)^{-1} Z_t = a_t$$

$$(1-\alpha_1 B)(1-\alpha_2 B)^{-1} Z_t = a_t$$

$$\pi(B) Z_t = a_t$$

$$\pi(B) = (1-\theta_1 B - \theta_2 B^2)^{-1}$$

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \cdots$$?

ARMA

Inverted form

$$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$$

$$(1-\theta_1 B - \theta_2 B^2)^{-1} Z_t = a_t$$

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$$\pi(B) = (1-\theta_1 B - \theta_2 B^2)^{-1}$$

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \cdots$$?

$$(1-\theta_1 B - \theta_2 B^2)(\pi_0 + \pi_1 B + \pi_2 B^2 + \pi_3 B^3 + \cdots) = 1$$

see page 119 of notes

see page 95~108 of notes

ARMA

why the inverted form

ARMA

why the inverted form

$$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$$

ARMA

why the inverted form

$$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$$

$$\pi(B)Z_t = a_t$$

$$\pi(B) = 1-\pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots$$

$$Z_t - \pi_1 Z_{t-1} - \pi_2 Z_{t-2} - \pi_3 Z_{t-3} - \cdots = a_t$$

$$Z_t = \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \pi_3 Z_{t-3} + \cdots + a_t$$

ARMA

$$Z_t = (1-\theta_1 B - \theta_2 B^2)a_t$$

Relations between Z_t and shocks a_t's

why the inverted form

$$\pi(B)Z_t = a_t$$

$$\pi(B) = 1-\pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots$$

$$Z_t - \pi_1 Z_{t-1} - \pi_2 Z_{t-2} - \pi_3 Z_{t-3} - \cdots = a_t$$

$$Z_t = \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \pi_3 Z_{t-3} + \cdots + a_t$$

Relations between Z_t and lag Z_t's

usefulness of the π form

relate Z_t to past values of Z_t plus current noise a_t

ACF &SACF

conceptual model

see page 121 of notes

ACF & SACF

conceptual model

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

stationary condition $\begin{cases} a_t \quad iid \quad N(0,\sigma_a^2) \\ \sum \psi_j^2 < \infty \end{cases}$

see page 121 of notes

ACF & SACF

conceptual model

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

stationary condition $\begin{cases} a_t \quad iid \quad N(0,\sigma_a^2) \\ \sum \psi_j^2 < \infty \end{cases}$

$$\gamma_0 = (1+\psi_1^2+\psi_2^2+\cdots)\sigma_a^2$$

$$\gamma_1 = E(a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\cdots)(a_{t-1}+\psi_1 a_{t-2}+\psi_2 a_{t-3}+\cdots)$$

$$= \sigma_a^2(\psi_1+\psi_1\psi_2+\psi_2\psi_3+\cdots)$$

see page 121 of notes

ACF & SACF

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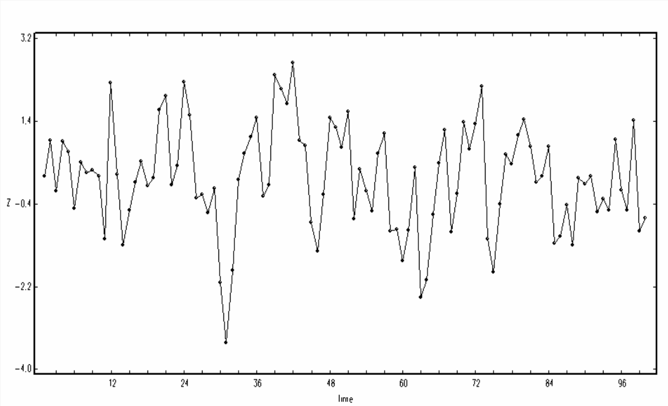
$\Rightarrow \gamma_k$

see page 121 of notes

ACF & SACF



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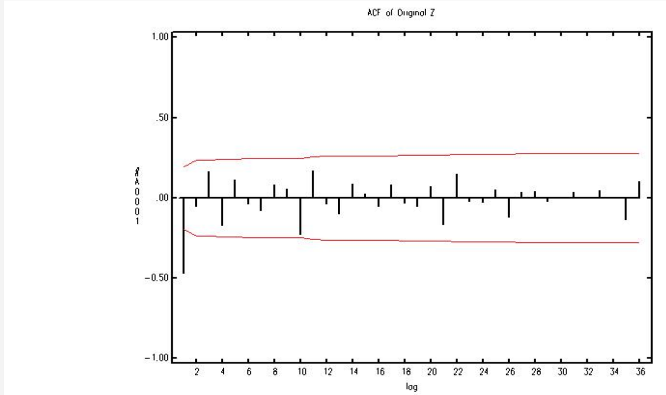
ACF & SACF



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see page 239~240 of notes

ACF & SACF

complex formulate to $\mathrm{Var}(r_k)$

ACF & SACF

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if all $\rho_k = 0$ $\quad \mathrm{Var}(r_k) \approx \dfrac{1}{n}$

see page 122 of notes

ACF & SACF

complex formulate to $\mathrm{Var}(r_k)$

if all $\rho_k = 0$ $\quad \mathrm{Var}(r_k) \approx \frac{1}{n}$

if $\begin{cases} \rho_l = 0 & \text{for } l > q \\ \rho_l \neq 0 & \text{for } l = q \end{cases}$

then for $k > q$ $\quad \mathrm{Var}(r_k) \doteq \frac{1}{n}[1 + 2(\rho_1^2 + \cdots + \rho_q^2)]$

see page 122 of notes

ACF & SACF

complex formulate to $\mathrm{Var}(r_k)$

if all $\rho_k = 0$ $\quad \mathrm{Var}(r_k) \approx \frac{1}{n}$

if $\begin{cases} \rho_l = 0 & \text{for } l > q \\ \rho_l \neq 0 & \text{for } l = q \end{cases}$

then for $k > q$ $\quad \mathrm{Var}(r_k) \doteq \frac{1}{n}[1 + 2(\rho_1^2 + \cdots + \rho_q^2)]$

see page 240 of notes

see page 122 of notes

ACF & SACF

practical use

1st assume MA(0) $\quad \mathrm{Var}(r_k) \approx \frac{1}{n}$

ACF & SACF

practical use

1st assume MA(0) $\quad \mathrm{Var}(r_k) \approx \frac{1}{n}$

then assume MA(1) for $k \geq 2$

$$\mathrm{Var}(r_k) \approx \frac{1}{n}[1 + 2 \times (0.47^2)] = \frac{1.44}{100}$$

ACF & SACF

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

ACF & SACF

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

e.g. for a long but fading memory the model might be

$$Z_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} + \cdots$$

ACF & SACF

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ACF & SACF

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$$-\phi Z_{t-1} = -\phi a_{t-1} - \phi^2 a_{t-2} - \phi^3 a_{t-3} - \cdots$$

$$Z_t - \phi Z_{t-1} = a_t$$

ACF & SACF

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

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$$-\phi Z_{t-1} = -\phi a_{t-1} - \phi^2 a_{t-2} - \phi^3 a_{t-3} - \cdots$$

$$Z_t - \phi Z_{t-1} = a_t$$ popular

$$Z_t = \phi Z_{t-1} + a_t$$ AR(1)

ACF & SACF

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

e.g. for a long but fading memory the model might be

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$$Z_t - \phi Z_{t-1} = a_t$$ popular

$$Z_t = \phi Z_{t-1} + a_t$$ AR(1)

$$Z_t = 5 + 0.7(Z_{t-1} - 5) + a_t$$

$$Z_t = 0.7 Z_{t-1} + a_t$$

see page 246 of notes

ACF & SACF

$$Z_t - \phi Z_{t-1} = a_t$$

$$(1 - \phi B) Z_t = a_t$$

$$Z_t = (1 - \phi B)^{-1} a_t$$

$$= (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots) a_t$$

时间序列分析

第三讲(上)

上次讲课内容回顾

- Conceptual Formulation
- Stochastic Process – stationary (p113-114)
- Stationary Gaussian
- ARMA
  - MA(1)
  - MA(2)
  - MA(1)
  - ACF + SACF
  - AR(1) Covariance Structure
  - AR(2)

本次讲课内容

- $\text{Var}(\Gamma_k)$
- AR(1)
- AR(2)
- AR(p)
- PACF + SPACF
- ARMA(p,q)
- Random Walk
- 1st difference as MA(1)

Variance of Γ_k

For MA(q) model

$$\because \ell > q \Longrightarrow \rho_\ell = 0$$

$$\therefore \text{ for } k > q, \quad \text{Var}(\Gamma_k) \doteq \frac{1}{n}[1 + 2(\rho_1^2 + \cdots + \rho_q^2)]$$

If Z_k is a white noise, then $\text{Var}(\Gamma_k) = \frac{1}{n}, \forall k$

• See Page 122 of Notes

Variance of Γ_k

$$Z_t = 5 + a_t - .7a_{t-1} \quad a_t \sim N(0,1)$$

```
          -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                      I
  1 -0.47                XXXXXXX+XXXXI     +
  2 -0.05                      +    XI      +
  3  0.16                      +     IXXXX +
  4 -0.17                      + XXXXI      +
  5  0.11                      +     IXXX   +
  6 -0.04                      +    XI      +
  7 -0.08                      +   XXI      +
  8  0.08                      +     IXX    +
  9  0.06                      +     IX     +
 10 -0.23                      XXXXXXI      +
 11  0.17                      +     IXXXX +
 12 -0.04                     +     XI       +
```

• See Page 240 of Notes

Variance of Γ_k

$$Z_t = 5 + a_t - .7a_{t-1} \quad a_t \sim N(0,1)$$

```
          -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                      I
  1 -0.47                XXXXXXX+XXXXI     +
  2 -0.05                      +    XI      +
  3  0.16                      +     IXXXX +
  4 -0.17                      + XXXXI      +
  5  0.11                      +     IXXX   +
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  7 -0.08                      +   XXI      +
  8  0.08                      +     IXX    +
  9  0.06                      +     IX     +
 10 -0.23                      XXXXXXI      +
 11  0.17                      +     IXXXX +
 12 -0.04                     +     XI       +
```

For $Z_t = a_t - \theta a_{t-1}$

$|\theta| < 1 \Longrightarrow |\rho_1| < 0.5$

• See Page 240 of Notes

Variance of Γ_k

$$Z_t = 5 + a_t - .7a_{t-1} \quad a_t \sim N(0,1)$$

```
          -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                      I
  1 -0.47                XXXXXXX+XXXXI     +
  2 -0.05                      +    XI      +
  3  0.16                      +     IXXXX +
  4 -0.17                      + XXXXI      +
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  7 -0.08                      +   XXI      +
  8  0.08                      +     IXX    +
  9  0.06                      +     IX     +
 10 -0.23                      XXXXXXI      +
 11  0.17                      +     IXXXX +
 12 -0.04                     +     XI       +
```

• See Page 240 of Notes

Variance of Γ_k

Assume $\forall j, \rho_j = 0$, then $\mathrm{Var}[\Gamma_k] = \frac{1}{100}$

Variance of Γ_k

Assume $\forall j, \rho_j = 0$, then $\mathrm{Var}[\Gamma_k] = \frac{1}{100}$

But $\mathrm{Var}(\Gamma_1) = .47 > 0.2 = 2\sqrt{0.01} \Longrightarrow \rho_1 \neq 0$

Variance of Γ_k

Assume $\forall j, \rho_j = 0$, then $\mathrm{Var}[\Gamma_k] = \frac{1}{100}$

But $\mathrm{Var}(\Gamma_1) = .47 > 0.2 = 2\sqrt{0.01} \Longrightarrow \rho_1 \neq 0$

Then try MA(1)

Variance of Γ_k

Assume $\forall j, \rho_j = 0$, then $\mathrm{Var}[\Gamma_k] = \frac{1}{100}$

But $\mathrm{Var}(\Gamma_1) = .47 > 0.2 = 2\sqrt{0.01} \Longrightarrow \rho_1 \neq 0$

Then try MA(1)

$$
\begin{aligned}
\mathrm{Var}(\Gamma_k) &\doteq \frac{1}{n}[1 + 2\rho_1^2] \\
&\doteq \frac{1}{n}[1 + 2\Gamma_1^2]
\end{aligned}
\qquad (\text{ k>1 })
$$

AR(1) Model

$$Z_t = 5 + a_t - .7a_{t-1} \quad a_t \sim N(0,1)$$

```
            -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
              +----+----+----+----+----+----+----+----+----+----+
                                       I
  1  -0.47                 XXXXXXX+XXXXI    +
  2  -0.05                       +    XI     +
  3   0.16                       +     IXXXX +
  4  -0.17                       + XXXXI     +
  5   0.11                       +     IXXX  +
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  9   0.06                       +     IX    +
 10  -0.23                       XXXXXXI     +
 11   0.17                       +     IXXXX +
 12  -0.04                      +     XI      +
```

1期以后的
SACF都比较小

• See Page 240 of Notes

AR(1) Model

• See Page 122 of Notes

AR(1) Model

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

• See Page 122 of Notes

AR(1) Model

$$\begin{aligned} Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\ &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} + \cdots \end{aligned}$$

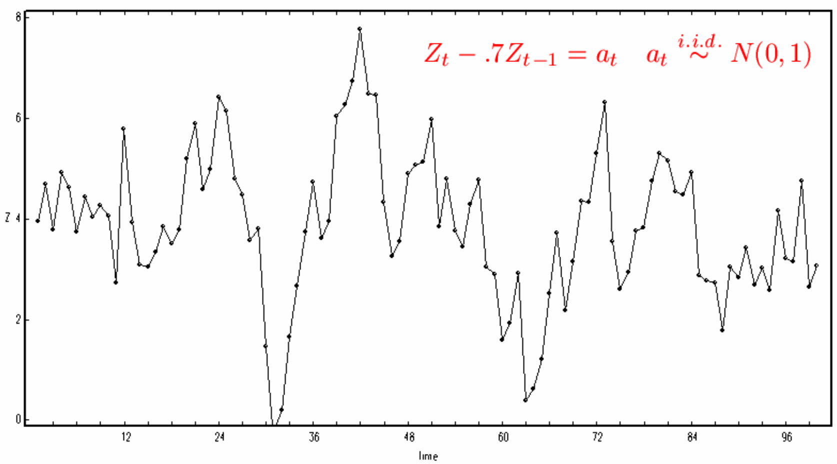
• See Page 122 of Notes

AR(1) Model

$$\begin{aligned} Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\ &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} + \cdots \\ &\Downarrow \\ Z_t &- \phi Z_{t-1} = a_t \qquad \text{Example : } Z_t - .7 Z_{t-1} = a_t \end{aligned}$$

• See Page 122 of Notes

Example of AR(1)



• See Page 244 of Notes

AR(1) Model

$$
\begin{aligned}
Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} + \cdots \\
&\Downarrow \\
Z_t &- \phi Z_{t-1} = a_t \qquad \text{Example : } Z_t - .7Z_{t-1} = a_t \\
&\Updownarrow \\
Z_t &= \phi Z_{t-1} + a_t
\end{aligned}
$$

• See Page 122 of Notes

AR(1) Model

$$
\begin{aligned}
Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} + \cdots \\
&\Downarrow \\
Z_t &- \phi Z_{t-1} = a_t \qquad \text{Example : } Z_t - .7Z_{t-1} = a_t \\
&\Updownarrow \\
Z_t &= \phi Z_{t-1} + a_t
\end{aligned}
$$

Yule Sunspot Data

$Z_t = \beta_1 Z_{t-1} + \beta_2 Z_{t-2} + \epsilon_t$

• See Page 122 of Notes

Covariance Structure of AR(1)

$$Z_t - \phi Z_{t-1} = a_t$$

$$Z_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots$$

What are γ_k and ρ_k ?

• See Page 122-123 on Notes

Covariance Structure of AR(1)

$$
\begin{aligned}
Z_t - \phi Z_{t-1} &= a_t \\
&\Downarrow \\
Z_{t-\ell}(Z_t - \phi Z_{t-1}) &= a_t Z_{t-\ell} \\
&\Downarrow \\
EZ_{t-\ell}(Z_t - \phi Z_{t-1}) &= Ea_t Z_{t-\ell}
\end{aligned}
$$

- See Page 122-123 on Notes

Covariance Structure of AR(1)

When $\ell = 0$:

$$
\begin{aligned}
EZ_{t-\ell}(Z_t - \phi Z_{t-1}) &= Ea_t Z_{t-\ell} \\
&\Downarrow \quad \ell = 0 \\
EZ_t(Z_t - \phi Z_{t-1}) &= Ea_t Z_t \\
&\Downarrow \\
\gamma_0 - \phi\gamma_1 &= \sigma_a^2
\end{aligned}
$$

- See Page 122-123 on Notes

Covariance Structure of AR(1)

$$
\begin{aligned}
Z_t &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots \\
a_t Z_t &= a_t(a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots) \\
Ea_t Z_t &= Ea_t(a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots)
\end{aligned}
$$

$$
\begin{aligned}
&= Ea_t Z_{t-\ell} \\
&\Downarrow \quad \ell = 0 \\
EZ_t(Z_t - \phi Z_{t-1}) &= Ea_t Z_t \\
&\Downarrow \\
\gamma_0 - \phi\gamma_1 &= \sigma_a^2
\end{aligned}
$$

- See Page 122-123 on Notes

Covariance Structure of AR(1)

$$EZ_{t-\ell}(Z_t - \phi Z_{t-1}) = Ea_t Z_{t-\ell}$$

$$
\begin{array}{ll}
\ell = 0 & \gamma_0 - \phi\gamma_1 = \sigma_a^2 \\
\ell = 1 & \gamma_1 - \phi\gamma_0 = 0 \\
\ell = 2 & \gamma_2 - \phi\gamma_1 = 0 \\
\vdots & \vdots \\
\ell > 1 & \gamma_l - \phi\gamma_{l-1} = 0
\end{array}
$$

- See Page 122-123 on Notes

Covariance Structure of AR(1)

$$EZ_{t-\ell}(Z_t - \phi Z_{t-1}) = Ea_t Z_{t-\ell}$$

$$\begin{array}{lll} \ell = 0 & \gamma_0 - \phi\gamma_1 = \sigma_a^2 & \\ \ell = 1 & \gamma_1 - \phi\gamma_0 = 0 & \\ \ell = 2 & \gamma_2 - \phi\gamma_1 = 0 & \\ \vdots & \vdots & \\ \ell > 1 & \gamma_l - \phi\gamma_{l-1} = 0 & \gamma_l = \phi\gamma_{l-1} = \phi^l\gamma_0 \end{array}$$

- See Page 122-123 on Notes

Covariance Structure of AR(1)

$$EZ_{t-\ell}(Z_t - \phi Z_{t-1}) = Ea_t Z_{t-\ell}$$

$$\begin{array}{lll} \ell = 0 & \gamma_0 - \phi\gamma_1 = \sigma_a^2 & \\ \ell = 1 & \gamma_1 - \phi\gamma_0 = 0 & \\ \ell = 2 & \gamma_2 - \phi\gamma_1 = 0 & \\ \vdots & \vdots & \rho_l = \phi^\ell \\ & & \Uparrow \\ \ell > 1 & \gamma_l - \phi\gamma_{l-1} = 0 & \gamma_l = \phi\gamma_{l-1} = \phi^l\gamma_0 \end{array}$$

- See Page 122-123 on Notes

Covariance Structure of AR(1)

$$EZ_{t-\ell}(Z_t - \phi Z_{t-1}) = Ea_t Z_{t-\ell}$$

$$\begin{array}{lll} \ell = 0 & \gamma_0 - \phi\gamma_1 = \sigma_a^2 \Rightarrow & \gamma_0 - \phi^2\gamma_0 = \sigma_a^2 \\ & & \Downarrow \\ \ell = 1 & \gamma_1 - \phi\gamma_0 = 0 & \gamma_0(1 - \phi^2) = \sigma_a^2 \\ \ell = 2 & \gamma_2 - \phi\gamma_1 = 0 & \\ \vdots & \vdots & \rho_l = \phi^\ell \\ & & \Uparrow \\ \ell > 1 & \gamma_l - \phi\gamma_{l-1} = 0 & \gamma_l = \phi\gamma_{l-1} = \phi^l\gamma_0 \end{array}$$

- See Page 122-123 on Notes

Covariance Structure of AR(1)

$$Z_t = \phi Z_{t-1} + a_t$$

$$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2) \text{ and } |\phi| < 1$$

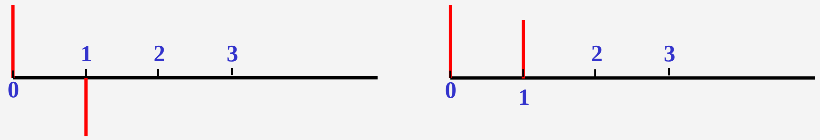
AR(1) 矩与系数的关系:

$$(\gamma_0, \gamma_1, \gamma_2, \cdots) \quad \rightleftarrows \quad (\phi, \sigma_a^2)$$

- See Page 122-123 on Notes

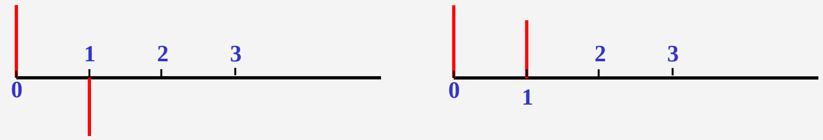
ACF : MA *v.s.* AR

MA(1)

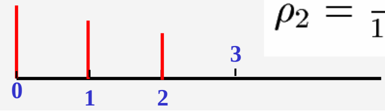


ACF : MA *v.s.* AR

MA(1)



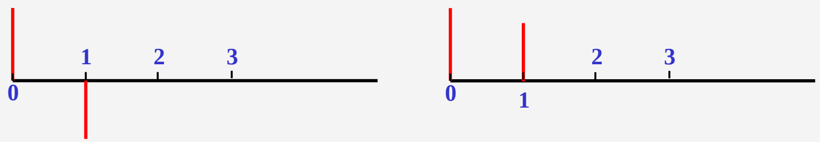
MA(2)



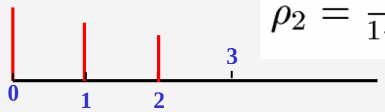
$$\rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

ACF : MA *v.s.* AR

MA(1)



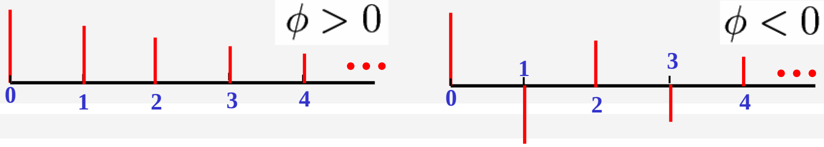
MA(2)



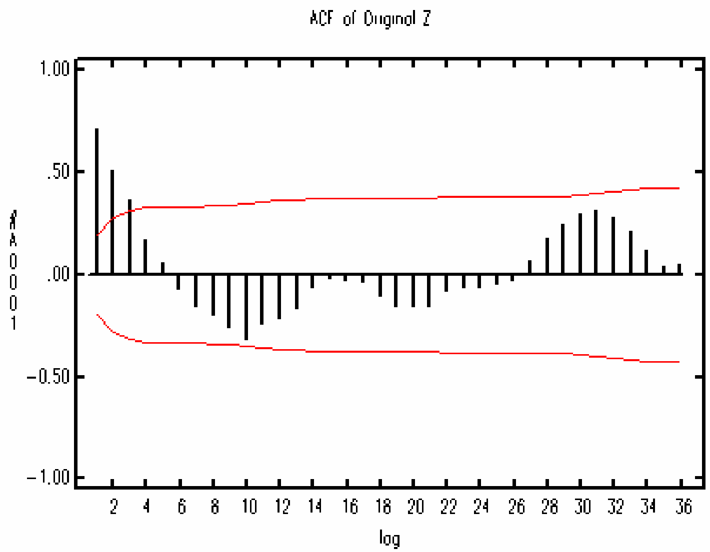
$$\rho_2 = \frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

AR(1) $\rho_k = \phi^k$

$\phi > 0$ $\phi < 0$



SACF of a Simulated Series - AR(1)



SACF of a Real Series - AR(1)

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Joint Distribution of AR(1)

$$
\begin{aligned}
P(Z_1, Z_2) &= P(Z_1)P(Z_2|Z_1) \\
P(Z_1, Z_2, Z_3) &= P(Z_1)P(Z_2|Z_1)P(Z_3|Z_1 Z_2) \\
\vdots &\quad \vdots \\
P(Z_1, \cdots, Z_t) &= P(Z_1)P(Z_2|Z_1)P(Z_3|Z_1 Z_2)\cdots P(Z_t|Z_1 \cdots Z_{t-1}) \\
&\Downarrow \ \mathrm{AR}(1) \\
P(Z_1, \cdots, Z_t) &= P(Z_1)P(Z_2|Z_1)P(Z_3|Z_2)\cdots P(Z_t|Z_{t-1})
\end{aligned}
$$

• See Page 122-123 on Notes

AR(1) and MA(1)

- AR(1)

$$(1 - \phi B)Z_t = a_t$$

- MA(1)

$$Z_t = (1 - \theta B)a_t$$

AR (2) Series

$$
\begin{aligned}
(1 - \phi_1 B - \phi_2 B^2)Z_t &= a_t \\
Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} &= a_t
\end{aligned}
$$

• See Page 124 on Notes

AR (2) Series

$$(1-\phi_1 B-\phi_2 B^2)Z_t = a_t$$
$$Z_t-\phi_1 Z_{t-1}-\phi_2 Z_{t-2} = a_t$$

$$Z_t = a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\cdots$$

• See Page 124 on Notes

AR (2) Series

$$(1-\phi_1 B-\phi_2 B^2)Z_t = a_t$$
$$Z_t-\phi_1 Z_{t-1}-\phi_2 Z_{t-2} = a_t$$

How to connect?

$$Z_t = a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\cdots$$

• See Page 124 on Notes

AR (2) Series

$$(1-\phi_1 B-\phi_2 B^2)Z_t = a_t$$
$$Z_t-\phi_1 Z_{t-1}-\phi_2 Z_{t-2} = a_t$$

How to connect?

- “硬”迭代
- Brute Force

$$Z_t = a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\cdots$$

• See Page 124 on Notes

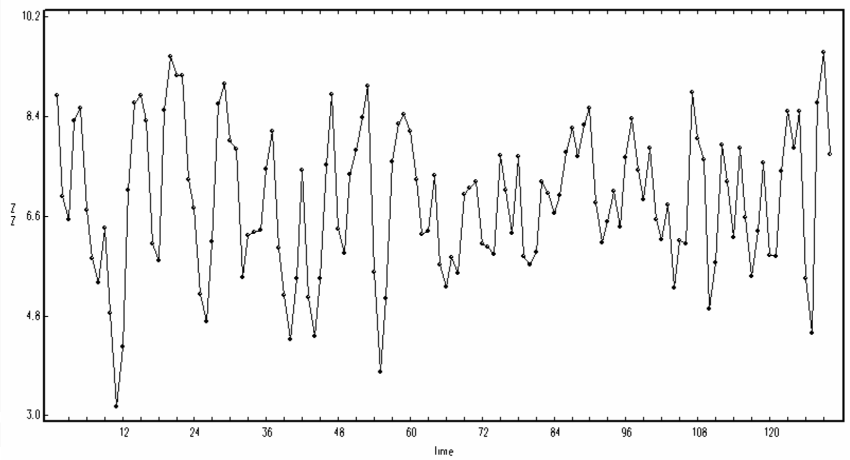
AR (2) Series

$$Z_t = (1-\phi B)^{-1}a_t$$
$$Z_t = \frac{1}{1-\phi_1 B-\phi_2 B^2}a_t$$
$$\Downarrow$$
$$(1-\alpha_1 B)(1-\alpha_2 B)Z_t = a_t$$
$$|\alpha_1|<1, |\alpha_2|<1$$

• See Page 124 on Notes

A Generated AR(2) Example

$$Z_t = .75Z_{t-1} - .5Z_{t-2} + a_t$$



Covariance Structure of AR(2)

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

where ψ_i is complicated.

• See Page 124 on Notes

Covariance Structure of AR(2)

$$Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} = a_t$$

$$\Downarrow$$

$$Z_{t-\ell}(Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2}) = a_t Z_{t-\ell}$$

$$\Downarrow$$

$$EZ_{t-\ell}(Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2}) = Ea_t Z_{t-\ell}$$

• See Page 124 on Notes

Covariance Structure of AR(2)

$$EZ_{t-\ell}(Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2}) = Ea_t Z_{t-\ell}$$

$$\Downarrow$$

$\ell = 0$	$\gamma_0 - \phi_1\gamma_1 - \phi_2\gamma_2 = \sigma_a^2$
$\ell = 1$	$\gamma_1 - \phi_1\gamma_0 - \phi_2\gamma_1 = 0$
$\ell = 2$	$\gamma_2 - \phi_1\gamma_1 - \phi_2\gamma_0 = 0$
$\ell = 3$	$\gamma_3 - \phi_1\gamma_2 - \phi_2\gamma_1 = 0$
$\ell \geq 2$	$\gamma_\ell - \phi_1\gamma_{\ell-1} - \phi_2\gamma_{\ell-2} = 0$

• See Page 124 on Notes

Covariance Structure of AR(2)

$$\ell \geq 2 \qquad \gamma_\ell - \phi_1\gamma_{\ell-1} - \phi_2\gamma_{\ell-2} = 0$$

$$\Updownarrow$$

$$\ell \geq 2 \qquad \gamma_\ell = \phi_1\gamma_{\ell-1} + \phi_2\gamma_{\ell-2}$$

$$\Updownarrow$$

$$\ell \geq 2 \qquad \frac{\gamma_\ell}{\gamma_0} = \phi_1\frac{\gamma_{\ell-1}}{\gamma_0} + \phi_2\frac{\gamma_{\ell-2}}{\gamma_0}$$

$$\Updownarrow$$

$$\ell \geq 2 \qquad \rho_\ell = \phi_1\rho_{\ell-1} + \phi_2\rho_{\ell-2}$$

• See Page 124 on Notes

Covariance Structure of AR(2)

$$\gamma_1 - \phi_1\gamma_0 - \phi_2\gamma_1 = 0$$

$$\gamma_2 - \phi_1\gamma_1 - \phi_2\gamma_0 = 0$$

• See Page 124 on Notes

Covariance Structure of AR(2)

$$\gamma_1 - \phi_1\gamma_0 - \phi_2\gamma_1 = 0$$

$$\gamma_2 - \phi_1\gamma_1 - \phi_2\gamma_0 = 0$$

$$\Downarrow$$

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} - \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

• See Page 124 on Notes

Covariance Structure of AR(2)

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} - \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

$$\Downarrow$$

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} - \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$$

Covariance Structure of AR(2)

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} - \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

$$\Downarrow$$

$$\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} - \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0$$

Yule-Walker Equation

$$\Updownarrow$$

$$\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$$

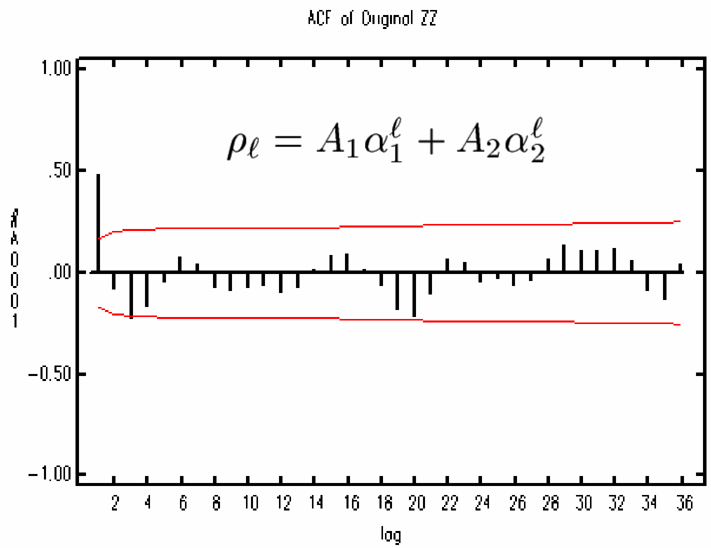
$$\Updownarrow$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$$

Covariance Structure of AR(2)

$$(\gamma_0 \ \gamma_1 \ \gamma_2 \ \gamma_3 \ \cdots) \leftrightarrows (\phi_1 \ \phi_2 \ \sigma_a^2)$$

Covariance Structure of AR(2)



$$\rho_\ell = A_1\alpha_1^\ell + A_2\alpha_2^\ell$$

时间序列分析

第三讲（下）

内容回顾

- Page 125 of Packets

AR(p)

AR(p)

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t, \quad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$$

AR(p)

AR(p)

$$a_t = (1 - \phi_1 B - \cdots - \phi_p B^p) Z_t$$

$$\Uparrow$$

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t, \quad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$$

AR(p)

AR(p)

$$
\begin{aligned}
a_t &= (1 - \phi_1 B - \cdots - \phi_p B^p) Z_t \\
&\Uparrow \\
Z_t &= \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t, \quad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)
\end{aligned}
$$

MA(q)

$$
Z_t = (1 - \theta_1 B - \cdots - \theta_q B^q) a_t
$$

AR(p)

AR(p)

$$
\begin{aligned}
a_t &= (1 - \phi_1 B - \cdots - \phi_p B^p) Z_t \\
&\Uparrow \\
Z_t &= \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t, \quad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)
\end{aligned}
$$

Finite Autoregression

MA(q)

$$
Z_t = (1 - \theta_1 B - \cdots - \theta_q B^q) a_t
$$

Finite Memory

AR(p)

AR(p)

$$
\begin{aligned}
a_t &= (1 - \phi_1 B - \cdots - \phi_p B^p) Z_t \\
&\Uparrow \\
Z_t &= \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t, \quad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)
\end{aligned}
$$

MA(q)

$$
\begin{aligned}
Z_t &= (1 - \theta_1 B - \cdots - \theta_q B^q) a_t \\
&\Downarrow \\
a_t &= (1 - \theta_1 B - \cdots - \theta_q B^q)^{-1} Z_t \\
&= (1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots) Z_t = \Pi(B) Z_t
\end{aligned}
$$

AR(p)

AR(p)

$$
\begin{aligned}
Z_t &= (1 - \phi_1 B - \cdots - \phi_p B^p)^{-1} a_t \\
&= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \Psi(B) a_t \\
&\Uparrow \\
a_t &= (1 - \phi_1 B - \cdots - \phi_p B^p) Z_t \\
&\Uparrow \\
Z_t &= \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \cdots + \phi_p Z_{t-p} + a_t, \quad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)
\end{aligned}
$$

$\Psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \cdots$

MA(q)

$$
\begin{aligned}
Z_t &= (1 - \theta_1 B - \cdots - \theta_q B^q) a_t \\
&\Downarrow \\
a_t &= (1 - \theta_1 B - \cdots - \theta_q B^q)^{-1} Z_t \\
&= (1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots) Z_t = \Pi(B) Z_t
\end{aligned}
$$

$\Pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \cdots$

AR(p)

AR(p)

$$
\begin{aligned}
Z_t &= (1-\phi_1 B-\cdots-\phi_p B^p)^{-1}a_t \\
&= a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\cdots=\Psi(B)a_t \\
&\Uparrow \\
a_t &= (1-\phi_1 B-\cdots-\phi_p B^p)Z_t \\
&\Uparrow \\
Z_t &= \phi_1 Z_{t-1}+\phi_2 Z_{t-2}+\cdots+\phi_p Z_{t-p}+a_t, \quad a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)
\end{aligned}
$$

差分方程

MA(q)

$$
\begin{aligned}
Z_t &= (1-\theta_1 B-\cdots-\theta_q B^q)a_t \\
&\Downarrow \\
a_t &= (1-\theta_1 B-\cdots-\theta_q B^q)^{-1}Z_t \\
&= (1-\pi_1 B-\pi_2 B^2-\pi_3 B^3-\cdots)Z_t=\Pi(B)Z_t
\end{aligned}
$$

差分方程

AR(p)

AR(p)

$$
\begin{aligned}
Z_t &= (1-\phi_1 B-\cdots-\phi_p B^p)^{-1}a_t \\
&= a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\cdots=\Psi(B)a_t \\
&\Uparrow \\
a_t &= (1-\phi_1 B-\cdots-\phi_p B^p)Z_t \\
&\Uparrow \\
Z_t &= \phi_1 Z_{t-1}+\phi_2 Z_{t-2}+\cdots+\phi_p Z_{t-p}+a_t, \quad a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)
\end{aligned}
$$

差分方程
Stationarity region of φ's

MA(q)

$$
\begin{aligned}
Z_t &= (1-\theta_1 B-\cdots-\theta_q B^q)a_t \\
&\Downarrow \\
a_t &= (1-\theta_1 B-\cdots-\theta_q B^q)^{-1}Z_t \\
&= (1-\pi_1 B-\pi_2 B^2-\pi_3 B^3-\cdots)Z_t=\Pi(B)Z_t
\end{aligned}
$$

差分方程
Invertibility region of π's

Covariance Structure of AR(p)

- See Page 126 of Notes

Covariance Structure of AR(p)

$$
Z_t-\phi_1 Z_{t-1}-\cdots-\phi_p Z_{t-p}=a_t
$$

$$
\Downarrow
$$

$$
EZ_{t-l}(Z_t-\phi_1 Z_{t-1}-\cdots-\phi_p Z_{t-p})=Ea_t Z_{t-l}
$$

- See Page 126 of Notes

Covariance Structure of AR(p)

l=0时：

$$EZ_{t-l}(Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p}) = Ea_t Z_{t-l}$$

$$\gamma_0 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_p = \sigma_a^2$$

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$$

• See Page 126 of Notes

Covariance Structure of AR(p)

$$EZ_{t-l}(Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p}) = Ea_t Z_{t-l}$$

$$\begin{array}{lll} l=0 & \gamma_0 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_p & = \sigma_a^2 \\ l=1 & \gamma_1 - \phi_1\gamma_0 - \cdots - \phi_p\gamma_{p-1} & = 0 \\ l=2 & \gamma_2 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_{p-2} & = 0 \\ & \vdots & \\ l=p & \gamma_p - \phi_1\gamma_{p-1} - \cdots - \phi_p\gamma_0 & = 0 \\ l>p & \gamma_l - \phi_1\gamma_{l-1} - \cdots - \phi_p\gamma_{l-p} & = 0 \end{array}$$

• See Page 126 of Notes

Covariance Structure of AR(p)

$$EZ_{t-l}(Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p}) = Ea_t Z_{t-l}$$

$$\begin{array}{lll} l=0 & \gamma_0 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_p & = \sigma_a^2 \\ l=1 & \gamma_1 - \phi_1\gamma_0 - \cdots - \phi_p\gamma_{p-1} & = 0 \\ l=2 & \gamma_2 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_{p-2} & = 0 \\ & \vdots & \\ l=p & \gamma_p - \phi_1\gamma_{p-1} - \cdots - \phi_p\gamma_0 & = 0 \\ l>p & \gamma_l - \phi_1\gamma_{l-1} - \cdots - \phi_p\gamma_{l-p} & = 0 \end{array}$$

差分方程

• See Page 126 of Notes

Covariance Structure of AR(p)

$$EZ_{t-l}(Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p}) = Ea_t Z_{t-l}$$

$$\begin{array}{lll} l=0 & \gamma_0 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_p & = \sigma_a^2 \\ l=1 & \gamma_1 - \phi_1\gamma_0 - \cdots - \phi_p\gamma_{p-1} & = 0 \\ l=2 & \gamma_2 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_{p-2} & = 0 \\ & \vdots & \\ l=p & \gamma_p - \phi_1\gamma_{p-1} - \cdots - \phi_p\gamma_0 & = 0 \\ l>p & \gamma_l - \phi_1\gamma_{l-1} - \cdots - \phi_p\gamma_{l-p} & = 0 \end{array}$$

General Yule-Walker Equation

• See Page 126 of Notes

$$l=1 \quad \gamma_1 - \phi_1\gamma_0 - \cdots - \phi_p\gamma_{p-1} = 0$$
$$l=2 \quad \gamma_2 - \phi_1\gamma_1 - \cdots - \phi_p\gamma_{p-2} = 0$$
$$\vdots$$
$$l=p \quad \gamma_p - \phi_1\gamma_{p-1} - \cdots - \phi_p\gamma_0 = 0$$

$$\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \vdots\\ \rho_p\end{pmatrix} - \begin{pmatrix}1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1}\\ \rho_1 & \ddots & \ddots & \ddots & \vdots\\ \rho_2 & \ddots & \ddots & \ddots & \rho_2\\ \vdots & \ddots & \ddots & \ddots & \rho_1\\ \rho_{p-1} & \cdots & \rho_2 & \rho_1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \vdots\\ \phi_p\end{pmatrix} = 0$$

• See Page 126 of Notes

$$\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \vdots\\ \rho_p\end{pmatrix} - \begin{pmatrix}1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1}\\ \rho_1 & \ddots & \ddots & \ddots & \vdots\\ \rho_2 & \ddots & \ddots & \ddots & \rho_2\\ \vdots & \ddots & \ddots & \ddots & \rho_1\\ \rho_{p-1} & \cdots & \rho_2 & \rho_1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \vdots\\ \phi_p\end{pmatrix} = 0$$

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \vdots\\ \phi_p\end{pmatrix} = \begin{pmatrix}1 & \rho_1 & \rho_2 & \cdots & \rho_{p-1}\\ \rho_1 & \ddots & \ddots & \ddots & \vdots\\ \rho_2 & \ddots & \ddots & \ddots & \rho_2\\ \vdots & \ddots & \ddots & \ddots & \rho_1\\ \rho_{p-1} & \cdots & \rho_2 & \rho_1 & 1\end{pmatrix}^{-1}\begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \vdots\\ \rho_p\end{pmatrix}$$

• See Page 126 of Notes

PACF

- 目的：识别AR模型的阶数
- 方法：通过Yule-Walker Equation

• See Page 127 of Notes

PACF

- AR(1) : $\wp_1 = \rho_1 \quad \text{and} \quad \rho_1 = \phi$
- AR(2) : $\wp_2 = \dfrac{\rho_2 - \rho_1^2}{1-\rho_1^2} = \phi_2$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}1 & \rho_1\\ \rho_1 & 1\end{pmatrix}^{-1}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix} = \begin{pmatrix}1 & -\rho_1\\ -\rho_1 & 1\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}\frac{1}{1-\rho_1^2}$$

$$\Longrightarrow \quad \phi_2 = \frac{1}{1-\rho_1^2}(\rho_2 - \rho_1^2)$$

• See Page 127 of Notes

PACF

$$\wp_3 = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} = \phi_3$$

Cramer's Rule!

- See Page 127 of Notes

PACF

$$\wp_\ell = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_1 \\ \rho_1 & 1 & \rho_1 & \ddots & \vdots \\ \rho_2 & \rho_1 & 1 & \ddots & \rho_{\ell-2} \\ \vdots & \ddots & \ddots & \ddots & \rho_{\ell-1} \\ \rho_{\ell-1} & \cdots & \rho_2 & \rho_1 & \rho_\ell \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{\ell-1} \\ \rho_1 & 1 & \rho_1 & \ddots & \vdots \\ \rho_2 & \rho_1 & 1 & \ddots & \rho_2 \\ \vdots & \ddots & \ddots & \ddots & \rho_1 \\ \rho_{\ell-1} & \cdots & \rho_2 & \rho_1 & 1 \end{vmatrix}}$$

Cramer's Rule!

- See Page 127 of Notes

PACF

- If true model is AR(1) :

$$\wp_1 = \rho_1 = \phi \neq 0$$

$$\wp_2 = \wp_3 = \wp_4 = \cdots = 0$$

- If true model is AR(2) :

$$\wp_2 \neq \phi_2$$

$$\wp_3 = \wp_4 = \wp_5 = \cdots = 0$$

- See Page 127 of Notes

SPACF的计算

- 假设有时间序列 $Z_1, \cdots, Z_n$:

1. Fit AR(1) : $Z_t = \phi Z_{t-1} + \epsilon_t$

   $\widehat{\phi} = \widehat{\wp_1}$

2. Fit AR(2) : $Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \epsilon_t$

   $\widehat{\phi_2} = \widehat{\wp_2}$

3. Fit AR(3) : $Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \phi_3 Z_{t-3} + \epsilon_t$

   $\widehat{\phi_3} = \widehat{\wp_3}$

4. Fit More… :

- See Page 127 of Notes

PACF Example

- 例1: A GENERATED AR(1) EXAMPLE

  Page 244-251 of Packets

- 例2: THE BATCH DATA

  Page 252-259 of Packets

• See Page 244 of Notes

PACF for MA(1) Model

$$Z_t = (1 - \theta B) a_t$$

$$\Leftrightarrow \quad (1 - \theta B)^{-1} Z_t = a_t$$

$$\Leftrightarrow \quad (1 + \theta B + \theta^2 B^2 + \theta^3 B^3 + \cdots) Z_t = a_t$$

$$\Leftrightarrow \quad Z_t = -\theta Z_{t-1} - \theta^2 Z_{t-2} - \theta^3 Z_{t-3} - \cdots + a_t$$

PACF Example

- 例3: SERIES A OF BOX-JENKINS AND REINSEL

  Page 274-282 of Packets

PACF Example

- 例3: SERIES A OF BOX-JENKINS AND REINSEL

  Page 274-282 of Packets

Mixed ARMA(p,q) Model

The mean of stationary AR model

- For MA(1) Model

$$
\begin{aligned}
& Z_t = \mu + a_t - \theta a_{t-1} \\
\Leftrightarrow \quad & Z_t - \mu = a_t - \theta a_{t-1} \\
\Leftrightarrow \quad & \mu = E(Z_t)
\end{aligned}
$$

$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$

The mean of stationary AR model

- For MA(1) Model

$$
\begin{aligned}
& Z_t = \mu + a_t - \theta a_{t-1} \\
\Leftrightarrow \quad & Z_t - \mu = a_t - \theta a_{t-1} \\
\Leftrightarrow \quad & \mu = E(Z_t)
\end{aligned}
$$

$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$

水平线

The mean of stationary AR model

- For AR(1) Model

$$
\begin{aligned}
Z_t &= \phi Z_{t-1} + a_t \\
& \Downarrow \quad E(Z_t) = \mu \\
Z_t - \mu &= \phi(Z_{t-1} - \mu) + a_t \\
& \Updownarrow \\
Z_t &= \mu + \phi(Z_{t-1} - \mu) + a_t \\
&= \mu + \phi Z_{t-1} - \phi\mu + a_t \\
& \Downarrow \quad C = \mu - \phi\mu = \mu(1-\phi) \\
&= C + \phi Z_{t-1} + a_t
\end{aligned}
$$

The mean of stationary AR model

- For AR(1) Model

$$
\begin{aligned}
Z_t &= \phi Z_{t-1} + a_t \\
& \Downarrow \quad E(Z_t) = \mu \\
Z_t - \mu &= \phi(Z_{t-1} - \mu) + a_t \\
& \Updownarrow \\
Z_t &= \mu + \phi(Z_{t-1} - \mu) + a_t \\
&= \mu + \phi Z_{t-1} - \phi\mu + a_t \\
& \Downarrow \quad C = \mu - \phi\mu = \mu(1-\phi) \left\{ \begin{array}{l} (C, \phi, \sigma_a^2) \\ (\mu, \phi, \sigma_a^2) \end{array} \right. \\
&= C + \phi Z_{t-1} + a_t
\end{aligned}
$$

mixed ARMA(p,q) model

$$(1-\phi_1 B-\cdots-\phi_p B^p)\dot{Z}_t=(1-\theta_1 B-\cdots-\theta_q B^q)a_t$$

其中：$\dot{Z}_t = Z_t - \mu$ and $a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)$

mixed ARMA(p,q) model

$$(1-\phi_1 B-\cdots-\phi_p B^p)\dot{Z}_t=(1-\theta_1 B-\cdots-\theta_q B^q)a_t$$

其中：$\dot{Z}_t = Z_t - \mu$ and $a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)$

p=q=1时：

ARMA(1,1) model $(1-\phi B)\dot{Z}_t=(1-\theta B)a_t$

• See Page 129-131 of Notes

mixed ARMA(p,q) model

$$(1-\phi_1 B-\cdots-\phi_p B^p)\dot{Z}_t=(1-\theta_1 B-\cdots-\theta_q B^q)a_t$$

其中：$\dot{Z}_t = Z_t - \mu$ and $a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)$

p=q=1时：

ARMA(1,1) model $(1-\phi B)\dot{Z}_t=(1-\theta B)a_t$

• See Page 129-131 of Notes

mixed ARMA(p,q) model

Three Forms of ARMA(1,1)：

1.
$$\begin{aligned}\dot{Z}_t &= (1-\phi B)^{-1}(1-\theta B)a_t\\ &= a_t+\psi_1 a_{t-1}+\psi_2 a_{t-2}+\psi_3 a_{t-3}+\cdots\\ &= \Psi(B)a_t\end{aligned}$$

2. $(1-\phi B)\dot{Z}_t=(1-\theta B)a_t$

3.
$$\begin{aligned}\underbrace{(1-\theta B)^{-1}(1-\phi B)}_{\Pi(B)}\dot{Z}_t &= a_t\\ \Pi(B)Z_t &= a_t\\ Z_t &= \pi_1 Z_{t-1}+\pi_2 Z_{t-2}+\cdots+a_t\end{aligned}$$

mixed ARMA(p,q) model

$$(1-\phi_1 B - \cdots - \phi_p B^p)\dot{Z}_t \quad = \quad (1-\theta_1 B - \cdots - \theta_q B^q)a_t$$

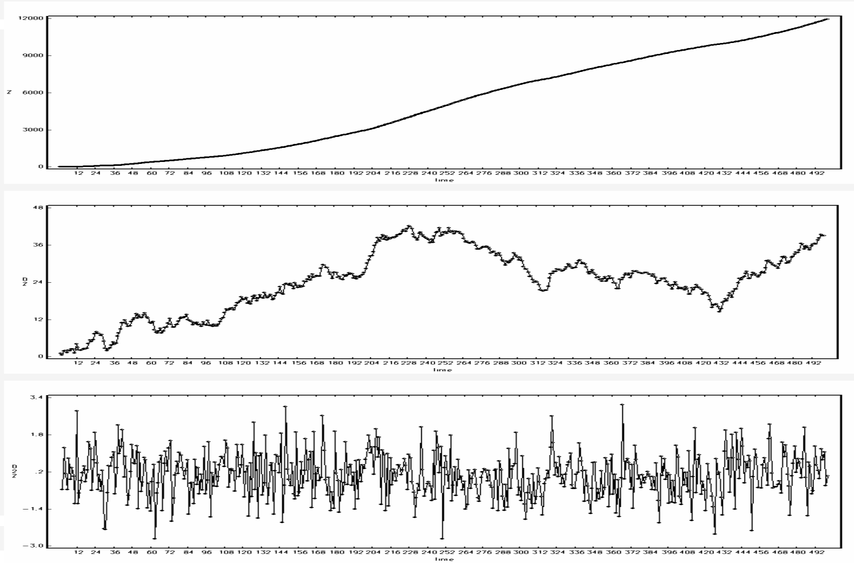
stationary if   invertible if

$$\sum \psi_j^2 < \infty \text{ and} \qquad \sum \pi_j^2 < \infty \text{ and}$$

$$1-\phi_1 B - \cdots - \phi_p B^p = 0 \qquad 1-\theta_1 B - \cdots - \theta_q B^q = 0$$

roots all outside unit circle   roots all outside unit circle

mixed ARMA(p,q) model



时间序列分析

第四讲(上)

内容回顾

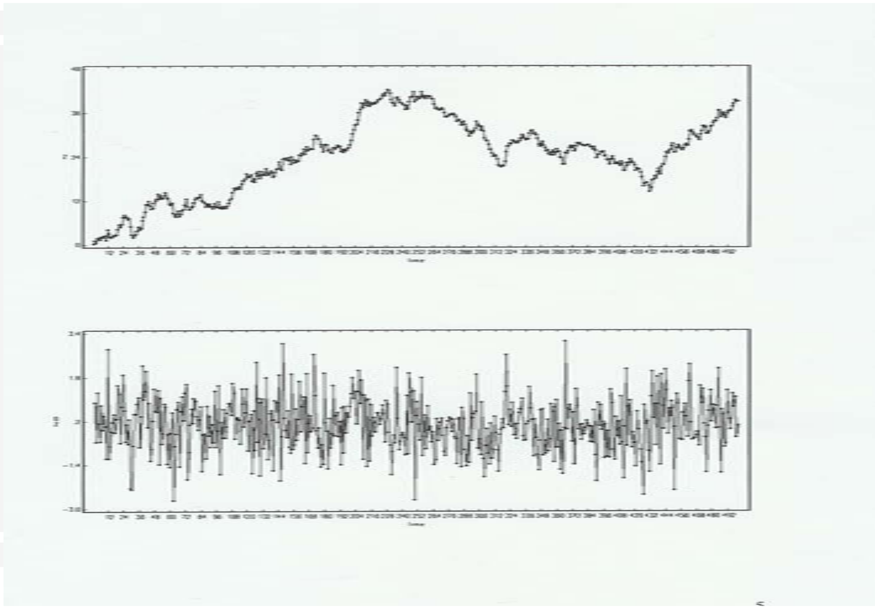
- ACF/PACF of MA/AR/ARMA model

本节课程内容

- Random Walk
- Differencing Data
- (0,1,1)
- (p,1,q)
- (0,2,2)
- (p,2,q)
- (p,d,q)

- Prediction:
- Basic
- AR(1)

Example of Random Walk

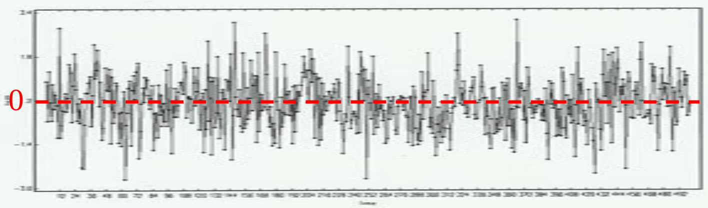


Example of Random Walk

Step1 : noise generation

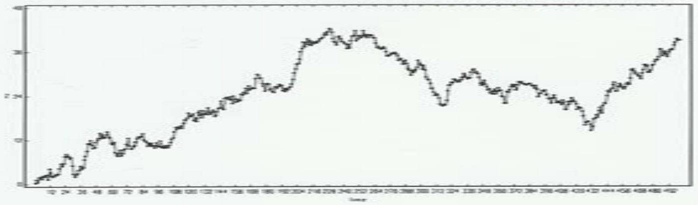
Let $a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$, where $\sigma_a^2 = 1$

Generate 500 data points: $a_1, a_2, \cdots, a_{500}$



Example of Random Walk

Step2 : series generation



$$\begin{aligned} Z_1 &= a_1 \\ Z_2 &= a_1 + a_2 \\ Z_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ Z_{500} &= a_1 + a_2 + a_3 + \cdots + a_{500} \end{aligned}$$

Example of Random Walk

Theory Exploration :

$$Z_t = a_t + a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1$$

Example of Random Walk

Theory Exploration :

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots$$

$$\Downarrow \quad \psi_1 = 1, \ \psi_2 = 1, \ \psi_3 = 1, \ \cdots$$

$$Z_t = a_t + a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1$$

Example of Random Walk

Theory Exploration :

$$
\begin{aligned}
Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&\Downarrow \;\; \psi_1 = 1,\ \psi_2 = 1,\ \psi_3 = 1,\ \cdots \\
Z_t &= a_t + a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
Z_{t-1} &= a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1
\end{aligned}
$$

Example of Random Walk

Theory Exploration :

$$
\begin{aligned}
Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&\Downarrow \;\; \psi_1 = 1,\ \psi_2 = 1,\ \psi_3 = 1,\ \cdots \\
Z_t &= a_t + a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
Z_{t-1} &= a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
&\Downarrow \\
Z_t &= Z_{t-1} + a_t
\end{aligned}
$$

Example of Random Walk

Theory Exploration :

$$
\begin{aligned}
Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&\Downarrow \;\; \psi_1 = 1,\ \psi_2 = 1,\ \psi_3 = 1,\ \cdots \\
Z_t &= a_t + a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
Z_{t-1} &= a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
&\Downarrow \\
Z_t &= Z_{t-1} + a_t \Longleftrightarrow \begin{cases} Z_t = \phi Z_{t-1} + a_t \\ \phi = 1 \end{cases}
\end{aligned}
$$

Example of Random Walk

Theory Exploration :

$$
\begin{aligned}
Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&\Downarrow \;\; \psi_1 = 1,\ \psi_2 = 1,\ \psi_3 = 1,\ \cdots \\
Z_t &= a_t + a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
Z_{t-1} &= a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
&\Downarrow \\
Z_t &= Z_{t-1} + a_t \Longleftrightarrow Z_t - Z_{t-1} = a_t
\end{aligned}
$$

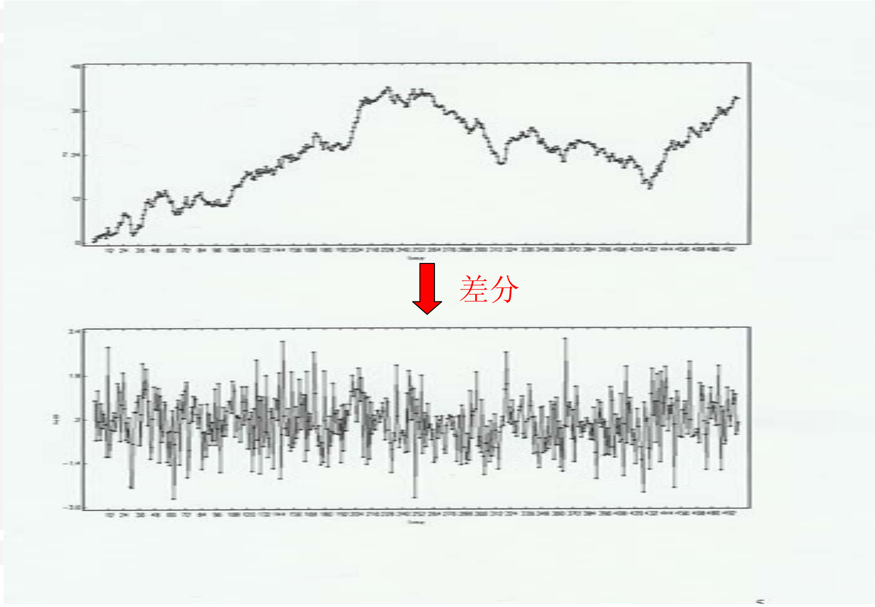
Example of Random Walk

Theory Exploration :

$$
\begin{aligned}
Z_t &= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \psi_3 a_{t-3} + \cdots \\
&\Downarrow \quad \psi_1 = 1,\ \psi_2 = 1,\ \psi_3 = 1,\ \cdots \\
Z_t &= a_t + a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
Z_{t-1} &= a_{t-1} + a_{t-2} + a_{t-3} + \cdots + a_1 \\
&\Downarrow \\
Z_t &= Z_{t-1} + a_t \Longleftrightarrow Z_t - Z_{t-1} = a_t
\end{aligned}
$$

差分

Example of Random Walk



AR(1) *v.s.* Random Walk

AR(1)

$$
\begin{aligned}
Z_t &= \phi Z_{t-1} + a_t \qquad |\phi| < 1 \\
&\Downarrow \\
Z_t &= \phi Z_{t-1} + \phi^2 Z_{t-2} + \cdots \\
&\Downarrow \\
\gamma_0 &= (1 + \phi^2 + \phi^4 + \phi^6 + \cdots)\sigma_a^2 = \frac{\sigma_a^2}{1-\phi^2}
\end{aligned}
$$

Random Walk

$$
\begin{aligned}
Z_t &= Z_{t-1} + a_t \\
&\Downarrow \\
Z_t &= a_t + a_{t-1} + a_{t-2} + \cdots + a_1 \\
&\Downarrow \\
\gamma_0 &= (1 + 1 + 1 + 1 + \cdots)\sigma_a^2
\end{aligned}
$$

Stationary与Nonstationary的关系

Time series starts at some fixed time point : t = m

• AR(1) :

$$
\begin{aligned}
t = m \qquad Z_m &= \omega_m + a_m \\
t = m+1 \qquad Z_{m+1} &= \phi Z_m + a_{m+1} \qquad |\phi| < 1 \\
&= \phi(\omega_m + a_m) + a_{m+1} \\
&= a_{m+1} + \phi a_m + \phi \omega_m \\
t = m+2 \qquad Z_{m+2} &= \phi Z_{m+1} + a_{m+2} \\
&= a_{m+2} + \phi a_{m+1} + \phi^2 a_m + \phi^2 \omega_m \\
&\vdots \\
Z_t &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots + \phi^{t-m} a_m + \phi^{t-m} \omega_m
\end{aligned}
$$

• See Page 142 of Notes

Stationary与Nonstationary的关系

Time series starts at some fixed time point : t = m

- AR(1) :

$$
\begin{aligned}
t=m \quad & Z_m = \omega_m + a_m \\
t=m+1 \quad & Z_{m+1} = \phi Z_m + a_{m+1} \qquad |\phi|<1 \\
& = \phi(\omega_m + a_m) + a_{m+1} \\
& = a_{m+1} + \phi a_m + \phi\omega_m \\
t=m+2 \quad & Z_{m+2} = \phi Z_{m+1} + a_{m+2} \\
& = a_{m+2} + \phi a_{m+1} + \phi^2 a_m + \phi^2\omega_m \\
& \vdots \\
& Z_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots + \phi^{t-m} a_m + \phi^{t-m}\omega_m
\end{aligned}
$$

- See Page 142 of Notes

Stationary与Nonstationary的关系

Time series starts at some fixed time point : t = m

- AR(1) :

$$
\begin{aligned}
Z_t &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots + \phi^{t-m} a_m + \phi^{t-m}\omega_m \\
&\Downarrow \\
\gamma_0(Z_t) &= \sigma_a^2[1+\phi^2+\cdots+\phi^{2(t-m)}] + \phi^{2(t-m)}\mathrm{Var}(\omega_m) \\
&\Downarrow \ \text{assume } \omega_m \perp a_t\, s \quad \text{and} \quad t \gg m \\
&= \frac{\sigma_a^2(1-\phi^{2(t-m+1)})}{1-\phi^2} + \phi^{2(t-m)}\mathrm{Var}(\omega_m)
\end{aligned}
$$

$$
\begin{aligned}
&(1) \quad |\phi|<1 \qquad \frac{\sigma_a^2}{1-\phi^2} \\
&(2) \quad |\phi|=1 \qquad \sigma_a^2[t-m+1] + \mathrm{Var}(\omega_m)
\end{aligned}
$$

- See Page 142 of Notes

Stationary与Nonstationary的关系

Time series starts at some fixed time point : t = m

- AR(1) :

$$
\begin{aligned}
Z_t &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \cdots + \phi^{t-m} a_m + \phi^{t-m}\omega_m \\
&\Downarrow \\
\gamma_0(Z_t) &= \sigma_a^2[1+\phi^2+\cdots+\phi^{2(t-m)}] + \phi^{2(t-m)}\mathrm{Var}(\omega_m) \\
&\Downarrow \ \text{assume } \omega_m \perp a_t\, s \quad \text{and} \quad t \gg m \\
&= \frac{\sigma_a^2(1-\phi^{2(t-m+1)})}{1-\phi^2} + \phi^{2(t-m)}\mathrm{Var}(\omega_m)
\end{aligned}
$$

$$
\begin{aligned}
&(1) \quad |\phi|<1 \qquad \frac{\sigma_a^2}{1-\phi^2} \\
&(2) \quad |\phi|=1 \qquad \sigma_a^2[t-m+1] + \mathrm{Var}(\omega_m)
\end{aligned}
$$

O(t)

- See Page 142 of Notes

Stationary与Nonstationary的关系

$$(1-B)Z_t = a_t$$

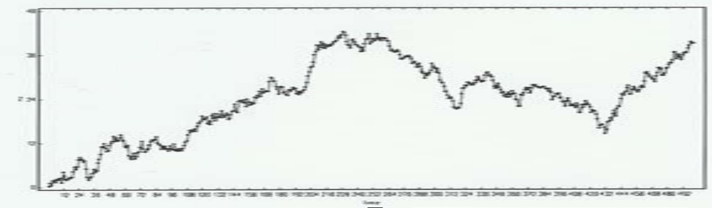
- 差分后stationary

$\phi = 1$

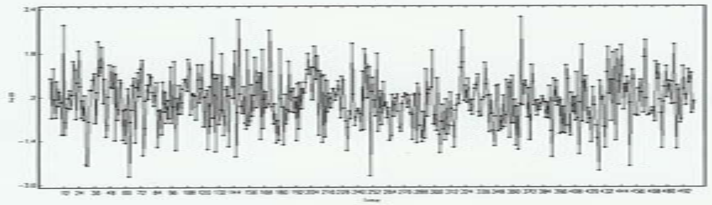
$$(1-\phi B)Z_t = a_t$$

- 单根 unit root
- 差分后stationary

Stationary与Nonstationary的关系



$$(1-B)Z_t = a_t$$



差分后stationary+MA(1)

$$w_t = (1-B)Z_t$$

$$\Downarrow$$

$$w_t = (1-\theta B)a_t$$

差分后stationary+MA(1)

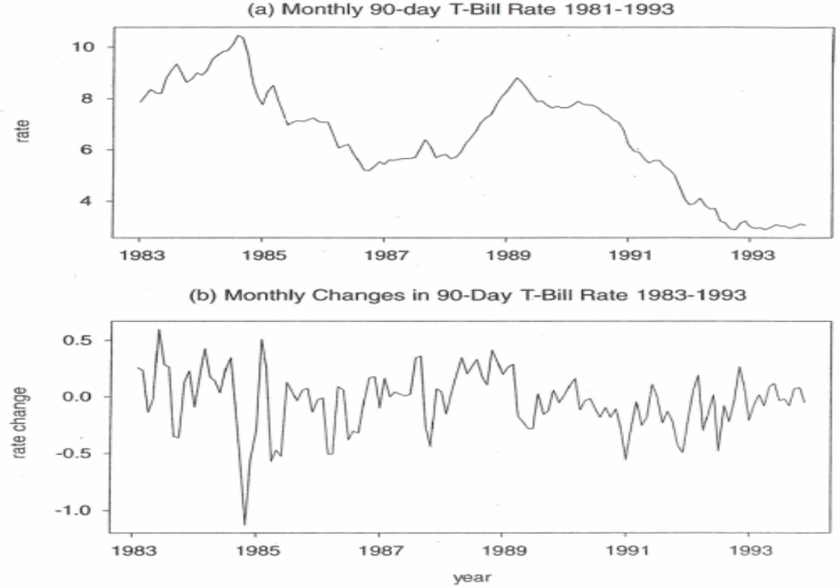


FIGURE 1.2 A nonstationary series and its first difference.

差分后stationary+MA(1) – Ψ-form

方法1 :

$$
\begin{aligned}
(1-B)Z_t &= (1-\theta B)a_t \\
&\Downarrow \\
Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\
Z_{t-1} - Z_{t-2} &= a_{t-1} - \theta a_{t-2} \\
Z_{t-2} - Z_{t-3} &= a_{t-2} - \theta a_{t-3} \\
&\vdots \\
Z_{m+1} - Z_m &= a_{m+1} - \theta a_m
\end{aligned}
\quad t > m\text{时}
$$

$$
\begin{aligned}
Z_m &= w_m + a_m \quad t = m\text{时} \\
&\Downarrow \\
Z_t &= a_t + (1-\theta)(a_{t-1} + \cdots + a_m) + \omega_m \\
&\Downarrow \\
\gamma_0(t) &= \sigma_a^2[1 + (t-m)(1-\theta)^2] + \mathrm{Var}(\omega_m) = O(t)
\end{aligned}
$$

- See Page 133 of Notes

差分后stationary+MA(1) – Ψ-form

方法1 :

$$(1-B)Z_t = (1-\theta B)a_t$$

$$\Downarrow$$

$t > m$时

$$\begin{aligned}
Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\
Z_{t-1} - Z_{t-2} &= a_{t-1} - \theta a_{t-2} \\
Z_{t-2} - Z_{t-3} &= a_{t-2} - \theta a_{t-3} \\
&\vdots \\
Z_{m+1} - Z_m &
\end{aligned}$$

$t = m$时

$$Z_m$$

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$$

$$\psi_1 = \psi_2 = \psi_3 = \cdots = (1-\theta) \quad \psi_j \not\to 0$$

$$Z_t = a_t + (1-\theta)(a_{t-1} + \cdots + a_m) + \omega_m$$

$$\Downarrow$$

$$\gamma_0(t) = \sigma_a^2[1 + (t-m)(1-\theta)^2] + \mathrm{Var}(\omega_m) = O(t)$$

- See Page 133 of Notes

差分后stationary+MA(1) – Ψ-form

方法1 :

$$(1-B)Z_t = (1-\theta B)a_t$$

$$\Downarrow$$

Muth (1960)

$t > m$时

$$\begin{aligned}
Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\
Z_{t-1} - Z_{t-2} &= a_{t-1} - \theta a_{t-2} \\
Z_{t-2} - Z_{t-3} &= a_{t-2} - \theta a_{t-3} \\
&\vdots \\
Z_{m+1} - Z_m &= a_{m+1} - \theta a_m
\end{aligned}$$

$t = m$时

$$Z_m = w_m + a_m$$

$$\Downarrow$$

$$Z_t = a_t + (1-\theta)(a_{t-1} + \cdots + a_m) + \omega_m$$

$$\Downarrow$$

$$\gamma_0(t) = \sigma_a^2[1 + (t-m)(1-\theta)^2] + \mathrm{Var}(\omega_m) = O(t)$$

- See Page 133 of Notes

差分后stationary+MA(1) – Ψ-form

方法2 :

$$(1-\phi B)Z_t = (1-\theta B)a_t$$

$$\Longleftrightarrow \quad Z_t = \frac{1-\theta B}{1-\phi B}a_t = \Psi(B)a_t, \qquad 其中\Psi(B) = \frac{1-\theta B}{1-\phi B}$$

$$(1-\phi B)(1 + \psi_1 B + \psi_2 B^2 + \cdots) = 1 - \theta B$$

其中，$\phi = 1,\ \psi's = 1-\theta$

- See Page 133 of Notes

差分后stationary+MA(1) – π-form

$$\begin{aligned}
Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\
Z_{t-1} - Z_{t-2} &= a_{t-1} - \theta a_{t-2} \\
Z_{t-2} - Z_{t-3} &= a_{t-2} - \theta a_{t-3} \\
&\vdots \\
Z_{m+1} - Z_m &= a_{m+1} - \theta a_m \\
Z_m &= \omega_m + a_m
\end{aligned}$$

- See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$\begin{aligned} Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\ \theta(Z_{t-1} - Z_{t-2}) &= (a_{t-1} - \theta a_{t-2})\theta \\ Z_{t-2} - Z_{t-3} &= a_{t-2} - \theta a_{t-3} \\ &\vdots \\ Z_{m+1} - Z_m &= a_{m+1} - \theta a_m \\ Z_m &= \omega_m + a_m \end{aligned}$$

- See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$\begin{aligned} Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\ \theta(Z_{t-1} - Z_{t-2}) &= (a_{t-1} - \theta a_{t-2})\theta \\ \theta^2(Z_{t-2} - Z_{t-3}) &= (a_{t-2} - \theta a_{t-3})\theta^2 \\ &\vdots \\ Z_{m+1} - Z_m &= a_{m+1} - \theta a_m \\ Z_m &= \omega_m + a_m \end{aligned}$$

- See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$\begin{aligned} Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\ \theta(Z_{t-1} - Z_{t-2}) &= (a_{t-1} - \theta a_{t-2})\theta \\ \theta^2(Z_{t-2} - Z_{t-3}) &= (a_{t-2} - \theta a_{t-3})\theta^2 \\ &\vdots \\ \theta^{t-m-1}(Z_{m+1} - Z_m) &= (a_{m+1} - \theta a_m)\theta^{t-m-1} \\ \theta^{t-m} Z_m &= (\omega_m + a_m)\theta^{t-m} \end{aligned}$$

- See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$\begin{aligned} Z_t - Z_{t-1} &= a_t - \theta a_{t-1} \\ \theta(Z_{t-1} - Z_{t-2}) &= (a_{t-1} - \theta a_{t-2})\theta \\ \theta^2(Z_{t-2} - Z_{t-3}) &= (a_{t-2} - \theta a_{t-3})\theta^2 \\ &\vdots \\ \theta^{t-m-1}(Z_{m+1} - Z_m) &= (a_{m+1} - \theta a_m)\theta^{t-m-1} \\ + \quad \theta^{t-m} Z_m &= (\omega_m + a_m)\theta^{t-m} \end{aligned}$$

$$\boxed{Z_t - (1-\theta)Z_{t-1} - \theta(1-\theta)Z_{t-2} - \theta^2(1-\theta)Z_{t-3} - \cdots - \theta^{t-m-1}(1-\theta)Z_m} = \boxed{a_t + \theta^{t-m}\omega_m}$$

- See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$Z_t - (1-\theta)Z_{t-1} - \theta(1-\theta)Z_{t-2} - \theta^2(1-\theta)Z_{t-3} - \cdots - \theta^{t-m-1}(1-\theta)Z_m = a_t + \theta^{t-m}\omega_m$$

$$\Updownarrow$$

$$Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots + \theta^{t-m-1}(1-\theta)Z_m + \theta^{t-m}\omega_m + a_t$$

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差分后stationary+MA(1) – π -form

$$Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots + \theta^{t-m-1}(1-\theta)Z_m + \theta^{t-m}\omega_m + a_t$$

例子：

t=month， Z_t=t-th month sales

θ and all Z_{t-j}'s are known and $|\theta| < 1$

$(1-\theta)$	Z_{t-1}	上个月
$\theta(1-\theta)$	Z_{t-2}	2个月前
$\theta^2(1-\theta)$	Z_{t-3}	3个月前
$\vdots$	$\vdots$	$\vdots$

• See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots + \theta^{t-m-1}(1-\theta)Z_m + \theta^{t-m}\omega_m + a_t$$

$|\theta| < 1$时，此项不要紧

例子：

t=month， Z_t=t-th month sales

θ and all Z_{t-j}'s are known and $|\theta| < 1$

$(1-\theta)$	Z_{t-1}	上个月
$\theta(1-\theta)$	Z_{t-2}	2个月前
$\theta^2(1-\theta)$	Z_{t-3}	3个月前
$\vdots$	$\vdots$	$\vdots$

• See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots + \theta^{t-m-1}(1-\theta)Z_m + \theta^{t-m}\omega_m + a_t$$

此部分用于预测

例子：

t=month， Z_t=t-th month sales

θ and all Z_{t-j}'s are known and $|\theta| < 1$

$(1-\theta)$	Z_{t-1}	上个月
$\theta(1-\theta)$	Z_{t-2}	2个月前
$\theta^2(1-\theta)$	Z_{t-3}	3个月前
$\vdots$	$\vdots$	$\vdots$

• See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots + \theta^{t-m-1}(1-\theta)Z_m + \theta^{t-m}\omega_m + a_t$$

例子：

t=month，Z_t=t-th month sales

θ and all Z_{t-j}'s are known and $|\theta| < 1$

$(1-\theta)$	Z_{t-1}	上个月
$\theta(1-\theta)$	Z_{t-2}	2个月前
$\theta^2(1-\theta)$	Z_{t-3}	3个月前
$\vdots$		

- 此部分和为1
- “加权平均”

• See Page 133 of Notes

差分后stationary+MA(1) – π -form

$$Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots + \theta^{t-m-1}(1-\theta)Z_m + \theta^{t-m}\omega_m + a_t$$

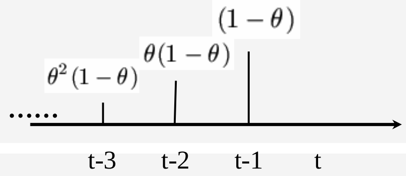
例子：

t=month，Z_t=t-th month sales

θ and all Z_{t-j}'s are known and $|\theta| < 1$

$(1-\theta)$	Z_{t-1}	上个月
$\theta(1-\theta)$	Z_{t-2}	2个月前
$\theta^2(1-\theta)$	Z_{t-3}	3个月前
$\vdots$		

- 此部分和为1
- “加权平均”



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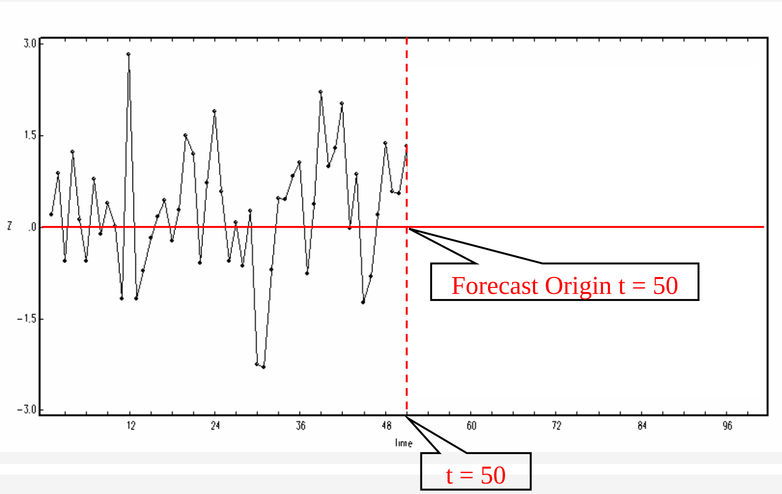
差分后stationary+MA(1) – π -form



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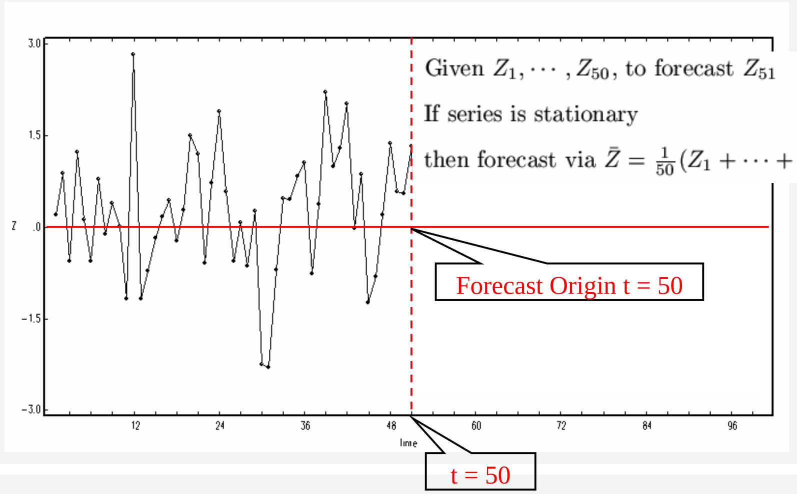
差分后stationary+MA(1) – π -form

Given $Z_1, \cdots, Z_{50}$, to forecast Z_{51}

If series is stationary

then forecast via $\bar{Z} = \frac{1}{50}(Z_1 + \cdots + Z_{50})$



• See Page 133 of Notes

时间序列分析

第四讲（下）

Nonstationary Series的处理

- Nonstationary Series
- Stationary

Nonstationary Series的处理

- Nonstationary Series

差分 $Z_t - Z_{t-1}$

- Stationary

Nonstationary Series的处理

t	Z_t
1	1.9
2	2.4
3	2.2
4	2.7
5	2.3
6	1.8
7	1.5

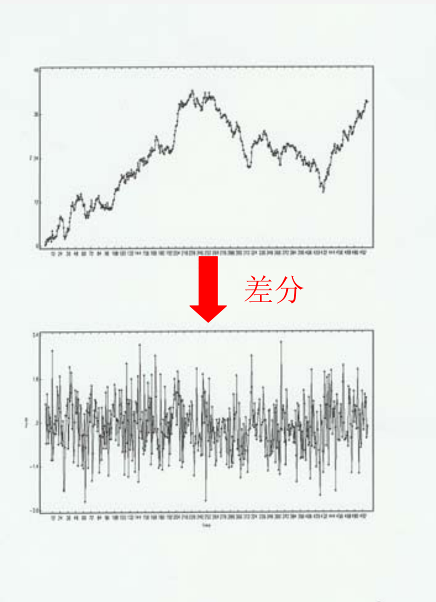
Nonstationary Series的处理

t	Z_t	Z_{t-1}
1	1.9	-
2	2.4	1.9
3	2.2	2.4
4	2.7	2.2
5	2.3	2.7
6	1.8	2.3
7	1.5	1.8

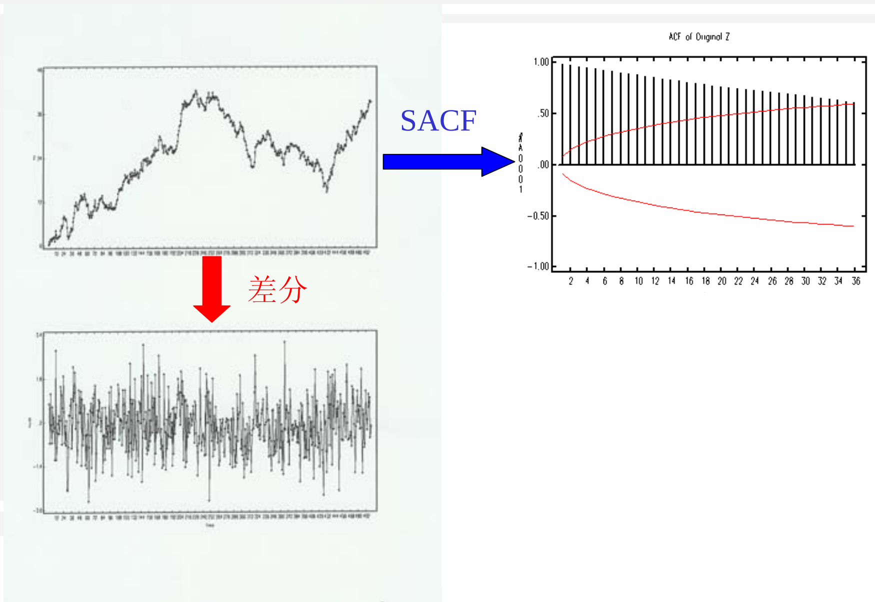
Nonstationary Series的处理

t	Z_t	Z_{t-1}	$Z_t - Z_{t-1}$
1	1.9	-	-
2	2.4	1.9	.5
3	2.2	2.4	-.2
4	2.7	2.2	.5
5	2.3	2.7	-.4
6	1.8	2.3	.5
7	1.5	1.8	-.3

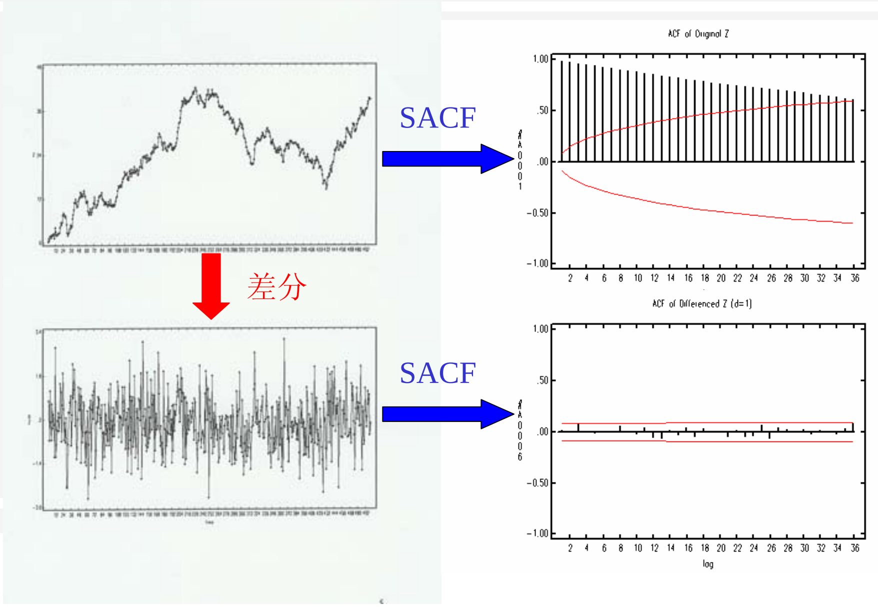
Nonstationary Series的处理



Nonstationary Series的处理



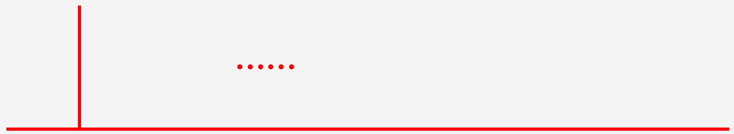
Nonstationary Series的处理



Nonstationary Series的处理

- SACF of Random Walk

- SACF of Differenced Random Walk



$Z_t - Z_{t-1} = a_t$ is reasonable!

Nonstationary Series的处理

- Example 2 $(1-B)Z_t = (1-\theta B)a_t$

- Very important in practice

  for Forecasting economic variables

- ψ_j weights $\nrightarrow 0$

Nonstationary Series的处理

$$(1-\phi B)Z_t \quad = \quad (1-\theta B)a_t$$

- ψ - form

$$(1-\phi B)(1+\psi_1 B+\psi_2 B^2+\psi_3 B^3+\cdots) = (1-\theta B)$$
$$\Downarrow \phi = 1$$
$$\psi_j = (1-\theta) \quad j = 1, 2, 3, \cdots$$

- π - form

$$\begin{aligned} (1-\phi B) &= (1-\theta B)(1-\pi_1 B-\pi_2 B^2-\cdots) \\ \Downarrow & \quad \phi = 1 \\ \pi_j &= (1-\theta)\theta^{j-1} \quad \text{for } j = 1, 2, \cdots \end{aligned}$$

Nonstationary Series的处理

$$
\begin{aligned}
& (1-B)Z_t = (1-\theta B)a_t \\
\Leftrightarrow \quad & \left(\frac{1-B}{1-\theta B}\right)Z_t = a_t \\
\Leftrightarrow \quad & \frac{1-\theta B+\theta B-B}{1-\theta B}Z_t = a_t \\
\Leftrightarrow \quad & \left\{1-\frac{(1-\theta)B}{1-\theta B}\right\}Z_t = a_t \\
\Leftrightarrow \quad & Z_t - \frac{(1-\theta)B}{1-\theta B}Z_t = a_t
\end{aligned}
$$

Nonstationary Series的处理

$$Z_t - \frac{(1-\theta)B}{1-\theta B}Z_t = a_t$$

$$\frac{(1-\theta)B}{1-\theta B} = (1-\theta)[B+\theta B^2+\theta^2 B^3+\theta^3 B^4+\cdots]$$

Nonstationary Series的处理

$$Z_t - \frac{(1-\theta)B}{1-\theta B}Z_t = a_t$$

$$
\begin{aligned}
\frac{(1-\theta)B}{1-\theta B} &= (1-\theta)[B+\theta B^2+\theta^2 B^3+\theta^3 B^4+\cdots] \\
\frac{(1-\theta)B}{1-\theta B}Z_t &= (1-\theta)Z_{t-1}+\theta(1-\theta)Z_{t-2}+\theta^2(1-\theta)Z_{t-3}+\cdots
\end{aligned}
$$

Nonstationary Series的处理

$$Z_t - \frac{(1-\theta)B}{1-\theta B}Z_t = a_t$$

$$
\begin{aligned}
\frac{(1-\theta)B}{1-\theta B} &= (1-\theta)[B+\theta B^2+\theta^2 B^3+\theta^3 B^4+\cdots] \\
\frac{(1-\theta)B}{1-\theta B}Z_t &= (1-\theta)Z_{t-1}+\theta(1-\theta)Z_{t-2}+\theta^2(1-\theta)Z_{t-3}+\cdots
\end{aligned}
$$

$$\frac{1-\theta}{1-\theta B}Z_{t-1}$$

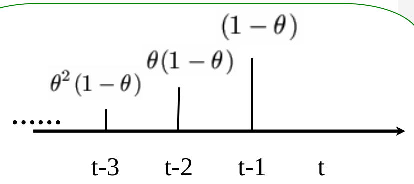
Nonstationary Series的处理

$$Z_t - \frac{(1-\theta)B}{1-\theta B} Z_t = a_t$$

$$\frac{(1-\theta)B}{1-\theta B} = (1-\theta)[B + \theta B^2 + \theta^2 B^3 + \theta^3 B^4 + \cdots]$$

$$\frac{(1-\theta)B}{1-\theta B} Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots$$

$$\frac{1-\theta}{1-\theta B} Z_{t-1}$$



Nonstationary Series的处理

$$Z_t - \frac{(1-\theta)B}{1-\theta B} Z_t = a_t$$

$$\frac{(1-\theta)B}{1-\theta B} = (1-\theta)[B + \theta B^2 + \theta^2 B^3 + \theta^3 B^4 + \cdots]$$

$$\frac{(1-\theta)B}{1-\theta B} Z_t = (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} + \cdots$$

$$(1-B)Z_t = (1-\theta B)a_t$$

"Exponential Smoothing"

ARIMA(p,1,q)

$$(1-B)Z_t = (1-\theta B)a_t$$

- 1st Diff MA(1)

ARIMA(p,1,q)

$$w_t = (1-B)Z_t$$

$$(1-\phi_1 B - \cdots - \phi_p B^p) w_t = (1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q) a_t$$

ARIMA(p,1,q)

$$w_t = (1 - B)Z_t$$

ARMA(p,q)

$$(1 - \phi_1 B - \cdots - \phi_p B^p)w_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t$$

ARIMA(p,1,q)

$$w_t = (1 - B)Z_t$$

ARMA(p,q)

$$(1 - \phi_1 B - \cdots - \phi_p B^p)w_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t$$

$$\Downarrow$$

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(1 - B)Z_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t$$

ARIMA(p,1,q)

$$w_t = (1 - B)Z_t$$

ARMA(p,q)

$$(1 - \phi_1 B - \cdots - \phi_p B^p)w_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t$$

$$\Downarrow$$

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(1 - B)Z_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t$$

- (p,1,q) model
- AutoRegressive Integrated Moving Average model

ARIMA(p,1,q)

$$w_t = (1 - B)Z_t$$

Key Features in Page 135 of Notes

$$(1 - \phi_1 B - \cdots - \phi_p B^p)w_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t$$

$$\Downarrow$$

$$(1 - \phi_1 B - \cdots - \phi_p B^p)(1 - B)Z_t = (1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t$$

- (p,1,q) model
- AutoRegressive Integrated Moving Average model

ARIMA(p,1,q)

$$\underbrace{(1-\phi_1 B-\cdots-\phi_p B^p)}_{\phi_p(B)}(1-B)Z_t = \underbrace{(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)}_{\theta_q(B)}a_t$$

$$\Downarrow$$

$$\phi_p(B)(1-B)Z_t = \theta_q(B)a_t$$

ARIMA(p,1,q)

$$\underbrace{(1-\phi_1 B-\cdots-\phi_p B^p)}_{\phi_p(B)}(1-B)Z_t = \underbrace{(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)}_{\theta_q(B)}a_t$$

$$\Downarrow$$

$$\phi_p(B)(1-B)Z_t = \theta_q(B)a_t$$

Two forms :
- ψ 's weight
- π 's weight

ARIMA(p,1,q)

$$\underbrace{(1-\phi_1 B-\cdots-\phi_p B^p)}_{\phi_p(B)}(1-B)Z_t = \underbrace{(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)}_{\theta_q(B)}a_t$$

$$\Downarrow$$

$$\phi_p(B)(1-B)Z_t = \theta_q(B)a_t$$

Two forms :
- ψ 's weight
- π 's weight

- variance $\gamma_0(t) = O(t)$
- property of π $\sum_{j=1}^{\infty} \pi_j = 1$

ARIMA(p,1,q)

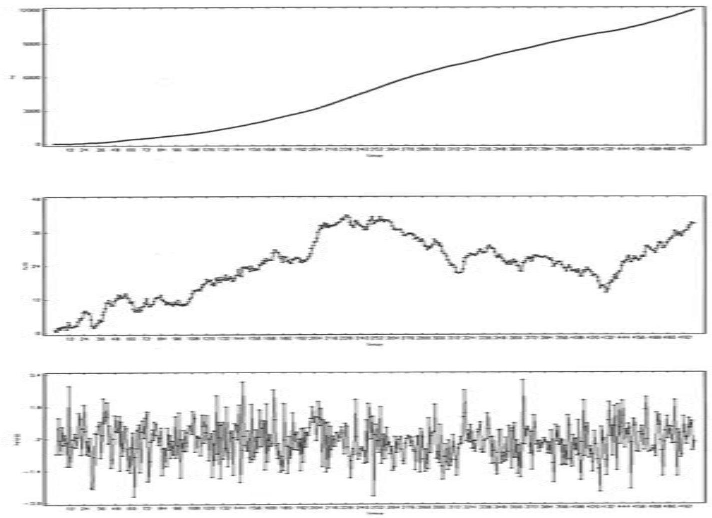
$$\underbrace{(1-\phi_1 B-\cdots-\phi_p B^p)}_{\phi_p(B)}(1-B)Z_t = \underbrace{(1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q)}_{\theta_q(B)}a_t$$

$$\Downarrow$$

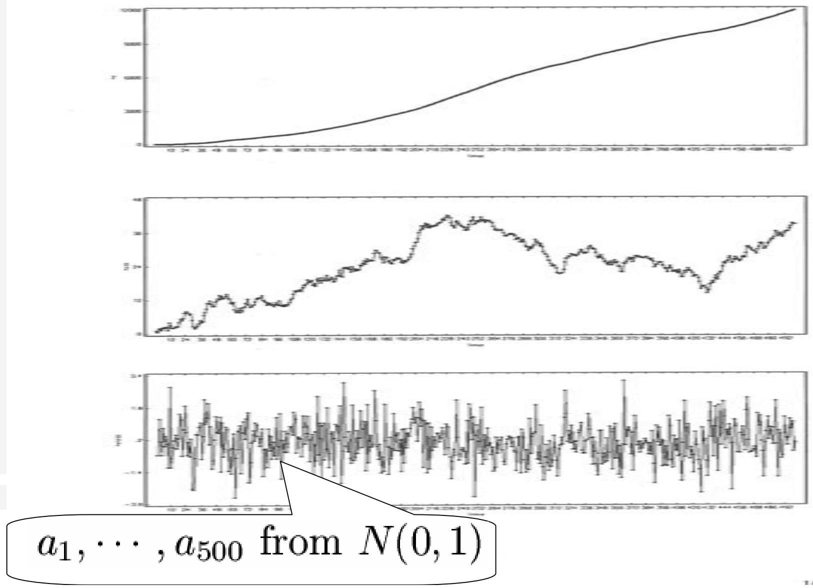
$$\phi_p(B)(1-B)Z_t = \theta_q(B)a_t$$

Two special forms :
- Random Walk : (0,1,0)
- 差分后为MA(1) : (0,1,1)

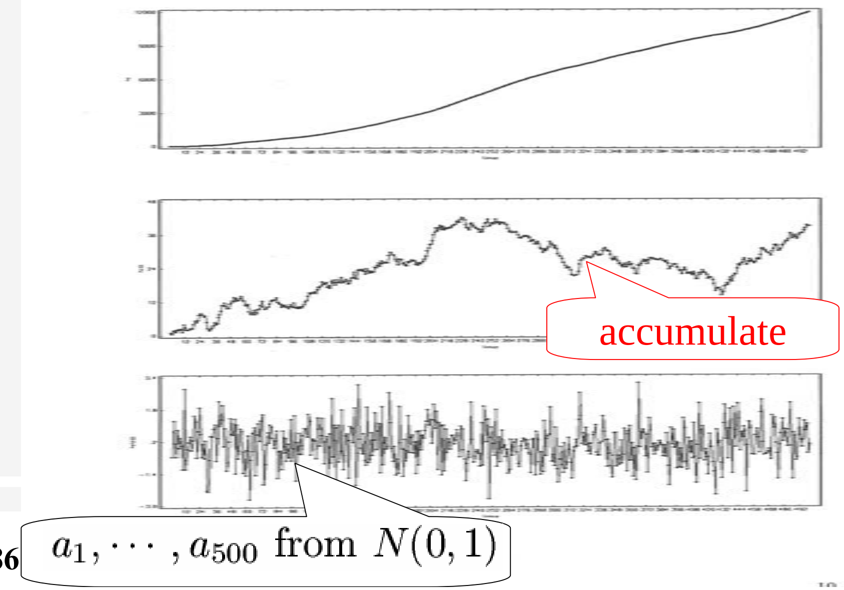
ARIMA(p,2,q)



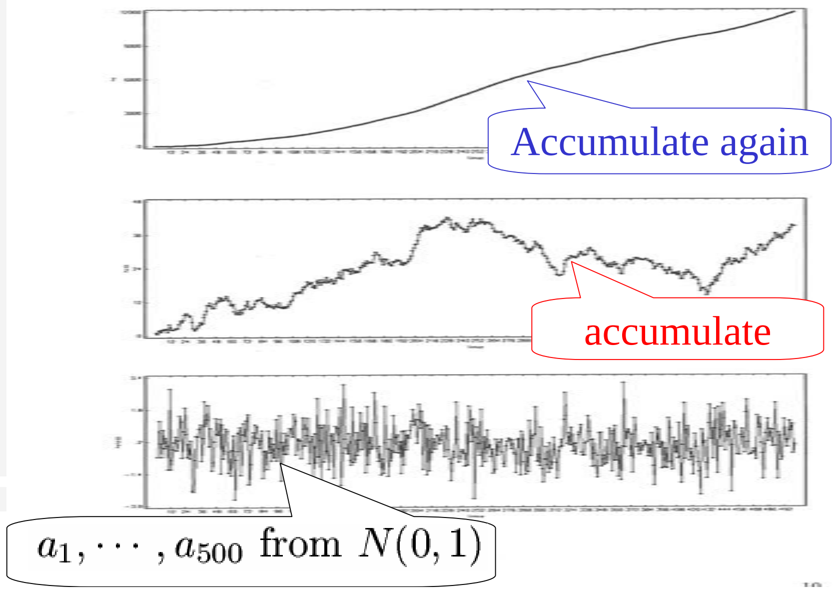
ARIMA(p,2,q)



ARIMA(p,2,q)



ARIMA(p,2,q)



ARIMA(p,2,q) p=q=0

$a_1, \cdots, a_{500}$ from $N(0,1)$

- See Page 136 of Notes

ARIMA(p,2,q) p=q=0

$a_1, \cdots, a_{500}$ from $N(0,1)$

$$\begin{aligned} u_1 &= a_1 \\ u_2 &= a_1 + a_2 \\ &\vdots \\ u_t &= a_1 + a_2 + \cdots + a_t \end{aligned}$$

- See Page 136 of Notes

ARIMA(p,2,q) p=q=0

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$$\begin{aligned} u_1 &= a_1 \\ u_2 &= a_1 + a_2 \\ &\vdots \\ u_t &= a_1 + a_2 + \cdots + a_t \end{aligned}$$

$$\begin{aligned} Z_1 &= u_1 \\ Z_2 &= u_1 + u_2 \\ Z_3 &= u_1 + u_2 + u_3 \\ &\vdots \end{aligned}$$

- See Page 136 of Notes

ARIMA(p,2,q) p=q=0

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ARIMA(p,2,q) p=q=0

$a_1, \cdots, a_{500}$ from $N(0,1)$

$$\begin{aligned} u_1 &= a_1 \\ u_2 &= a_1 + a_2 \\ &\vdots \\ u_t &= a_1 + a_2 + \cdots + a_t \end{aligned}$$

$$(1-B)u_t = a_t$$

$$(1-B)Z_t = u_t$$

$$\Downarrow$$

$$(1-B)^2 Z_t = a_t$$

$$\begin{aligned} Z_1 &= u_1 \\ Z_2 &= u_1 + u_2 \\ Z_3 &= u_1 + u_2 + u_3 \\ &\vdots \end{aligned}$$

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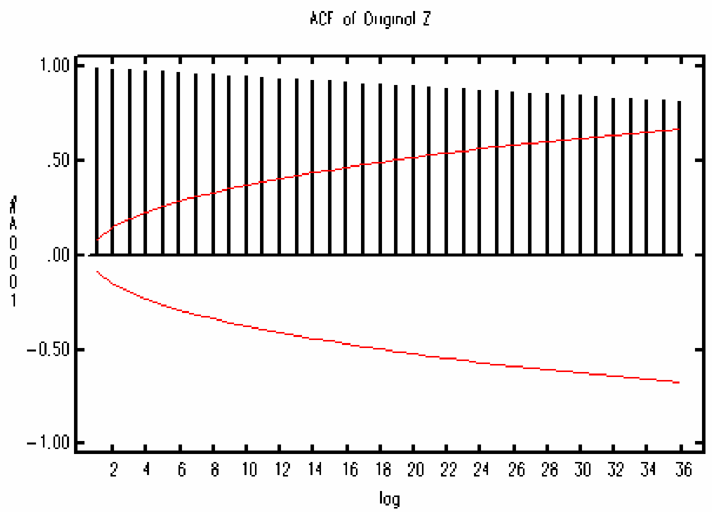
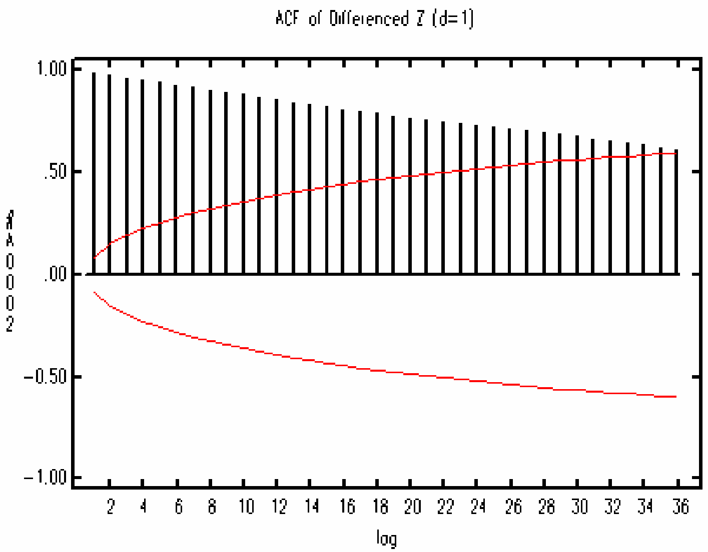
ARIMA(p,2,q) p=q=0

原始序列的
SACF



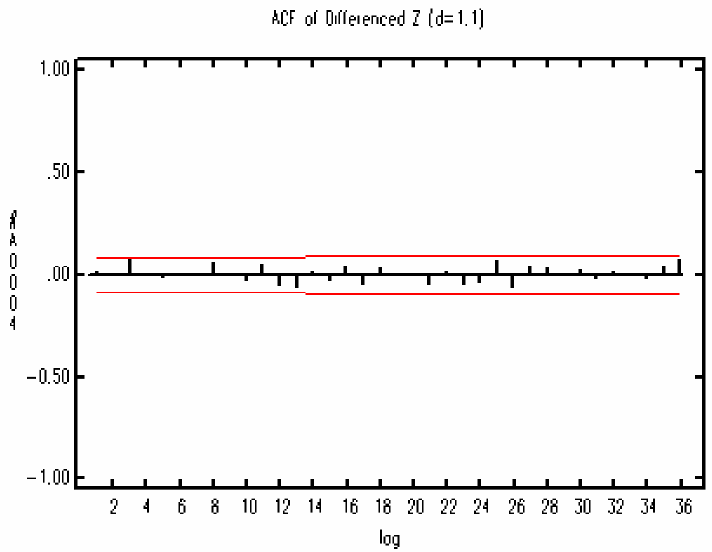
ARIMA(p,2,q) p=q=0

原始序列一次差分后的SACF



ARIMA(p,2,q) p=q=0

原始序列两次差分后的SACF



ARIMA(p,2,q) p=q=0

$$(1-B)^2 Z_t = a_t$$

$$\Leftrightarrow \quad Z_t = a_t + 2a_{t-1} + \cdots + (t-1)a_2 + ta_1$$

ARIMA(p,2,q) p=q=0

$$(1-B)^2 Z_t = a_t$$

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- m=1, Series start at t=1

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- m=1, Series start at t=1

$$\begin{aligned} \gamma_0(t) &= \sigma_a^2[1+2^2+3^2+\cdots+t^2] && > O(t) \\ &= \frac{\sigma_a^2}{6} t(t+1)(2t+1) && = O(t^3) \end{aligned}$$

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If series is stationary, then $\gamma_0(t) \to Constant$

ARIMA(p,2,q)

- Generalization :

$$\left.\begin{array}{l}(1-B)^2 Z_t = w_t \\ \phi_p(B) w_t = \theta_q(B)\end{array}\right\} \text{(p,2,q) model}$$

ARIMA(p,2,q)

- Generalization :

$$\left.\begin{array}{l}(1-B)^2 Z_t = w_t \\ \phi_p(B) w_t = \theta_q(B)\end{array}\right\} \text{(p,2,q) model}$$

- For all (p,2,q) models: $\gamma_0(t) = O(t^3)$

ARIMA(p,d,q)

Entire Class of (p,d,q) Model:

- stationary + invertible

$$\phi_p(B) Z_t = \theta_q(B) a_t$$

- nonstationary

$$\begin{aligned}\phi_p(B)(1-B)^d Z_t &= \theta_q(B) a_t \\ &\Downarrow \quad (1-B)^d Z_t = w_t \\ \phi_p(B)\omega_t &= \theta_q(B) a_t\end{aligned}$$

ARIMA(p,d,q)

Entire Class of (p,d,q) Model:

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p-th polynomial

d-th polynomial

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t$$

ARIMA(p,d,q)

Entire Class of (p,d,q) Model:

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t$$

•Writing:

$$\Phi(B) = \phi(B)(1-B)^d$$

•Then:

$$\Phi(B)Z_t = \theta_q(B)a_t$$

zeros can be **ON** or outside unit circle

ARIMA(p,d,q)

Entire Class of (p,d,q) Model:

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t$$

(p,d,q)

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ARIMA(p,d,q)

Entire Class of (p,d,q) Model:

$$\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t$$

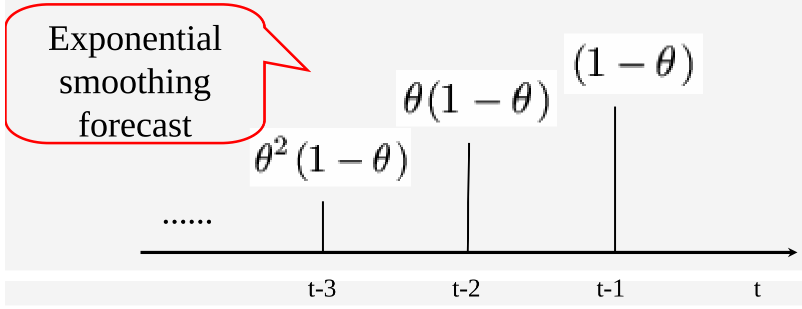
• See Page 136-140 of Packets

Example: ARIMA(0,2,2)

$$(1-B)^2 X_t \quad = \quad (1-\theta_1 B-\theta_2 B^2)a_t$$

• For ARIMA(0,1,1) :

$$(1-B)Z_t = (1-\theta B)a_t$$

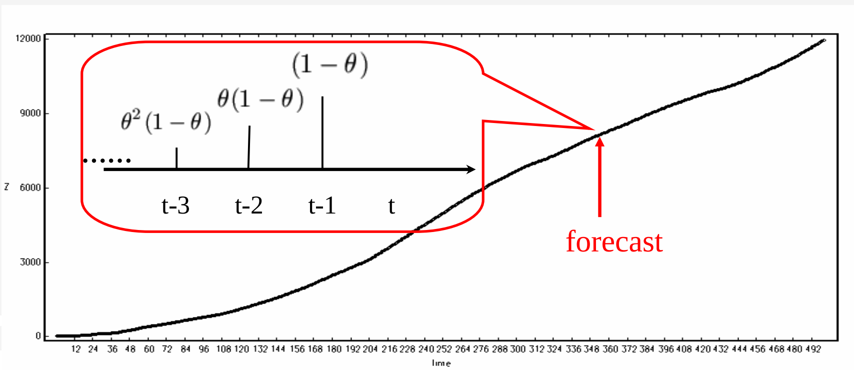


Example: ARIMA(0,2,2)

• ARIMA(0,1,1) 的缺陷:

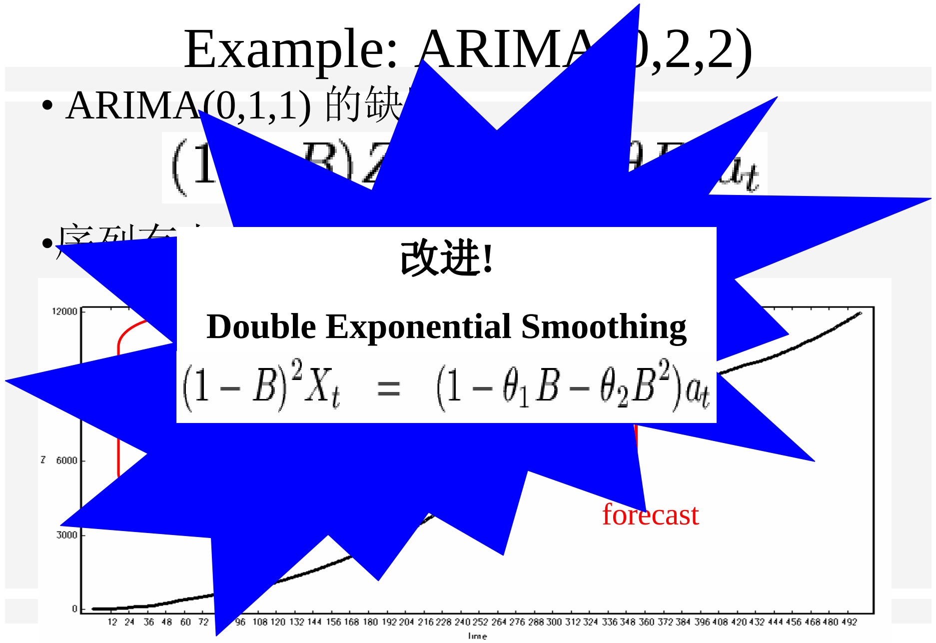
$$(1-B)Z_t = (1-\theta B)a_t$$

•序列有上升趋势时预测不好:



Example: ARIMA(0,2,2)

• ARIMA(0,1,1) 的缺



Example: ARIMA(0,2,2)

Double Exponential Smoothing:

- If the model is of the form:

$$(1-B)^2 Z_t = (1-\alpha B)^2 a_t$$

Example: ARIMA(0,2,2)

Double Exponential Smoothing:

- If the model is of the form:

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对比

$$\begin{aligned}(1-B)^2 X_t &= (1-\theta_1 B-\theta_2 B^2)a_t \\ &= (1-\alpha_1 B)(1-\alpha_2 B)a_t\end{aligned}$$

Example: ARIMA(0,2,2)

Double Exponential Smoothing:

Double exponential smoothing Formula

- If the model is of the form:

$$(1-B)^2 Z_t = (1-\alpha B)^2 a_t$$

对比

$$\begin{aligned}(1-B)^2 X_t &= (1-\theta_1 B-\theta_2 B^2)a_t \\ &= (1-\alpha_1 B)(1-\alpha_2 B)a_t\end{aligned}$$

课程内容回顾

- Time Series
- Pattern
- Conceptual Formulation
- Dynamical memory
- Linear Dynamical Model
- Parsimonious Models
- ARMA Models (stationary)
- ACF, PACF; Sample Version
- Nonstationary Models (Differencing)
- (p,d,q) class

• See Page 109-140 of Notes

时间序列分析

第五讲(上)

Today's Topic

1. Review of ARIMA
   - (p,d,q) models
2. Prediction
   - General MSE
   - AR(1)
   - AR(2)
   - MA(1)
   - ARIMA(0,1,1)
   - ARIMA(p,d,q)
3. Model Building
   - 3-stage iterative procedure
   - Tentative Specification : SACF, SPACF, SEACF

Review of ARIMA(p,d,q)

Z_t : obs. time series

Z_t	• 不差分,原始序列
$(1-B)Z_t$	• 一次差分
$(1-B)^2 Z_t$	• 二次差分
$(1-B)^d Z_t$	• d次差分

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差分后变为 stationary series

Review of ARIMA(p,d,q)

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$(1-B)^d Z_t$	• d次差分

差分后变为 stationary series

$$(1-\phi_1 B - \cdots - \phi_p B^p)(1-B)^d Z_t = (1-\theta_1 B - \cdots - \theta_q B^q) a_t$$

$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$

Review of ARIMA(p,d,q)

$$\textbf{ARIMA Representation :} \ \phi_p(B)(1-B)^d Z_t = \theta_q(B) a_t$$

$$\psi \textbf{ form :} \quad Z_t = \psi(B) a_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$$

$$\pi \textbf{ form :} \quad Z_t = \pi_1 Z_{t-1} + \pi_2 Z_{t-2} + \cdots + \pi_k Z_{t-k} + a_t$$

Today's Topic

1. Review of ARIMA
   - (p,d,q) models
2. Prediction
   - General MSE
   - AR(1)
   - AR(2)
   - MA(1)
   - ARIMA(0,1,1)
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Today's Topic

1. Review of ARIMA
   - (p,d,q) models
2. Prediction

   Based on the feature of the model
   - General MSE
   - AR(1)
   - AR(2)
   - MA(1)
   - ARIMA(0,1,1)
   - ARIMA(p,d,q)
3. Model Building
   - 3 stage iterative procedure
   - Tentative Specification : SACF, SPACF, SEACF

Prediction

Mean Square Error (MSE)

• See Page 140 of Notes

Prediction

Mean Square Error (MSE)

问题 :

Given 2 *r.v.*s x and y, find a h(y) to estimate/predict x

• See Page 140 of Notes

Prediction

Mean Square Error (MSE)

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误差平方(loss function) :

$$E[x - h(y)]^2$$

• See Page 140 of Notes

Prediction

Mean Square Error (MSE)

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Given 2 *r.v.*s x and y, find a h(y) to estimate/predict x

误差平方(loss function) :

$$E[x - h(y)]^2$$

最优估计 :

$$\min_x E[x - h(y)]^2 \Longrightarrow \widehat{x} = E(x|y)$$

• See Page 140 of Notes

Prediction

Mean Square Error (MSE)

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Given 2 r.v.s x and y, find a h(y) to estimate/predict x

误差平方(loss function) :

$$E[x - h(y)]^2$$

最优估计 :

$$\min_x E[x - h(y)]^2 \Longrightarrow \widehat{x} = \underbrace{E(x|}_{Z_t}\underbrace{y)}_{Z_{t-1}, Z_{t-2}, \cdots}$$

• See Page 140 of Notes

Prediction – AR(1)

模型:

$$Z_t = \phi Z_{t-1} + a_t$$

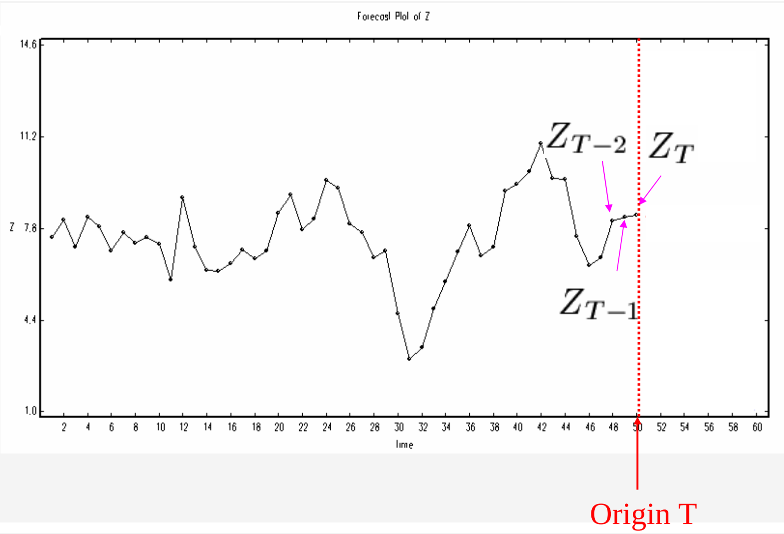
问题:

$Z_1, \cdots, Z_T$ known, predict $Z_{T+\ell}, \ell = 1, 2, 3, \cdots$

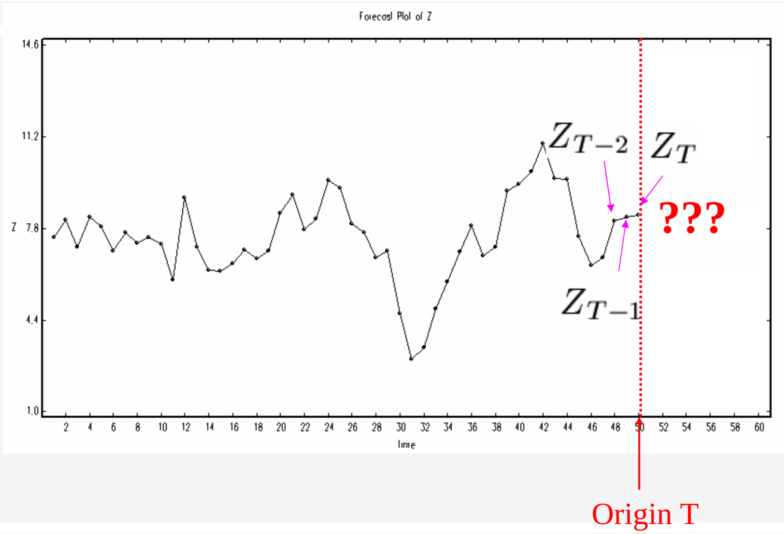
其中T: forecast origin

• See Page 140 of Notes

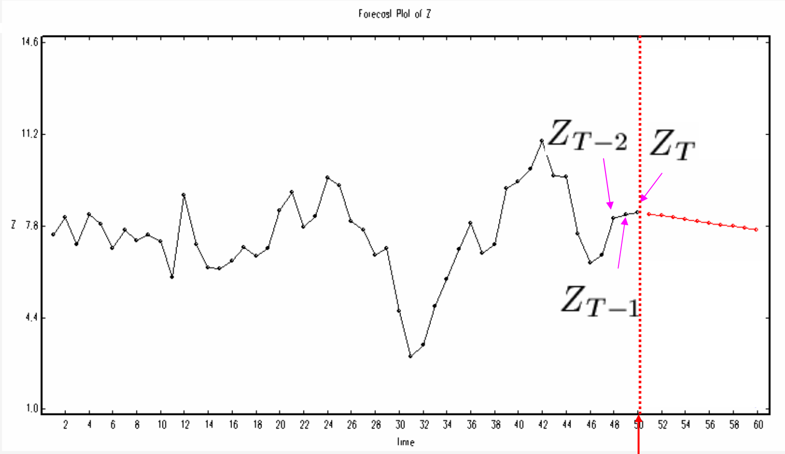
Prediction – AR(1)



Prediction – AR(1)

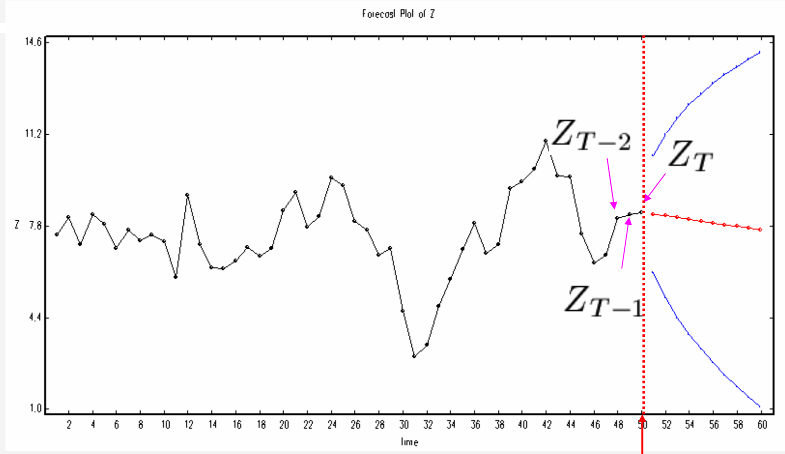


Prediction – AR(1)



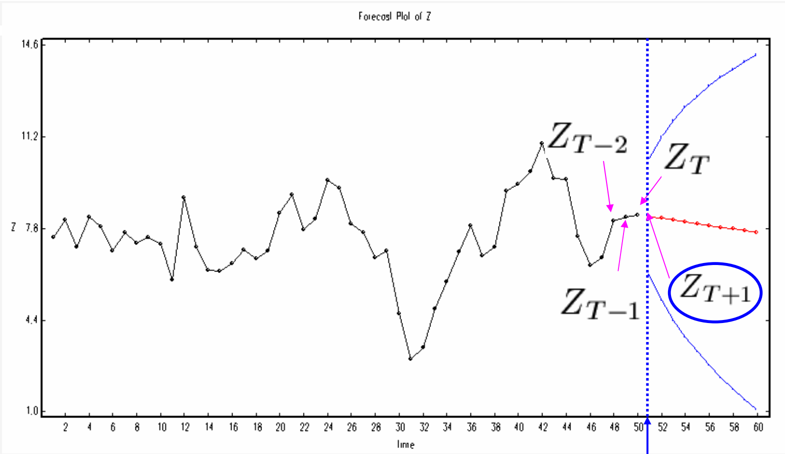
- best point forecast

Prediction – AR(1)



- best point forecast
- uncertainty

Prediction – AR(1)



- best point forecast
- uncertainty
- updating

Prediction – AR(1)

问题：

$Z_1, \cdots, Z_T$ known, predict $Z_{T+\ell}, \ell = 1, 2, 3, \cdots$

其中T: forecast origin

$\ell = 1$时, model $Z_{T+1} = \phi Z_T + a_{T+1}$

答案：

the expectation of Z_{T+1}, given $Z_T, Z_{T-1}, Z_{T-2}, \cdots$

Prediction – AR(1)

问题：

$Z_1, \cdots, Z_T$ known, predict $Z_{T+\ell}, \ell = 1, 2, 3, \cdots$

其中T: forecast origin

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答案：

the expectation of Z_{T+1}, given $Z_T, Z_{T-1}, Z_{T-2}, \cdots$

$\underset{T}{E}$

Prediction – AR(1)

模型：

$\ell = 1$时, model $Z_{T+1} = \phi Z_T + a_{T+1}$

答案：

$\underset{T}{E}$: conditional expectation given $Z_T, Z_{T-1}, Z_{T-2}, \cdots$

$$\begin{aligned}\underset{T}{E}(Z_{T+1}) &= \phi \underset{T}{E}(Z_T) + \underset{T}{E}(a_{T+1}) \\ &= \phi Z_T + 0\end{aligned}$$

$N(0, \sigma_a^2)$

Prediction – AR(1)

模型：

$\ell = 1$时, model $Z_{T+1} = \phi Z_T + a_{T+1}$

答案：

$\underset{T}{E}$: conditional expectation given $Z_T, Z_{T-1}, Z_{T-2}, \cdots$

$$\begin{aligned}\underset{T}{E}(Z_{T+1}) &= \phi \underset{T}{E}(Z_T) + \underset{T}{E}(a_{T+1}) \\ &= \phi Z_T + 0\end{aligned}$$

$$\widehat{Z_T(1)} = \phi Z_T$$

Prediction – AR(1)

1步向前预测:

Model	$Z_{T+1} = \phi Z_T + a_{T+1}$
MSE Forecast	$\widehat{Z_T(1)} = \phi Z_T$
Forecast Error	$e_T(1) = a_{T+1}$
Variance	$\mathrm{Var}(e_T(1)) = \sigma_a^2$
S.E.	σ_a

Prediction – AR(1)

2步向前预测:

Model	$Z_{T+2} = \phi Z_{T+1} + a_{T+2}$
MSE Forecast	$\widehat{Z_T(2)} = \phi \widehat{Z_T(1)} + 0$
Forecast Error	$e_T(2) = \phi e_T(1) + a_{T+2}$

Prediction – AR(1)

l 步向前预测:

MSE Forecast	$\widehat{Z_T(\ell)} = \phi Z_T\widehat{(\ell-1)} = \phi^\ell Z_t$
	$\longrightarrow 0$, when ℓ large
Forecast Error	$e_T(\ell) = \phi e_T(\ell-1) + a_{T+\ell}$

Prediction – AR(1)

l 步向前预测:

$$e_T(\ell) = a_{T+1} \qquad \ell = 1$$

$$e_T(\ell) = \phi e_T(\ell-1) + a_{T+\ell}$$

Prediction – AR(1)

l 步向前预测:

$$e_T(\ell) = a_{T+1} \qquad \ell = 1$$

$$e_T(\ell) = \phi e_T(\ell-1) + a_{T+\ell}$$

$$\begin{pmatrix} 1 & & & \\ -\phi & \ddots & & \\ & \ddots & \ddots & \\ & & -\phi & 1 \end{pmatrix} \begin{pmatrix} e_T(1) \\ e_T(2) \\ \vdots \\ e_T(\ell) \end{pmatrix} = \begin{pmatrix} a_{T+1} \\ a_{T+2} \\ \vdots \\ a_{T+\ell} \end{pmatrix}$$

$$e_T(\ell) = a_{T+\ell} + \phi a_{T+\ell-1} + \cdots + \phi^{\ell-1} a_{T+1}$$

Prediction – AR(1)

l 步向前预测:

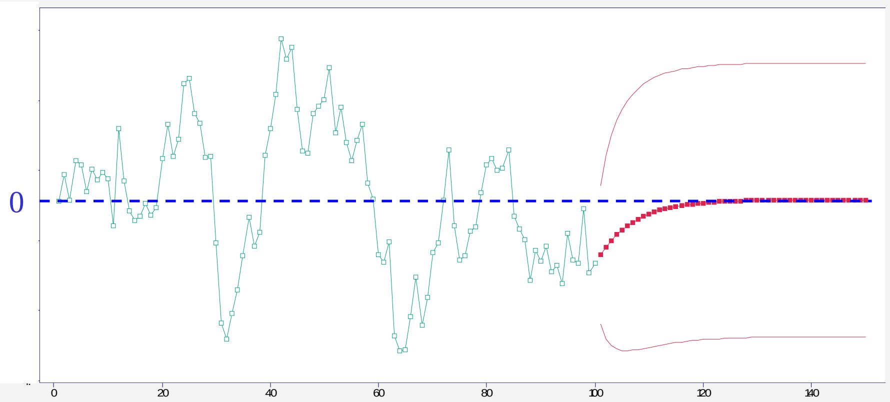
$$e_T(\ell) = a_{T+1} \qquad \ell = 1$$

$$e_T(\ell) = \phi e_T(\ell - 1) + a_{T+\ell}$$

$$\mathrm{Var}(e_T(\ell)) = \begin{cases} \sigma_a^2 & \ell = 1 \\ \frac{1-\phi^{2\ell}}{1-\phi^2}\sigma_a^2 & \ell > 1 \end{cases}$$

Prediction – AR(1)

向前预测的方差变化:



Prediction – AR(1)

向前预测的方差变化:

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Prediction – AR(2)

模型:

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

问题:

$Z_1, \cdots, Z_T$ known

Forecast $Z_{t+\ell}$, $\ell = 1, 2, 3, \cdots$

• See Page 143 of Notes

Prediction – AR(2)

模型:

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

l=1:

$$\begin{aligned} Z_{T+1} &= \phi_1 Z_T + \phi_2 Z_{T-1} + a_{T+1} \\ \widehat{Z_T(1)} &= \phi_1 Z_T + \phi_2 Z_{T-1} \\ e_T(1) &= a_{T+1} \end{aligned}$$

• See Page 143 of Notes

Prediction – AR(2)

模型:

$$Z_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

l=2:

$$\begin{aligned} Z_{T+2} &= \phi_1 Z_{T+1} + \phi_2 Z_T + a_{T+2} \\ \widehat{Z_T(2)} &= \phi_1 \widehat{Z_T(1)} + \phi_2 Z_T \\ e_T(2) &= \phi_1 e_T(1) + a_{T+2} \end{aligned}$$

• See Page 143 of Notes

Prediction – AR(2)

Stationary AR(2) 的共同性质:

模型:

$$(1 - \phi_1 B - \phi_2 B^2)\widehat{Z_T(\ell)} = 0$$

估计:

$$\begin{aligned} \widehat{Z_T(\ell)} &= A_1 \alpha_1^\ell + A_2 \alpha_2^\ell \\ \widehat{Z_T(\ell)} &\longrightarrow 0 \quad \ell \text{ large} \end{aligned}$$

误差:

$$\begin{aligned} e_T(\ell) &= a_{T+\ell} + \psi_1 a_{T+\ell-1} + \cdots + \psi_{\ell-1} a_{T+1} \\ \mathrm{Var}(e_T(\ell)) &= \sigma_a^2 (1 + \psi_1^2 + \cdots + \psi_{\ell-1}^2) \longrightarrow \gamma_0 \quad (\ell \text{ large}) \end{aligned}$$

Prediction – MA(1)

模型:

$$Z_t = a_t - \theta a_{t-1}$$

问题:

$Z_1, \cdots, Z_T$ known

Forecast $Z_{t+\ell}$, $\ell = 1, 2, 3, \cdots$

• See Page 143 of Notes

Prediction – MA(1)

模型:

$$Z_t = a_t - \theta a_{t-1}$$

1步向前预测:

$$\begin{aligned} Z_{T+1} &= a_{T+1} - \theta a_T \\ \widehat{Z_T(1)} &= \qquad - \theta \underset{T}{E}(a_T) \end{aligned}$$

• See Page 142 of Notes

Prediction – MA(1)

模型:

$$Z_t = a_t - \theta a_{t-1}$$

1步向前预测:

$$\begin{aligned} Z_{T+1} &= a_{T+1} - \theta a_T \\ \widehat{Z_T(1)} &= \qquad - \theta \underset{T}{E}(a_T) \end{aligned}$$

• See Page 142 of Notes

Prediction – MA(1)

模型:

$$Z_t = a_t - \theta a_{t-1}$$

1步向前预测:

$$\begin{aligned} Z_{T+1} &= a_{T+1} - \theta a_T \\ \widehat{Z_T(1)} &= \qquad - \theta \underset{T}{E}(a_T) \end{aligned}$$

2步向前预测:

$$\begin{aligned} Z_{T+2} &= a_{T+2} - \theta a_{T+1} \\ \widehat{Z_T(2)} &= 0 \end{aligned}$$

• See Page 142 of Notes

Prediction – MA(1)

模型:

$$Z_t = a_t - \theta a_{t-1}$$

Invertible $\Rightarrow a_T = \overbrace{Z_T - \theta Z_{T-1} - \theta^2 Z_{T-2} - \theta^3 Z_{T-3} - \cdots}^{\text{Everything known}}$

$$\widehat{Z_T(1)} = \qquad - \theta \underset{T}{E}(a_T)$$

2步向前预测:

$$\begin{aligned} Z_{T+2} &= a_{T+2} - \theta a_{T+1} \\ \widehat{Z_T(2)} &= 0 \end{aligned}$$

• See Page 142 of Notes

Prediction – MA(1)

模型:

$$Z_t = a_t - \theta a_{t-1}$$

1步向前预测:

$$\begin{aligned} Z_{T+1} &= a_{T+1} - \theta a_T \\ \widehat{Z_T(1)} &= -\theta \underset{T}{E}(a_T) \end{aligned}$$

$$\left\{ \begin{aligned} \underset{T}{E}(a_T) &= a_T \\ e_T(1) &\doteq a_{T+1} \end{aligned} \right.$$

2步向前预测:

$$\begin{aligned} Z_{T+2} &= a_{T+2} - \theta a_{T+1} \\ \widehat{Z_T(2)} &= 0 \end{aligned}$$

- See Page 142 of Notes

Prediction – MA(1)

1步向前预测:

$$\begin{aligned} \widehat{Z_T(1)} &= -\theta a_T \\ e_T(1) &\doteq a_{T+1} \\ \mathrm{Var}(e_T(1)) &= \sigma_a^2 \end{aligned}$$

*l*步向前预测:

$$\begin{aligned} e_T(\ell) &= a_{T+\ell} - \theta a_{T+\ell-1} \\ \mathrm{Var}(e_T(\ell)) &= (1+\theta^2)\sigma_a^2 = \gamma_0 \end{aligned}$$

- See Page 142 of Notes

时间序列分析

第五讲(下)

Prediction – MA(1)

1步向前预测:

$$\begin{aligned}\widehat{Z_T(1)} &= -\theta a_T \quad \text{(if invertible)}\\ e_T(1) &\doteq a_{T+1}\\ \mathrm{Var}(e_T(1)) &= \sigma_a^2\end{aligned}$$

l步向前预测:

$$\begin{aligned}\widehat{Z_T(\ell)} &= 0 \qquad \ell > 1\\ e_T(\ell) &= a_{T+\ell} - \theta a_{T+\ell-1}\\ \mathrm{Var}(e_T(\ell)) &= (1+\theta^2)\sigma_a^2 = \gamma_0\end{aligned}$$

• See Page 142 of Notes

Prediction – MA(1)

$$Z_t = a_t - \theta a_{t-1}$$

$|\theta| < 1$ invertible form

$|\theta| > 1$

$|\theta| = 1$?

• See Page 142 of Notes

Prediction – MA(1)

$$Z_t = a_t - \theta a_{t-1}$$

$|\theta| < 1$ invertible form

$|\theta| > 1$

$|\theta| = 1$?

$$\mathrm{Cov}\begin{pmatrix} Z_1 \\ \vdots \\ Z_T \\ Z_{T+1}\end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & 1+\theta^2 \end{pmatrix}\sigma_a^2$$

• See Page 144 of Notes

Prediction – MA(1)

$$Z_t = a_t - \theta a_{t-1}$$

$|\theta| < 1$ invertible form

Invertible $\Rightarrow a_T = Z_T - \theta Z_{T-1} - \theta^2 Z_{T-2} - \theta^3 Z_{T-3} - \cdots$

$$\mathrm{Cov}\begin{pmatrix} Z_1 \\ \vdots \\ Z_T \\ Z_{T+1} \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & 1+\theta^2 \end{pmatrix} \sigma_a^2$$

- See Page 144 of Notes

本课时内容

1. (0,1,1) model forecast
2. (p,d,q) model forecast in general

Prediction – ARIMA(0,1,1)

模型:

$$Z_t = Z_{t-1} + a_t - \theta a_{t-1}$$

问题:

$Z_1, \cdots, Z_T$ known, predict $Z_{T+\ell}, \ell = 1, 2, 3, \cdots$

其中T: forecast origin

Prediction – ARIMA(0,1,1)

$$\begin{aligned} (1-B)Z_t &= (1-\theta B)a_t \\ Z_t &= (1-\theta)Z_{t-1} + \theta(1-\theta)Z_{t-2} + \theta^2(1-\theta)Z_{t-3} \\ &\quad + \cdots + a_t \end{aligned}$$

Prediction – ARIMA(0,1,1)

l=1时 (invertible):

$$Z_{T+1} = (1-\theta)Z_T + \theta(1-\theta)Z_{T-1} + \cdots + a_{T+1}$$

Prediction – ARIMA(0,1,1)

l=1时 (invertible):

$$Z_{T+1} = (1-\theta)Z_T + \theta(1-\theta)Z_{T-1} + \cdots + a_{T+1}$$

Invertible when | θ | < 1

Prediction – ARIMA(0,1,1)

l=1时 (invertible):

$$\begin{aligned} Z_{T+1} &= (1-\theta)Z_T + \theta(1-\theta)Z_{T-1} + \cdots + a_{T+1} \\ &\Downarrow \\ \widehat{Z_T(1)} &= (1-\theta)Z_T \qquad + \underset{T}{E}(a_{T+1}) \\ &+ \theta(1-\theta)Z_{T-1} \\ &+ \theta^2(1-\theta)Z_{T-2} \\ &+ \theta^3(1-\theta)Z_{T-3} \\ &+ \vdots \end{aligned}$$

Invertible when | θ | < 1

Prediction – ARIMA(0,1,1)

l=1时 (invertible):

$$\begin{aligned} Z_{T+1} &= (1-\theta)Z_T + \theta(1-\theta)Z_{T-1} + \cdots + a_{T+1} \\ &\Downarrow \\ \widehat{Z_T(1)} &= (1-\theta)Z_T \qquad + \underset{T}{E}(a_{T+1}) \\ &+ \theta(1-\theta)Z_{T-1} \\ &+ \theta^2(1-\theta)Z_{T-2} \\ &+ \theta^3(1-\theta)Z_{T-3} \\ &+ \vdots \end{aligned}$$

Invertible when | θ | < 1

0

Prediction – ARIMA(0,1,1)

l=1时 (invertible):

$$Z_{T+1} = (1-\theta)Z_T + \theta(1-\theta)Z_{T-1} + \cdots + a_{T+1}$$

Invertible when | θ | < 1

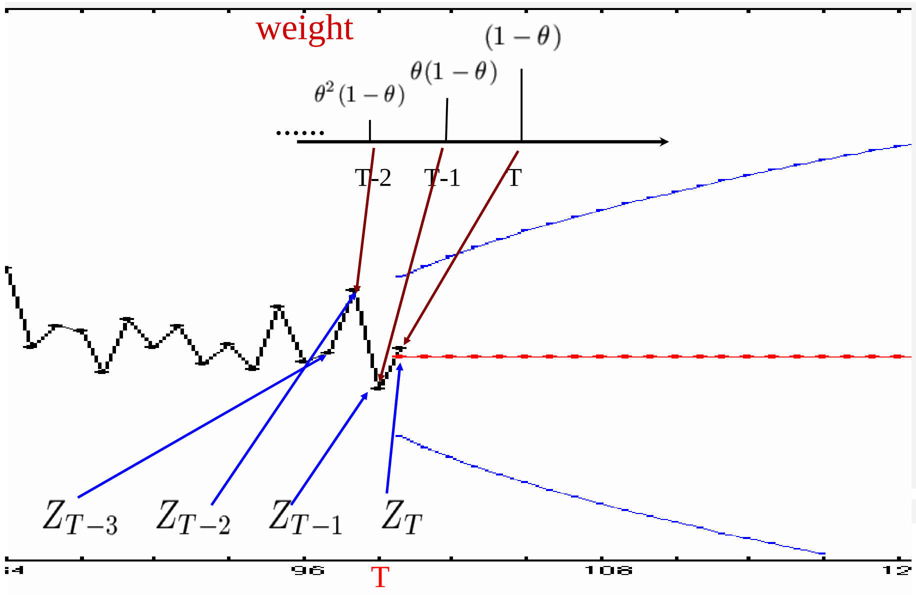
$$\Downarrow$$

$$\begin{aligned}
\widehat{Z_T(1)} &= (1-\theta)Z_T \qquad + \underset{T}{E}(a_{T+1}) \\
&+ \theta(1-\theta)Z_{T-1} \\
&+ \theta^2(1-\theta)Z_{T-2} \\
&+ \theta^3(1-\theta)Z_{T-3} \\
&+ \vdots
\end{aligned}$$

0

$$e_T(1) = 0 + \qquad a_{T+1}$$

Prediction – ARIMA(0,1,1)



Prediction – ARIMA(0,1,1)

l=2时 :

$$Z_{T+2} = Z_{T+1} + a_{T+2} - \theta a_{T+1}$$

Prediction – ARIMA(0,1,1)

l=2时 :

$$Z_{T+2} = Z_{T+1} + a_{T+2} - \theta a_{T+1}$$

$$\Rightarrow \widehat{Z_T(2)} = \widehat{Z_T(1)}$$

l=3时 :

$$Z_{T+3} = Z_{T+2} + a_{T+3} - \theta a_{T+2}$$

$$\Rightarrow \widehat{Z_T(3)} = \widehat{Z_T(2)}$$

Prediction – ARIMA(0,1,1)

l=1时 (invertible):

$$
\begin{aligned}
Z_{T+1} &= (1-\theta)Z_T + \theta(1-\theta)Z_{T-1} + \cdots + a_{T+1} \\
&\Downarrow \\
\widehat{Z_T(1)} &= (1-\theta)Z_T \qquad + \underset{T}{E}(a_{T+1}) \\
&+ \theta(1-\theta)Z_{T-1} \\
&+ \theta^2(1-\theta)Z_{T-2} \\
&+ \theta^3(1-\theta)Z_{T-3} \\
&+ \vdots
\end{aligned}
$$

$$\mathrm{Var}(e_T(1)) = \sigma_a^2$$

$$e_T(1) = 0 + \qquad a_{T+1}$$

Prediction – ARIMA(0,1,1)

l=2时：

$$
\begin{aligned}
Z_{T+2} &= Z_{T+1} + a_{T+2} - \theta a_{T+1} \\
\widehat{Z_T(2)} &= \widehat{Z_T(1)} \\
e_T(2) &= e_T(1) + a_{T+2} - \theta a_{T+1} \\
&= a_{T+1} + a_{T+2} - \theta a_{T+1} \\
&= a_{T+2} + (1-\theta)a_{T+1} \\
\mathrm{Var}(e_T(2)) &= \sigma_a^2[1 + (1-\theta)^2]
\end{aligned}
$$

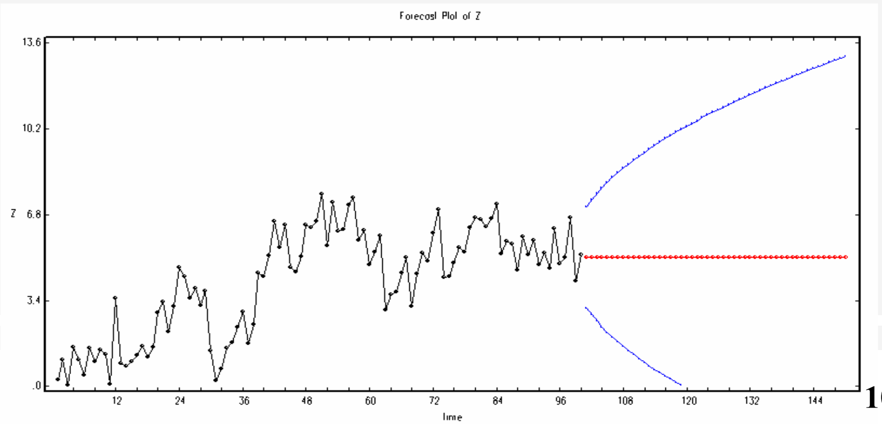
Prediction – ARIMA(0,1,1)

l=3时：

$$
\begin{aligned}
Z_{T+3} &= Z_{T+2} + a_{T+3} - \theta a_{T+2} \\
\widehat{Z_T(3)} &= \widehat{Z_T(2)} \\
e_T(3) &= e_T(2) + a_{T+3} - \theta a_{T+2} \\
&= a_{T+3} + (1-\theta)a_{T+2} + (1-\theta)a_{T+1} \\
\mathrm{Var}(e_T(3)) &= \sigma_a^2[1 + 2(1-\theta)^2]
\end{aligned}
$$

Prediction – ARIMA(0,1,1)

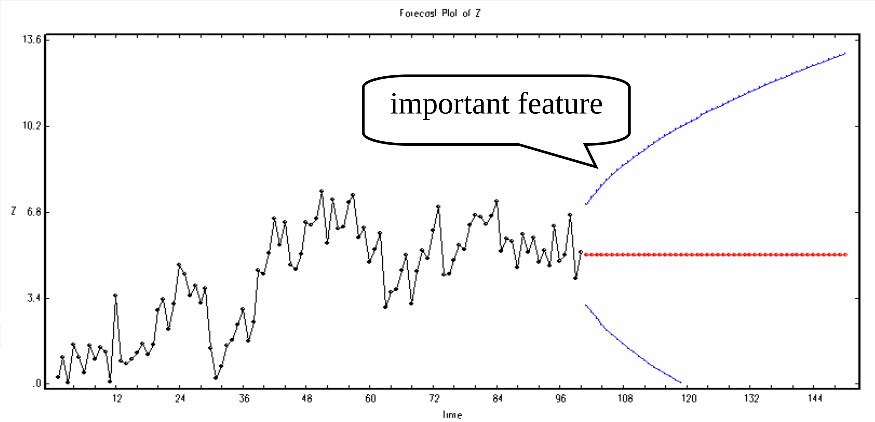
$$
\begin{aligned}
\widehat{Z_T(\ell)} &= \widehat{Z_T(1)} \\
\mathrm{Var}(e_T(\ell)) &= \sigma_a^2[1 + (\ell-1)(1-\theta)^2]
\end{aligned}
$$



Prediction – ARIMA(0,1,1)

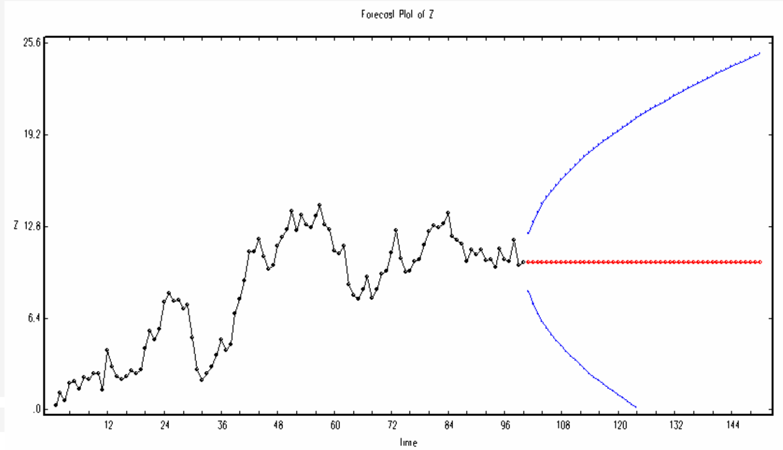
$$\widehat{Z_T(\ell)} = \widehat{Z_T(1)}$$

$$\mathrm{Var}(e_T(\ell)) = \sigma_a^2[1 + (\ell - 1)(1 - \theta)^2]$$



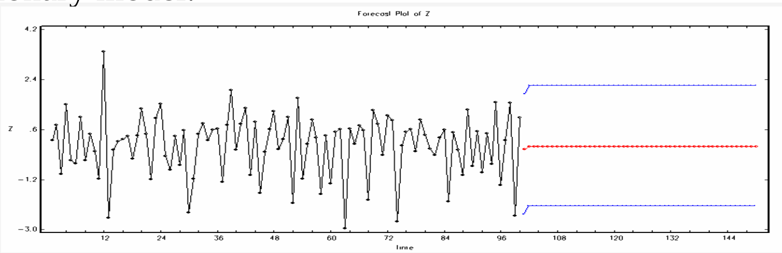
Prediction – ARIMA(0,1,1)

Random Walk: $(1 - B)Z_t = a_t$

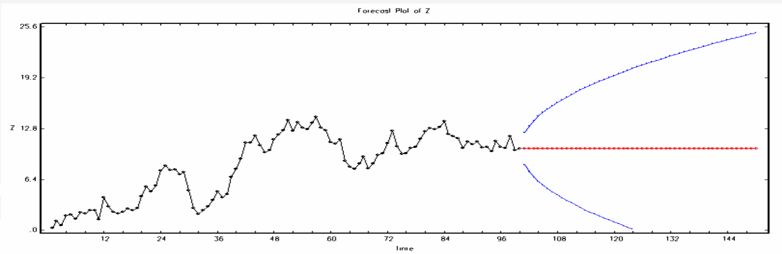


Prediction – ARIMA(0,1,1)

Stationary model:

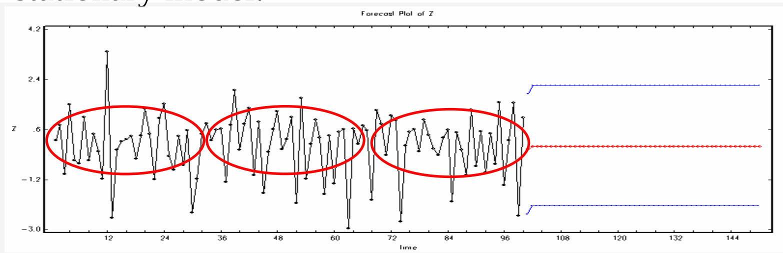


Nonstationary model: (0,1,1)

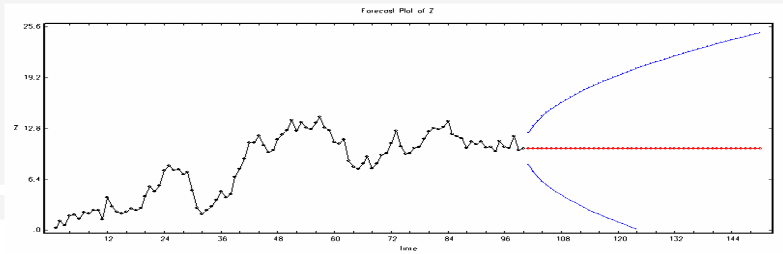


Prediction – ARIMA(0,1,1)

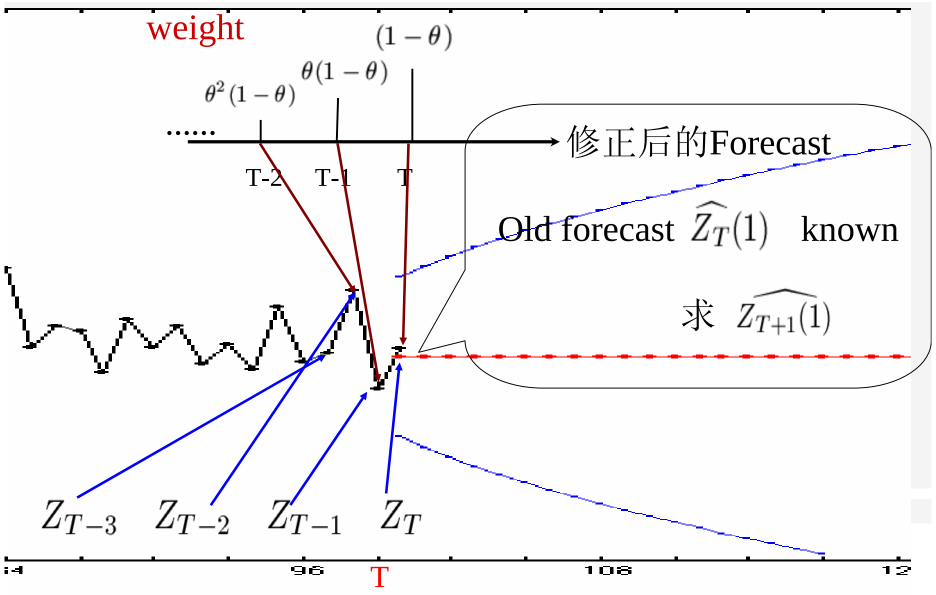
Any stationary model:



Nonstationary model: (0,1,1)



Prediction – ARIMA(0,1,1)



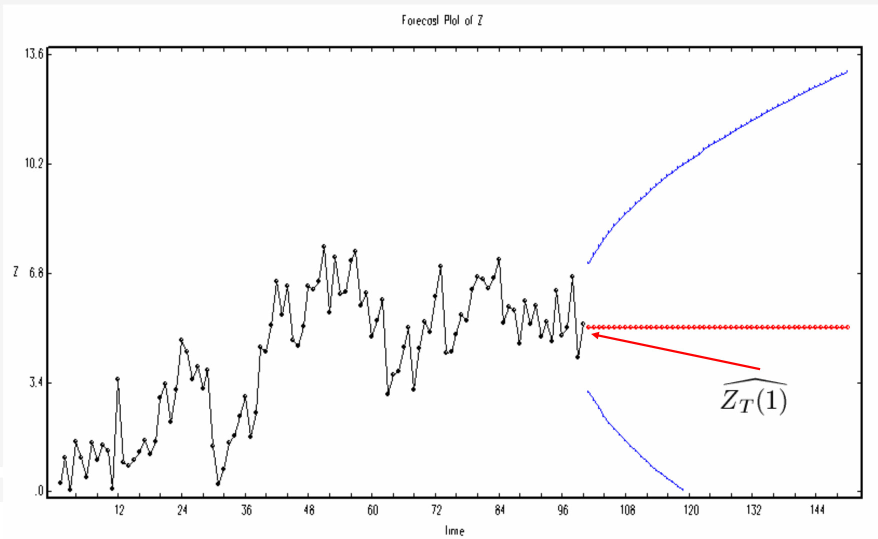
Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}

$$
\begin{aligned}
Z_{T+2} &= Z_{T+1} + a_{T+2} - \theta a_{T+1} \\
\widehat{Z_{T+1}(1)} &= Z_{T+1} + 0 - \theta[Z_{T+1} - \widehat{Z_T(1)}] \\
&= (1-\theta)Z_{T+1} + \theta\widehat{Z_T(1)}
\end{aligned}
$$

Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}



Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}

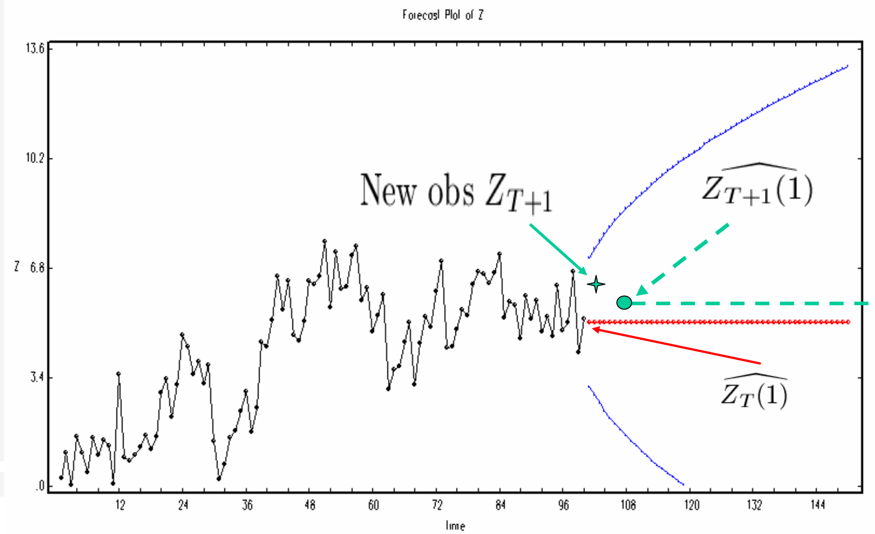
Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}

$$
\begin{aligned}
Z_{T+2} &= Z_{T+1} + a_{T+2} - \theta a_{T+1} \\
\widehat{Z_{T+1}(1)} &= Z_{T+1} + 0 - \theta[Z_{T+1} - \widehat{Z_T(1)}] \\
&= (1-\theta)Z_{T+1} + \theta\widehat{Z_T(1)}
\end{aligned}
$$

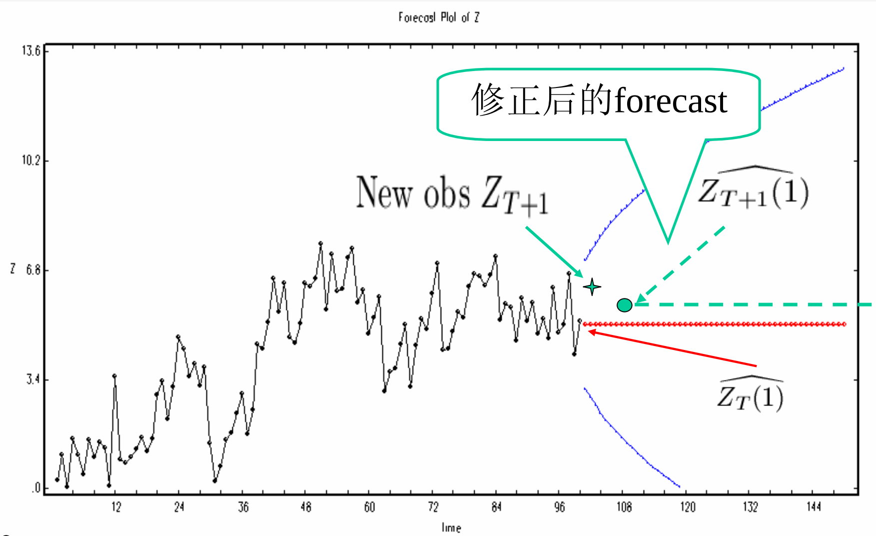
Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}



Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}



Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}

$$
\begin{aligned}
Z_{T+2} &= Z_{T+1} + a_{T+2} - \theta a_{T+1} \\
\widehat{Z_{T+1}(1)} &= Z_{T+1} + 0 - \theta[Z_{T+1} - \widehat{Z_T(1)}] \\
&= (1-\theta)Z_{T+1} + \theta\widehat{Z_T(1)}
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\widehat{Z_{T+1}(1)} &= \widehat{Z_T(1)} + (1-\theta)Z_{T+1} - (1-\theta)\widehat{Z_T(1)} \\
\widehat{Z_{T+1}(1)} &= \widehat{Z_T(1)} + (1-\theta)[Z_{T+1} - \widehat{Z_T(1)}]
\end{aligned}
}
$$

$$
\text{new forecast} = \text{old forecast} + (1-\theta) \times \text{forecast error}
$$

Prediction – ARIMA(0,1,1)

At time T+1, Z_{T+1} known, forecast Z_{T+2}

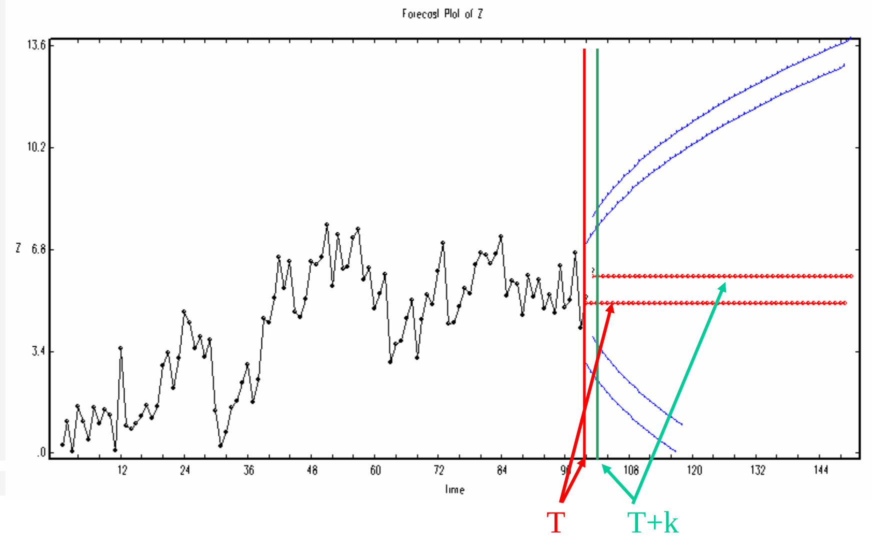
$$\begin{aligned}
Z_{T+2} &= Z_{T+1} + a_{T+2} - \theta a_{T+1} \\
\widehat{Z_{T+1}(1)} &= Z_{T+1} + 0 - \theta[Z_{T+1} - \widehat{Z_T(1)}] \\
&= (1-\theta) Z_{T+1} + \theta \widehat{Z_T(1)} \\
\widehat{Z_{T+1}(1)} &= \widehat{Z_T(1)} + (1-\theta) Z_{T+1} - (1-\theta)\widehat{Z_T(1)} \\
\widehat{Z_{T+1}(1)} &= \widehat{Z_T(1)} + (1 -
\end{aligned}$$

Updating formula

$$\boxed{\text{new forecast} = \text{old forecast} + (1-\theta) \times \text{forecast error}}$$

Prediction – ARIMA(0,1,1)

Nonstationary, Simple Example (0,1,1) :



Prediction – ARIMA(0,1,1)

Example: Page 260 of Packets

Prediction – ARIMA(p,d,q)

模型: $\phi_p(B)(1-B)^d Z_t = \theta_q(B) a_t$

其中:

$$\begin{aligned}
\phi_p(B) &= 1 - \phi_1 B - \cdots - \phi_p B^p \\
\text{stationary condition} &: \text{all roots outside unit circle} \\
\theta_q(B) &= 1 - \theta_1 B - \cdots - \theta_q B^q \\
\text{invertible condition} &: \text{all roots outside unit circle}
\end{aligned}$$

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

模型: $\phi_p(B)(1-B)^d Z_t = \theta_q(B)a_t$

$$\Phi_{p+d}(B) = 1 - \Phi_1 B - \cdots - \Phi_{p+d}B^{p+d}$$

$$\Phi_{p+d}(B)Z_t = \theta_q(B)a_t$$

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

模型:

$$\begin{aligned} & Z_t - \Phi_1 Z_{t-1} - \cdots - \Phi_{p+d} Z_{t-(p+d)} \\ = \ & a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q} \end{aligned}$$

问题:

$Z_1, \cdots, Z_T$ known, predict $Z_{T+\ell}, \ell = 1, 2, 3, \cdots$

其中T: forecast origin

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

l=1:

$$\begin{aligned} Z_{T+1} &= \Phi_1 Z_T + \cdots + \Phi_{p+d} Z_{(T+1)-(p+d)} \\ &\quad + a_{T+1} - \theta_1 a_T - \cdots - \theta_q a_{T+1-q} \\ \widehat{Z_T(1)} &= \Phi_1 Z_T + \cdots + \Phi_{p+d} Z_{(T+1)-(p+d)} \\ &\quad + 0 - \theta_1 a_T - \cdots - \theta_q a_{T+1-q} \\ e_T(1) &= a_{T+1} \end{aligned}$$

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

l=2:

$$\begin{aligned} Z_{T+2} &= \Phi_1 Z_{T+1} + \cdots + \Phi_{p+d} Z_{(T+2)-(p+d)} \\ &\quad + a_{T+2} - \theta_1 a_{T+1} - \cdots - \theta_q a_{T+2-q} \\ \widehat{Z_T(2)} &= \phi_p(B)(1-B)^d + \cdots \end{aligned}$$

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

预测时参数值：

$$E(Z_{T+j}) = \begin{cases} Z_{T+j} & j \le 0 \\ \widehat{Z_T(j)} & j > 0 \end{cases}$$

$$E(a_{T+j}) = \begin{cases} 0 & j > 0 \\ a_{T+j} & j \le 0 \end{cases}$$

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

For $\ell > \max(p+d, q)$:

$$(1 - \Phi_1 B - \cdots - \Phi_{p+d} B^{p+d})\widehat{Z_T(\ell)} = 0$$

Solution:

$$\widehat{Z_T(\ell)} = b_0 + \cdots + b_{d-1}\ell^{d-1} + c_1\alpha_1^\ell + \cdots + c_p\alpha_p^\ell$$

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

For $\ell > \max(p+d, q)$:

$$(1 - \Phi_1 B - \cdots - \Phi_{p+d} B^{p+d})\widehat{Z_T(\ell)} = 0$$

Solution:

$$\widehat{Z_T(\ell)} = b_0 + \cdots + b_{d-1}\ell^{d-1} + c_1\alpha_1^\ell + \cdots + c_p\alpha_p^\ell$$

non-stationary

Transient when | α |<1

• See Page 147-149 of Notes

Prediction – ARIMA(p,d,q)

For $\ell > \max(p+d, q)$:

$$(1 - \Phi_1 B - \cdots - \Phi_{p+d} B^{p+d})\widehat{Z_T(\ell)} = 0$$

Solution:

$$\widehat{Z_T(\ell)} = b_0 + \cdots + b_{d-1}\ell^{d-1} + c_1\alpha_1^\ell + \cdots + c_p\alpha_p^\ell$$

non-stationary

Transient when | α |<1

$d=1$	b_0	horizontal line
$d=2$	$b_0 + b_1\ell$	straight line
$d=3$	$b_0 + b_1\ell + b_2\ell^2$	quadratic line

• See Page 147-149 of Notes

下节课时内容

Model Building

How to pick a member from the ARIMA(p,d,q) class to fit your data at hand?

$\Downarrow$

Iterative Process:

Trail+Error

• See Page 149-159 of Notes

时间序列分析

第六讲(上)

本课时内容

Model Building : Conceptual Framework

1. Tentative Specification
   - SACF
   - SPACF
   - SEACF (EACF)
2. Estimation (MLE)
   - AR(1)
   - AR(p)
3. Diagnostic Checking

ARIMA models

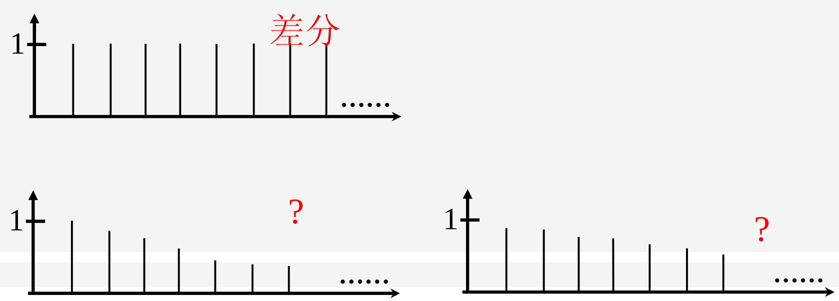
ACF	MA	Cut-off property
		Differencing
SACF		Decide on MA order
PACF	AR	Cut-off property
SPACF		Decide on AR order

最常用的方法! 通过SACF和SPACF来判定模型

• See Page 149-150 of Notes

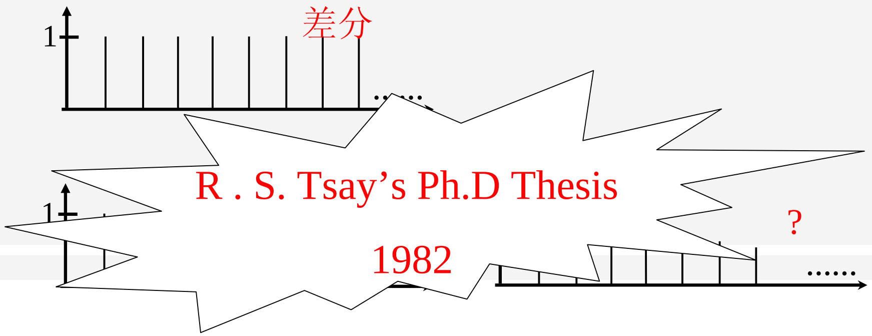
通过SACF和SPACF判断模型的缺点

1. mixed model
   - Difficult to specify mixed model
2. non-stationary model
   - Often hard to decide on differencing



通过SACF和SPACF判断模型的缺点

1. mixed model
   - Difficult to specify mixed model
2. non-stationary model
   - Often hard to decide on differencing



ARIMA(p,d,q) 回顾

$$(1-\phi_1 B - \cdots - \phi_p B^p)(1-B)^d Z_t = (1-\theta_1 B - \cdots - \theta_q B^q)a_t$$

ARIMA(p,d,q) 回顾

$$\underbrace{(1-\phi_1 B - \cdots - \phi_p B^p)(1-B)^d}\, Z_t = (1-\theta_1 B - \cdots - \theta_q B^q)a_t$$

$$\Phi(B) = (1-\Phi_1 B - \cdots - \Phi_{p+d} B^{p+d})$$

order: (p+d)

order : (p+d, q)

ARMA(P,d) = ARMA(p+d,q) ⇔ ARIMA(p,d,q)

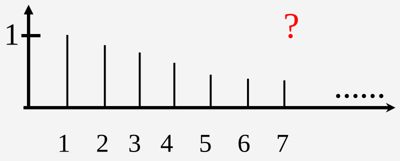
Stationary ARMA(1,1)

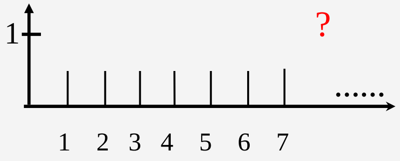
Example : From Page 233 of Packets

- See Page 151 of Notes

Stationary ARIMA(1,1)

Example : 是否差分？





EACF for Stationary ARMA(1,1)

$$Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}$$

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + +\psi_3 a_{t-3} + \cdots$$

Covariance Structure:

$$EZ_{t-k}(Z_t - \phi Z_{t-1}) = EZ_{t-k}(a_t - \theta a_{t-1})$$

$$k=1 \qquad \rho_1 - \phi\rho_0 = -\theta\sigma_a^2/\gamma_0 \qquad \rho_1 \begin{cases} = \phi & \text{if } \theta = 0 \\ \neq \phi & \text{if } \theta \neq 0 \end{cases}$$

$$k=2 \qquad \rho_2 - \phi\rho_1 = 0 \qquad \phi = \rho_2/\rho_1$$

$$k=3 \qquad \rho_3 - \phi\rho_2 = 0 \qquad \phi = \rho_3/\rho_2$$

EACF for Stationary ARMA(1,1)

$$Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}$$

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + +\psi_3 a_{t-3} + \cdots$$

Covariance Structure:

$$EZ_{t-k}(Z_t - \phi Z_{t-1}) = EZ_{t-k}(a_t - \theta a_{t-1})$$

$\neq$0 unless θ =0

$$k=1 \qquad \rho_1 - \phi\rho_0 = -\theta\sigma_a^2/\gamma_0 \qquad \rho_1 \begin{cases} = \phi & \text{if } \theta = 0 \\ \neq \phi & \text{if } \theta \neq 0 \end{cases}$$

$$k=2 \qquad \rho_2 - \phi\rho_1 = 0 \qquad \phi = \rho_2/\rho_1$$

$$k=3 \qquad \rho_3 - \phi\rho_2 = 0 \qquad \phi = \rho_3/\rho_2$$

EACF for Stationary ARMA(1,1)

$$Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}$$

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + +\psi_3 a_{t-3} + \cdots$$

Covariance Structure:

$$EZ_{t-k}(Z_t - \phi Z_{t-1}) = EZ_{t-k}(a_t - \theta a_{t-1})$$

$$k=1 \qquad \rho_1 - \phi\rho_0 = -\theta\sigma_a^2/\gamma_0 \qquad \rho_1 \begin{cases} = \phi & \text{if } \theta = 0 \\ \neq \phi & \text{if } \theta \neq 0 \end{cases}$$

$$k=2 \qquad \rho_2 - \phi\rho_1 = 0 \qquad \phi = \rho_2/\rho_1$$

$$k=3 \qquad \rho_3 - \phi\rho_2 = 0 \qquad \phi = \rho_3/\rho_2$$

EACF for Stationary ARMA(1,1)

If we know ϕ =.7, then let :

$$w_t = Z_t - .7Z_{t-1}$$

Note: w_t can be calculated from data !

$$\{Z_t\} \to \{w_t\}$$

Then $w_t = a_t - \theta a_{t-1}$ is a MA(1) model !

EACF for Stationary ARMA(1,1)

$$k=1 \qquad \rho_1 - \phi\rho_0 = -\theta\sigma_a^2/\gamma_0 \qquad \rho_1 \begin{cases} = \phi & \text{if } \theta = 0 \\ \neq \phi & \text{if } \theta \neq 0 \end{cases}$$

$$k=2 \qquad \rho_2 - \phi\rho_1 = 0 \qquad \phi = \rho_2/\rho_1$$

$$k=3 \qquad \rho_3 - \phi\rho_2 = 0 \qquad \phi = \rho_3/\rho_2$$

$$k=4 \qquad \rho_4 - \phi\rho_3 = 0 \qquad \phi = \rho_4/\rho_3$$

…

Consider : $\rho_{\ell+1} - \phi^{(\ell)}\rho_\ell = 0$

Define : $\phi^{(\ell)}$ solution of the above equation

EACF for Stationary ARMA(1,1)

Consider : $\rho_{\ell+1} - \phi^{(\ell)}\rho_\ell = 0$

Define : $\phi^{(\ell)}$ solution of the above equation

For ARMA(1,1) :

$$\rho^{(\ell)} \begin{cases} = \phi & \text{for } \ell = 1,2,3,\cdots \\ \neq \phi & \text{for } \ell = 0 \end{cases}$$

Consider : $w_t^{(\ell)} = Z_t - \phi^{(\ell)}Z_{t-1}$

Then, for $\ell = 1,2,3,\cdots$ of $\mathrm{ARMA}(1,1)$

$$\Rightarrow w_t^{(\ell)} \sim \mathrm{MA}(1)$$

EACF for Stationary ARMA(1,1)

Compute : $\widehat{\phi^{(\ell)}} = \widehat{\rho_{\ell+1}}/\widehat{\rho_\ell}$

$$w_t^{(\ell)} = Z_t - \phi^{(\ell)}Z_{t-1}$$

SACF of $w_t^{(\ell)} \sim \mathrm{MA}(1)$

理论上：

ℓ \ lag	1	2	3	4	5	6
1	X	O	O	O	O	$\cdots$
2	X	O	O	O	O	$\cdots$
3	X	O	O	O	O	$\cdots$
4	X	O	O	O	O	$\cdots$

EACF for Stationary ARMA(1,1)

Compute : $\widehat{\phi^{(\ell)}} = \widehat{\rho_{\ell+1}} / \widehat{\rho_\ell}$

$w_t^{(\ell)} = Z_t - \phi^{(\ell)} Z_{t-1}$

SACF of $w_t^{(\ell)}$ ~MA(1)

理论上：

ℓ \ lag					6
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2	[illegible]	[illegible]	[illegible]	[illegible]	$\cdots$
3	X	[illegible]	O	O	O $\cdots$
4	X	O	O	O	O $\cdots$

If the model is ARMA(1,2),
Then?

EACF for Stationary ARMA(1,2)

$$Z_t - \phi Z_{t-1} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

EACF for Stationary ARMA(1,1)

$$Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}$$

$$Z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + +\psi_3 a_{t-3} + \cdots$$

Covariance Structure:

$$EZ_{t-k}(Z_t - \phi Z_{t-1}) = EZ_{t-k}(a_t - \theta a_{t-1})$$

$k=1 \qquad \rho_1 - \phi\rho_0 = -\theta\sigma_a^2/\gamma_0 \qquad \rho_1 \begin{cases} = \phi & \text{if } \theta = 0 \\ \neq \phi & \text{if } \theta \neq 0 \end{cases}$

$k=2 \qquad \rho_2 - \phi\rho_1 = 0 \qquad \phi = \rho_2/\rho_1$

$k=3 \qquad \rho_3 - \phi\rho_2 = 0 \qquad \phi = \rho_3/\rho_2$

EACF for Stationary ARMA(1,2)

$$Z_t - \phi Z_{t-1} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

$$EZ_{t-k}(Z_t - \phi Z_{t-1}) = EZ_{t-k}(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})$$

$k=2 \qquad \rho_2 - \phi\rho_1 \neq 0$

$k=3 \qquad \rho_3 - \phi\rho_2 = 0$

$k=4 \qquad \rho_4 - \phi\rho_3 = 0$

Transform : $w_t^{(\ell)} = Z_t - \phi^{(\ell)} Z_{t-1}$

For $\ell = 2, 3, \cdots$, $w_t^{(\ell)}$ ~MA(2)

时间序列分析

第六讲(下)

EACF for Stationary ARMA(1,2)

$$Z_t - \phi Z_{t-1} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

$$\Downarrow$$

$$EZ_{t-k}(Z_t - \phi Z_{t-1}) = EZ_{t-k}(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})$$

$$\Downarrow$$

$$\begin{array}{ll} k=2 & \rho_2 - \phi\rho_1 \neq 0 \\ k=3 & \rho_3 - \phi\rho_2 = 0 \\ k=4 & \rho_4 - \phi\rho_3 = 0 \end{array}$$

Transform : $w_t^{(\ell)} = Z_t - \phi^{(\ell)} Z_{t-1}$

$\Rightarrow$ For $\ell = 2, 3, \cdots, w_t^{(\ell)}$ ~MA(2)

- See Page 151 of Notes

EACF for Stationary ARMA(1,q)

ARMA(1,2)

\	1	2	3	4	5	6
1						⋯
2		X	O	O	O	⋯
3		X	O	O	O	⋯
4		X	O	O	O	⋯

ARMA(1,1)

ℓ \ lag	1	2	3	4	5	6
1	X	0	0	0	0	⋯
2	X	0	0	0	0	⋯
3	X	0	0	0	0	⋯
4	X	0	0	0	0	⋯

EACF for Stationary ARMA(1,q)

$$EZ_{t-k}(Z_t - \phi Z_{t-1}) = EZ_{t-k}(a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q})$$

$$\begin{array}{ll} k=q & \rho_q - \phi\rho_{q-1} \neq 0 \\ k=q+1 & \rho_{q+1} - \phi\rho_q = 0 \\ \vdots & \vdots \end{array}$$

$$\Downarrow$$

$$\omega_t^{(\ell)} = Z_t - \phi^{(\ell)} Z_{t-1} \Rightarrow \omega_t^{(\ell)} \sim \text{MA(q)}$$

- See Page 151 of Notes

EACF for Stationary ARMA(1,q)

ARMA(1,2)

\	1	2	3	4	5	6
1						$\cdots$
2		X	O	O	O	$\cdots$
3		X	O	O	O	$\cdots$
4		X	O	O	O	$\cdots$

ARMA(1,1)

$\ell \setminus$ lag	1	2	3	4	5	6
1	X	0	0	0	0	$\cdots$
2	X	0	0	0	0	$\cdots$
3	X	0	0	0	0	$\cdots$
4	X	0	0	0	0	$\cdots$

EACF for Stationary ARMA(1,q)

\	1	$\cdots$	q	q+1	q+2	$\cdots$
1						
$\vdots$						
q			X	O	O	$\cdots$
q+1			X	O	O	$\cdots$
q+2			X	O	O	$\cdots$
$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\cdots$

EACF for Stationary ARMA(1,q)

\	1	$\cdots$	q	q+1	q+2	$\cdots$
1						
$\vdots$						
q			X	O	O	$\cdots$
q+1			X	O	O	$\cdots$
q+2			X	O	O	$\cdots$
$\vdots$			$\vdots$	$\vdots$	$\vdots$	$\cdots$

1) Define $\phi^{(\ell)}$ as solution of $\rho_{\ell+1} - \phi^{(\ell)}\rho_\ell = 0$

2) Transform Z_t: $\omega_t^{(\ell)} = Z_t - \phi^{(\ell)} Z_{t-1}$

3) Take $\rho_\ell(1)$ as the ℓ-th autocorrelation of $\omega_t^{(\ell)}$

EACF for Stationary ARMA(1,q)

ARMA(1,1)

$\ell \setminus$ lag	1	2	3	4	5	6
1	X	0	0	0	0	$\cdots$
2	X	0	0	0	0	$\cdots$
3	X	0	0	0	0	$\cdots$
4	X	0	0	0	0	$\cdots$

EACF for Stationary ARMA(1,q)

ARMA(1,2)

\	1	2	3	4	5	6
1						⋯
2		X	O	O	O	⋯
3		X	O	O	O	⋯
4		X	O	O	O	⋯

EACF for Stationary ARMA(1,q)

ARMA(1,q)

\	1	⋯	q	q+1	q+2	⋯
1						
⋮						
q			X	O	O	⋯
q+1			X	O	O	⋯
q+2			X	O	O	⋯
⋮			⋮	⋮	⋮	⋯

EACF for Stationary ARMA(1,q)

ARMA(1,q)

\	1	⋯	q	q+1	q+2	⋯
1						
⋮						
q						
q+1						⋯
q+2				O	O	⋯
⋮			⋮	⋮	⋮	⋯

How about (2,q)?

EACF for Stationary ARMA(1,q)

ARMA(1,2)

\	1	2	3	4	5	6
1						⋯
2		X	O	O	O	⋯
3		X	O	O	O	⋯
4		X	O	O	O	⋯

ARMA(1,1)

ℓ \ lag	1	2	3	4	5	6
1	X	O	O	O	O	⋯
2	X	O	O	O	O	⋯
3	X	O	O	O	O	⋯
4	X	O	O	O	O	⋯

EACF for Stationary ARMA(1,q)

ARMA(1,q)

\	1	⋯	q	q+1	q+2	⋯
1						
⋮						
q			X	O	O	⋯
q+1			X	O	O	⋯
q+2			X	O	O	⋯
⋮			⋮	⋮	⋮	⋯

EACF for Stationary ARMA(1,q)

1) Define $\phi^{(\ell)}$ as solution of $\rho_{\ell+1} - \phi^{(\ell)}\rho_\ell = 0$

2) Transform Z_t: $\omega_t^{(\ell)} = Z_t - \phi^{(\ell)} Z_{t-1}$

3) Take $\rho_\ell(1)$ as the ℓ-th autocorrelation of $\omega_t^{(\ell)}$

Then $\rho_\ell(1)$ has the following property :

$$\rho_\ell(1) \neq 0 \qquad \ell = q$$

$$\rho_\ell(1) = 0 \qquad \ell > q$$

$\rho_\ell(1)$: 1st Extended AutoCorrelation Function (EACF)

EACF for Stationary ARMA(2,q)

$$(Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2}) = (a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q})$$

(1,q) $\qquad \widehat{\rho_{\ell+1}} - \phi^{(\ell)}\widehat{\rho_\ell} = 0$

(2,q) $\qquad \phi_1, \phi_2 \qquad \underset{\sim}{\phi}^{(\ell)} = \begin{pmatrix} \phi_1^{(\ell)} \\ \phi_2^{(\ell)} \end{pmatrix}$

• See Page 152 of Notes

EACF for Stationary ARMA(p,q)

MA order →

AR order ↓

m \ j	0	1	2	3	4	5	
0	$\rho_1(0)$	$\rho_2(0)$	$\rho_3(0)$	$\rho_4(0)$	$\rho_5(0)$	⋯	ordinary ACF
1	$\rho_1(1)$	$\rho_2(1)$	$\rho_3(1)$	$\rho_4(1)$	$\rho_5(1)$	⋯	1st EACF
2	$\rho_1(2)$	$\rho_2(2)$	$\rho_3(2)$	$\rho_4(2)$	$\rho_5(2)$	⋯	2nd EACF
3	$\rho_1(3)$	$\rho_2(3)$	$\rho_3(3)$	$\rho_4(3)$	$\rho_5(3)$	⋯	3rd EACF
4	⋯	⋯	⋯	⋯	⋯	⋯	
5	⋯	⋯	⋯	⋯	⋯	⋯	

• See Page 153 of Notes

EACF for Stationary ARMA(2,q)

$$(Z_t - \phi_1 Z_{t-1} - \phi_2 Z_{t-2}) = (a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q})$$

From sample

(1,q)		$\widehat{\rho_{\ell+1}} - \phi^{(\ell)} \widehat{\rho_\ell} = 0$
(2,q)	ϕ_1, ϕ_2	$\underset{\sim}{\phi}^{(\ell)} = \begin{pmatrix} \phi_1^{(\ell)} \\ \phi_2^{(\ell)} \end{pmatrix}$

• See Page 152 of Notes

EACF for Stationary ARMA(p,q)

MA order →

AR order ↓

m \ j	0	1	2	3	4	5	
0	$\rho_1(0)$	$\rho_2(0)$	$\rho_3(0)$	$\rho_4(0)$	$\rho_5(0)$	$\cdots$	ordinary ACF
1	$\rho_1(1)$	$\rho_2(1)$	$\rho_3(1)$	$\rho_4(1)$	$\rho_5(1)$	$\cdots$	1st EACF
2	$\rho_1(2)$	$\rho_2(2)$	$\rho_3(2)$	$\rho_4(2)$	$\rho_5(2)$	$\cdots$	2nd EACF
3	$\rho_1(3)$	$\rho_2(3)$	$\rho_3(3)$	$\rho_4(3)$	$\rho_5(3)$	$\cdots$	3rd EACF
4	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
5	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

MA order →

AR order ↓

m \ j	0	1	2	3	4	5	
0	$\rho_1(0)$	$\rho_2(0)$	$\rho_3(0)$	$\rho_4(0)$	$\rho_5(0)$	$\cdots$	ARMA(0,q)
1	$\rho_1(1)$	$\rho_2(1)$	$\rho_3(1)$	$\rho_4(1)$	$\rho_5(1)$	$\cdots$	ARMA(1,q)
2	$\rho_1(2)$	$\rho_2(2)$	$\rho_3(2)$	$\rho_4(2)$	$\rho_5(2)$	$\cdots$	ARMA(2,q)
3	$\rho_1(3)$	$\rho_2(3)$	$\rho_3(3)$	$\rho_4(3)$	$\rho_5(3)$	$\cdots$	ARMA(3,q)
4	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
5	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

True Model MA(1)

\	0	1	2	3	4	5
0	X	O	O	O	O	$\cdots$
1						
2						
3						
4						

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

True Model MA(1)

\	0	1	2	3	4	5
0	X	O	O	O	O	⋯
1	?	X?	O	O	O	⋯
2						
3						
4						

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

True Model MA(1)

\	0	1	2	3	4	5
0	X	O	O	O	O	⋯
1	?	X?	O	O	O	⋯
2	?	?	X?	O	O	⋯
3						
4						

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

True Model MA(1)

\	0	1	2	3	4	5
0	X	O	O	O	O	⋯
1	?	X?	O	O	O	⋯
2	?	?	X?	O	O	⋯
3						
4						

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

True Model MA(1)

\	0	$\boxed{1}$	2	3	4	5
$\boxed{0}$	X	O	O	O	O	⋯
1	?	X?	O	O	O	⋯
2	?	?	X?	O	O	⋯
3						
4						

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

True Model ARMA(1,1)

AR order \ MA Order	0	$\boxed{1}$	2	3	4	5
0						
$\boxed{1}$		O	O	O	O	$\cdots$
2			O	O	O	$\cdots$
3				O	O	$\cdots$
4					O	$\cdots$

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

Simplest Example : $Z_t = a_t$

$\rho_1 = \rho_2 = \rho_3 = \cdots = 0$

\	0	1	2	3	4	5	
0	O	O	O	O	O	$\cdots$	ACF
1	X?	O	O	O	O	$\cdots$	1st EACF
2	X	X?	O	O	O	$\cdots$	2nd EACF
3							

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

Simplest Example : $Z_t = a_t$

$\rho_1 = \rho_2 = \rho_3 = \cdots = 0$

\	0	1	2	3	4	5	
0	O	O	O	O	O	$\cdots$	ACF
1	X?	O	O	O	O	$\cdots$	1st EACF
2	X	X?	O	O	O	$\cdots$	2nd EACF

$$\rho_2 - \phi^{(1)}\rho_1 = 0$$

$$\omega_t^{(1)} = Z_t - \phi^{(1)} Z_{t-1}$$

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

Simplest Example : $Z_t = a_t$

$\rho_1 = \rho_2 = \rho_3 = \cdots = 0$

\	0	1	2	3	4	5	
0	O	O	O	O	O	$\cdots$	ACF
1	X?	O	O	O	O	$\cdots$	1st EACF
2	X	X?	O	O	O	$\cdots$	2nd EACF

$$\rho_2 - \phi^{(1)}\rho_1 = 0$$

$$\omega_t^{(1)} = Z_t - \phi^{(1)} Z_{t-1}$$

$\boxed{\phi^{(1)} \text{ arbitrary}}$

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

Simplest Example : $Z_t = a_t$

$\rho_1 = \rho_2 = \rho_3 = \cdots = 0$

\	0	1	2	3	4	5	
0	O	O	O	O	O	$\cdots$	ACF
1	X?	O	O	O	O	$\cdots$	1st EACF
2	X	X?	O	O	O	$\cdots$	2nd EACF

$$\boxed{\begin{aligned}&\rho_2 - \phi^{(1)}\rho_1 = 0\\ &\omega_t^{(1)} = Z_t - \phi^{(1)}Z_{t-1}\end{aligned}} \rightarrow \boxed{\phi^{(1)} \text{ arbitrary}}$$

How?

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

Simplest Example : $Z_t = a_t$

$\rho_1 = \rho_2 = \rho_3 = \cdots = 0$

$$\boxed{\begin{aligned}&\rho_2 - \phi^{(1)}\rho_1 = 0\\ &\omega_t^{(1)} = Z_t - \phi^{(1)}Z_{t-1}\end{aligned}} \rightarrow \boxed{\phi^{(1)} \text{ arbitrary}}$$

How?

If $\phi^{(1)} = 1.2$,then $W_t^{(1)} = Z_t - 1.2Z_{t-1}$

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

Simplest Example : $Z_t = a_t$

$\rho_1 = \rho_2 = \rho_3 = \cdots = 0$

$$\boxed{\begin{aligned}&\rho_2 - \phi^{(1)}\rho_1 = 0\\ &\omega_t^{(1)} = Z_t - \phi^{(1)}Z_{t-1}\end{aligned}} \rightarrow \boxed{\phi^{(1)} \text{ arbitrary}}$$

How?

If $\phi^{(1)} = 1.2$,then $\omega_t^{(1)} = Z_t - 1.2Z_{t-1}$

+

$Z_t = a_t$

$$\boxed{\omega_t^{(1)} = a_t - 1.2a_{t-1}}$$

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

Simplest Example : $Z_t = a_t$

$\rho_1 = \rho_2 = \rho_3 = \cdots = 0$

\	0	1	2	3	4	5	
0	O	O	O	O	O	$\cdots$	ACF
1	X?	O	O	O	O	$\cdots$	1st EACF
2	X	X?	O	O	O	$\cdots$	2nd EACF
3							

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

True Model ARMA(1,1) (JASA 1984)

AR order \ MA Order	0	1	2	3	4	5
0						
1		O	O	O	O	...
2			O	O	O	...
3				O	O	...
4					O	...

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

True Model ARMA(1,1) (JASA 1984)

AR order \ MA Order	0	1	2	3	4	5
0						
1		O	O	O	O	...
2			O	O	O	...
3				O	O	...
4					O	...

Sample EACF!

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

True Model ARMA(1,1) (JASA 1984)

AR order \ MA Order	0	1	2	3	4	5
0						
1		O	O	O	O	...
2			O	O	O	...
3				O	O	...
4					O	...

确定部分!

• See Page 154 of Notes

EACF for Stationary ARMA(p,q)

True Model AR(1)

\	0	1	2	3	4	5
0						
1	O	O	O	O	O	...
2		O	O	O	O	...
3			O	O	O	...
4				O	O	...

• See Page 153 of Notes

EACF for Stationary ARMA(p,q)

True Model ARMA(1,1)
(nonstationary)

$$Z_t - \Phi Z_{t-1} = a_t - \theta a_{t-1}$$

其中: $\Phi = 1$

• See Page 154 of Notes

本课时内容回顾

Model Building

1. Tentative Specification
   - SACF MA
   - SPACF AR
   - SEACF Mix (including (1-B))
2. Estimation (MLE)
3. Diagnostic Checking

时间序列分析

第七讲(上)

ARIMA(p,d,q) - Model Building

1. Tentative Model Specification
   - SACF
   - SPACF
   - SEACF
2. Model Fitting or Model Estimation
3. Diagnostic Checking

Today's Topic

Likelihood Function ＋ MLE

1. AR Models
2. MA Models
   - Conditional Likelihood
   - Exact Likelihood
   - Extensions

• See Page 156-166 of Notes

Likelihood Function

Data : $Z_1, \cdots, Z_n$

Likelihood Function : $P(Z_1, \cdots, Z_n | \underset{\sim}{\theta})$ (p.d.f. of data)

Likelihood Function

Data : $Z_1, \cdots, Z_n$

参数 θ 未知

Likelihood Function : $P(Z_1, \cdots, Z_n|\theta)$ (p.d.f. of data)

Data 未知

Likelihood Function

Data : $Z_1, \cdots, Z_n$

参数 θ 未知

Likelihood Function : $P(Z_1, \cdots, Z_n|\theta)$ (p.d.f. of data)

假设资料分布为
高斯分布 P

Data 未知

Sample n points from $N(\mu, \sigma_a^2)$

Joint p.d.f. :

$$P(Z_1, \cdots, Z_n|\mu, \sigma_a^2) = (\frac{1}{\sqrt{2\pi}})^n \sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(Z_t-\mu)^2\}$$

Likelihood Function

Suppose : $\mu = 5,\ \sigma_a = 1$

Data : 5.1, 4.7, 5.32. 4.85, ……

What is the likelihood function？ (assume the obs. are from $N(\mu, \sigma_a^2)$)

Likelihood Function

观察值
(Data已知)

Suppose : $\mu = 5,\ \sigma_a = 1$

Data : 5.1, 4.7, 5.32. 4.85, ……

What is the likelihood function？ (assume the obs. are from $N(\mu, \sigma_a^2)$)

分布已知

参数未知

Likelihood Function

Suppose : $\mu = 5,\ \sigma_a = 1$

Data : 5.1, 4.7, 5.32. 4.85, ……

观察值 (Data已知)

What is the likelihood function? (assume the obs. are from $N(\mu, \sigma_a^2)$)

分布已知

参数未知

Likelihood Function : function of unknown coefficients for given data

$$\begin{aligned} & \ell(\mu, \sigma_a^2 | 5.1, 4.7, \cdots) \\ \propto \ & P(Z_1, \cdots, Z_n | \theta) \\ \propto \ & \sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(Z_t - \mu)^2\} \end{aligned}$$

Likelihood Function

Likelihood Function :

$$\begin{aligned} & \ell(\mu, \sigma_a^2 | Z_1, \cdots, Z_n) \\ \propto \ & \sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(Z_t - \mu)^2\} \end{aligned}$$

maximize

$$\begin{aligned} &\text{MLE of } \mu & \widehat{\mu} = \bar{Z} \\ &\text{MLE of } \sigma_a^2 & \widehat{\sigma_a^2} = \frac{1}{n}\sum(Z_i - \bar{Z})^2 \end{aligned}$$

MLE is one of the **most basic and popular** estimation methods !

Likelihood Function for AR(1)

AR(1) :

$$\begin{aligned} Z_t - \mu &= \phi(Z_{t-1} - \mu) + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2) \\ &\Downarrow \\ Z_t &= c + \phi Z_{t-1} + a_t \qquad c = \mu(1 - \phi) \end{aligned}$$

• See Page 156 of Notes

Likelihood Function for AR(1)

AR(1) :

$$\begin{aligned} Z_t - \mu &= \phi(Z_{t-1} - \mu) + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2) \\ &\Downarrow \\ Z_t &= c + \phi Z_{t-1} + a_t \qquad c = \mu(1 - \phi) \end{aligned}$$

Stationary AR(1) :

$$\begin{aligned} & P(Z_1, Z_2, \cdots, Z_n) \\ = \ & P(Z_1)P(Z_2|Z_1)P(Z_3|Z_2)\cdots P(Z_n|Z_{n-1}) \end{aligned}$$

• See Page 156 of Notes

Likelihood Function for AR(1)

AR(1) :

$$
\begin{aligned}
Z_t - \mu &= \phi(Z_{t-1} - \mu) + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2) \\
&\Downarrow \\
Z_t &= c + \phi Z_{t-1} + a_t \qquad c = \mu(1 - \phi)
\end{aligned}
$$

$\boxed{N(\phi Z_1, \sigma_a^2)}$ $\boxed{N(\phi Z_2, \sigma_a^2)}$

Stationary

AR(1) :

$$
\begin{aligned}
& P(Z_1, Z_2, \cdots, Z_n) \\
= \; & P(Z_1) \underbrace{\boxed{P(Z_2|Z_1)}\boxed{P(Z_3|Z_2)} \cdots P(Z_n|Z_{n-1})}_{\boxed{\sigma_a^{-(n-1)} \exp\{-\frac{1}{2\sigma_a^2}\sum(Z_t - \phi Z_{t-1})^2\}}}
\end{aligned}
$$

• See Page 156 of Notes

Likelihood Function for AR(1)

AR(1) :

$$
\begin{aligned}
Z_t - \mu &= \phi(Z_{t-1} - \mu) + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2) \\
&\Downarrow \\
Z_t &= c + \phi Z_{t-1} + a_t \qquad c = \mu(1 - \phi)
\end{aligned}
$$

$\boxed{N(\phi Z_1, \sigma_a^2)}$ $\boxed{N(\phi Z_2, \sigma_a^2)}$

Stationary

AR(1) :

$$
\begin{aligned}
& P(Z_1, Z_2, \cdots, Z_n) \\
= \; & \underset{??}{P(Z_1)} \underbrace{\boxed{P(Z_2|Z_1)}\boxed{P(Z_3|Z_2)} \cdots P(Z_n|Z_{n-1})}_{\boxed{\sigma_a^{-(n-1)} \exp\{-\frac{1}{2\sigma_a^2}\sum(Z_t - \phi Z_{t-1})^2\}}}
\end{aligned}
$$

• See Page 156 of Notes

Likelihood Function for AR(1)

AR(1) :

$$
\begin{aligned}
Z_t - \mu &= \phi(Z_{t-1} - \mu) + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2) \\
&\Downarrow \\
Z_t &= c + \phi Z_{t-1} + a_t \qquad c = \mu(1 - \phi)
\end{aligned}
$$

Assume c = 0

$\boxed{N(\phi Z_1, \sigma_a^2)}$ $\boxed{N(\phi Z_2, \sigma_a^2)}$

Stationary

AR(1) :

$$
\begin{aligned}
& P(Z_1, Z_2, \cdots, Z_n) \\
= \; & \underset{??}{P(Z_1)} \underbrace{\boxed{P(Z_2|Z_1)}\boxed{P(Z_3|Z_2)} \cdots P(Z_n|Z_{n-1})}_{\boxed{\sigma_a^{-(n-1)} \exp\{-\frac{1}{2\sigma_a^2}\sum(Z_t - \phi Z_{t-1})^2\}}}
\end{aligned}
$$

• See Page 156 of Notes

Likelihood Function for AR(1)

Stationary

AR(1) :

$$
\begin{aligned}
& P(Z_1, Z_2, \cdots, Z_n) \\
= \; & P(Z_1) P(Z_2|Z_1) P(Z_3|Z_2) \cdots P(Z_n|Z_{n-1})
\end{aligned}
$$

$$Z_1 = c + \phi Z_0 + a_1$$

$$Z_t = a_t + \phi a_{t-1} + \cdots + \phi^{t-m} a_m + \cdots$$

If Stationary ($|\phi| < 1$)

$$\mathrm{Var}(Z_t) = \frac{\sigma_a^2}{1 - \phi^2}$$

$$P(Z_1) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sigma_a}{\sqrt{1 - \phi^2}}\right)^{-1} \exp\left\{-\frac{1 - \phi^2}{2\sigma_a^2}(Z_1 - \mu)^2\right\}$$

• See Page 156 of Notes

Likelihood Function for AR(1)

$$\begin{aligned} & P(Z_1, Z_2, \cdots, Z_n) \\ = \ & P(Z_1)P(Z_2|Z_1)P(Z_3|Z_2)\cdots P(Z_n|Z_{n-1}) \\ \propto \ & P(Z_1)\sigma_a^{-(n-1)} \exp\{-\frac{1}{2\sigma_a^2}\sum(Z_t - c - \phi Z_{t-1})^2\} \end{aligned}$$

+

$$P(Z_1) = \frac{1}{\sqrt{2\pi}}(\frac{\sigma_a}{\sqrt{1-\phi^2}})^{-1} \exp\{-\frac{1-\phi^2}{2\sigma_a^2}(Z_1-\mu)^2\}$$

$$\Downarrow$$

$$\begin{aligned} \ell(\mu, \phi, \sigma_a^2 | \underset{\sim}{Z}) \ \propto \ & \frac{1}{\sqrt{2\pi}}(\frac{\sigma_a}{\sqrt{1-\phi^2}})^{-1} \exp\{-\frac{1-\phi^2}{2\sigma_a^2}(Z_1-\mu)^2\} \\ & \sigma_a^{-(n-1)} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}(Z_t - c - \phi Z_{t-1})^2\} \end{aligned}$$

Likelihood Function for AR(1)

$$\begin{aligned} \ell(\mu, \phi, \sigma_a^2 | \underset{\sim}{Z}) \ \propto \ & \boxed{\frac{1}{\sqrt{2\pi}}(\frac{\sigma_a}{\sqrt{1-\phi^2}})^{-1} \exp\{-\frac{1-\phi^2}{2\sigma_a^2}(Z_1-\mu)^2\}} \\ & \boxed{\sigma_a^{-(n-1)} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}(Z_t - c - \phi Z_{t-1})^2\}} \end{aligned}$$

• See Page 156 of Notes

Likelihood Function for AR(1)

$$\begin{aligned} \ell(\mu, \phi, \sigma_a^2 | \underset{\sim}{Z}) \ \propto \ & \boxed{\frac{1}{\sqrt{2\pi}}(\frac{\sigma_a}{\sqrt{1-\phi^2}})^{-1} \exp\{-\frac{1-\phi^2}{2\sigma_a^2}(Z_1-\mu)^2\}} \\ & \boxed{\sigma_a^{-(n-1)} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}(Z_t - c - \phi Z_{t-1})^2\}} \end{aligned}$$

ℓ_1

ℓ_*

$$\Downarrow$$

$$\ell \propto \ell_1 \ell_*$$

n ↑ ℓ_* dominate

• See Page 156 of Notes

Likelihood Function for AR(1)

$$\ell \propto \ell_1 \ell_*$$

n ↑ ℓ_* dominate

Ignore ℓ_1 or $P(Z_1)$

$$\ell_* \propto P(Z_2|Z_1)P(Z_3|Z_2)\cdots P(Z_n|Z_{n-1})$$

Note: Both for stationary $|\phi| < 1$ or

nonstationary case $|\phi| = 1$

• See Page 156 of Notes

Likelihood Function for AR(1)

Approximately(ignore ℓ_1)

$$\begin{aligned}\ell(\mu,\phi,\sigma_a^2|\underset{\sim}{Z}) &\propto \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}[Z_t-\mu-\phi(Z_{t-1}-\mu)]^2\}\\ &= \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}(Z_t-c-\phi Z_{t-1})^2\}\end{aligned}$$

Note : $c=\mu(1-\phi)$

• See Page 156 of Notes

Likelihood Function for AR(1)

Approximately(ignore ℓ_1)

$$\begin{aligned}\ell(\mu,\phi,\sigma_a^2|\underset{\sim}{Z}) &\propto \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}[Z_t-\mu-\phi(Z_{t-1}-\mu)]^2\}\\ &= \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}(Z_t-c-\phi Z_{t-1})^2\}\end{aligned}$$

Note : $c=\mu(1-\phi)$

• See Page 156 of Notes

Likelihood Function for AR(1)

Approximately(ignore ℓ_1)

$$\begin{aligned}\ell(\mu,\phi,\sigma_a^2|\underset{\sim}{Z}) &\propto \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}[Z_t-\mu-\phi(Z_{t-1}-\mu)]^2\}\\ &= \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}(Z_t-c-\phi Z_{t-1})^2\}\end{aligned}$$

Note : $c=\mu(1-\phi)$

• See Page 156 of Notes

Likelihood Function for AR(1)

Approximately(ignore ℓ_1)

$$\begin{aligned}\ell(\mu,\phi,\sigma_a^2|\underset{\sim}{Z}) &\propto \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}[Z_t-\mu-\phi(Z_{t-1}-\mu)]^2\}\\ &= \sigma_a^{-(n-1)}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=2}^{n}(Z_t-c-\phi Z_{t-1})^2\}\end{aligned}$$

Note : $c=\mu(1-\phi)$

Based on linear regression model

$$Z_t = c+\phi Z_{t-1}+a_t$$

• See Page 157 of Notes

Likelihood Function for AR(2)

AR(2) : $Z_t - C - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} = a_t$

$t = 1, 2, 3, \cdots, n$

$$\begin{aligned} P(Z_1, \cdots, Z_n) &= P(Z_1, Z_2) P(Z_3, \cdots, Z_n | Z_1, Z_2) \\ &= P(Z_1, Z_2) \prod_{t=3}^{n} P(Z_t | Z_{t-1}, Z_{t-2}) \end{aligned}$$

• See Page 158 of Notes

Likelihood Function for AR(2)

AR(2) : $Z_t - C - \phi_1 Z_{t-1} - \phi_2 Z_{t-2} = a_t$

$t = 1, 2, 3, \cdots, n$

$$\begin{aligned} P(Z_1, \cdots, Z_n) &= P(Z_1, Z_2) P(Z_3, \cdots, Z_n | Z_1, Z_2) \\ &= P(Z_1, Z_2) \prod_{t=3}^{n} P(Z_t | Z_{t-1}, Z_{t-2}) \end{aligned}$$

Complex but limited effect if n is large

$\ell(\phi_1, \phi_2, \mu, \sigma_a^2 | \underset{\sim}{Z}) \propto \ell_*$

• See Page 158 of Notes

Likelihood Function for AR

Example :

• BATCH Data, P252 of Notes

(1,2)z = c + noise

• See Page 158 of Notes

Likelihood Function for MA(1)

MA(1) : $Z_t = c + a_t - \theta a_{t-1}$

已知 : $Z_1, \cdots, Z_n$

求 : Likelihood Function ℓ

$\ell \propto P(Z_1, \cdots, Z_n)$

求 : $P(Z_1, \cdots, Z_n)$

- $E(Z_t) = c$
- $\text{Cov}\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \sigma_a^2 \begin{pmatrix} 1+\theta^2 & -\theta & & \\ -\theta & \ddots & \ddots & \\ & \ddots & \ddots & -\theta \\ & & -\theta & 1+\theta^2 \end{pmatrix} = \sigma_a^2 \Omega$

• See Page 159 of Notes

Likelihood Function for MA(1)

MA(1) : $Z_t = c + a_t - \theta a_{t-1}$

已知 : $Z_1, \cdots, Z_n$

Joint Distribution :

$$P(\underset{\sim}{Z}) = \sigma_a^{-n} |\Omega|^{1/2} \exp\{-\frac{1}{2\sigma_a^2} \dot{\underset{\sim}{Z}}^\top \Omega^{-1} \dot{\underset{\sim}{Z}}\}$$

$$\Omega = \begin{pmatrix} 1+\theta^2 & -\theta & & \\ -\theta & \ddots & \ddots & \\ & \ddots & \ddots & -\theta \\ & & -\theta & 1+\theta^2 \end{pmatrix} \qquad \dot{\underset{\sim}{Z}} = \begin{pmatrix} Z_1 - c \\ Z_2 - c \\ \vdots \\ Z_n - c \end{pmatrix}$$

• See Page 159 of Notes

Likelihood Function for MA(1)

MA(1) : $Z_t = c + a_t - \theta a_{t-1}$

已知 : $Z_1, \cdots, Z_n$

Joint Distribution :

very complex

$$P(\underset{\sim}{Z}) = \sigma_a^{-n} |\Omega|^{1/2} \exp\{-\frac{1}{2\sigma_a^2} \dot{\underset{\sim}{Z}}^\top \Omega^{-1} \dot{\underset{\sim}{Z}}\}$$

very simple

$$\Omega = \begin{pmatrix} 1+\theta^2 & -\theta & & \\ -\theta & \ddots & \ddots & \\ & \ddots & \ddots & -\theta \\ & & -\theta & 1+\theta^2 \end{pmatrix} \qquad \dot{\underset{\sim}{Z}} = \begin{pmatrix} Z_1 - c \\ Z_2 - c \\ \vdots \\ Z_n - c \end{pmatrix}$$

• See Page 159 of Notes

Likelihood Function for MA(1)

MA(1) : $Z_t = c + a_t - \theta a_{t-1}$

c = 0

$$Z_t = a_t - \theta a_{t-1}$$

$$Z_1 = a_1 - \theta a_0$$

$$Z_2 = \cdots$$

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} \theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

• See Page 159 of Notes

Likelihood Function for MA(1)

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} \theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{Z} = D_\theta \underset{\sim}{a} - \begin{pmatrix} \theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{其中：} \quad D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}_{n \times n}$$

• See Page 159 of Notes

Likelihood Function for MA(1)

$$D_\theta = \begin{pmatrix} 1 & & & & \\ -\theta & 1 & & & \\ & \ddots & \ddots & & \\ & & -\theta & 1 & \\ & & & -\theta & 1 \end{pmatrix}_{n\times n}$$

$$D_\theta^{-1} = \begin{pmatrix} 1 & & & & \\ \theta & 1 & & & \\ \theta^2 & \theta & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \theta^{n-1} & \cdots & \theta^2 & \theta & 1 \end{pmatrix}_{n\times n}$$

时间序列分析

第七讲(下)

Today's Topic

Likelihood Function ＋ MLE

1. AR Models
2. MA Models
   - Conditional Likelihood
   - Exact Likelihood
   - Extensions

• See Page 156-166 of Notes

Likelihood Function for MA(1)

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} \theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

• See Page 159 of Notes

Likelihood Function for MA(1)

n点

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} \theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

n+1点

• See Page 159 of Notes

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} a_0 \\ -\Delta C a_0 + D_\theta \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} 1 & 0 \\ -\Delta C_\theta & D_\theta \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}$$

其中：$C_\theta = \theta$

$$-\Delta C_\theta a_0 = -\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \theta a_0 = \begin{pmatrix} -\theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \underset{\sim}{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Likelihood Function for MA(1)

all a_t's i.i.d. $N(0, \sigma_a^2)$

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} a_0 \\ -\Delta C a_0 + D_\theta \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} 1 & 0 \\ -\Delta C_\theta & D_\theta \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}$$

其中：$C_\theta = \theta$

$$-\Delta C_\theta a_0 = -\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \theta a_0 = \begin{pmatrix} -\theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \underset{\sim}{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Likelihood Function for MA(1)

linear transformation

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} a_0 \\ -\Delta C a_0 + D_\theta \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1} = \boxed{\begin{pmatrix} 1 & 0 \\ -\Delta C_\theta & D_\theta \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}}$$

其中：$C_\theta = \theta$

$$-\Delta C_\theta a_0 = -\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \theta a_0 = \begin{pmatrix} -\theta a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \underset{\sim}{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Likelihood Function for MA(1)

$$\boxed{\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}_{(n+1)\times 1}} = \begin{pmatrix} a_0 \\ -\Delta C a_0 + D_\theta \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} 1 & 0 \\ -\Delta C_\theta & D_\theta \end{pmatrix}_{(n+1)\times(n+1)} \boxed{\begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}}$$

其中：$P(a_0, \underset{\sim}{Z}) \xleftrightarrow{\text{linear transformation}} P(a_0, \underset{\sim}{a})$

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} a_0 \\ -\Delta C a_0 + D_\theta \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}$$

$$= \begin{pmatrix} 1 & 0 \\ -\Delta C_\theta & D_\theta \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}$$

其中 :$P(a_0, \underset{\sim}{Z})$ linear transformation $P(a_0, \underset{\sim}{a})$

$$D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}_{n\times n}$$

$|J| = 1$

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}_{(n+1)\times 1} = \begin{pmatrix} a_0 \\ -\Delta C a_0 + D_\theta \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}$$

$$= \begin{pmatrix} 1 & 0 \\ -\Delta C_\theta & D_\theta \end{pmatrix}_{(n+1)\times(n+1)} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}_{(n+1)\times 1}$$

其中 :

$P(a_0, \underset{\sim}{Z}) = P(a_0, \underset{\sim}{a})$

$|J| = 1$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) = P(a_0, \underset{\sim}{Z}) \Rightarrow P(\underset{\sim}{Z}) = \int P(a_0, \underset{\sim}{Z}) da_0$$

proportional to likelihood

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) = P(a_0, \underset{\sim}{Z}) \Rightarrow P(\underset{\sim}{Z}) = \int P(a_0, \underset{\sim}{Z}) da_0$$

$$P(\underset{\sim}{Z}) = \sigma_a^{-n} |\Omega|^{1/2} \exp\{-\frac{1}{2\sigma_a^2} \dot{\underset{\sim}{Z}}^\top \Omega^{-1} \dot{\underset{\sim}{Z}}\}$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) = P(a_0, \underset{\sim}{Z}) \Rightarrow P(\underset{\sim}{Z}) = \int P(a_0, \underset{\sim}{Z}) da_0$$

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$P(a_0, \underset{\sim}{a})$ = ?

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) = P(a_0, \underset{\sim}{Z}) \Rightarrow P(\underset{\sim}{Z}) = \int P(a_0, \underset{\sim}{Z}) da_0$$

$$P(\underset{\sim}{Z}) = \sigma_a^{-n} |\Omega|^{1/2} \exp\{-\frac{1}{2\sigma_a^2} \dot{\underset{\sim}{Z}}^\top \Omega^{-1} \dot{\underset{\sim}{Z}}\}$$

$$P(a_0, \underset{\sim}{a}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2} \sum_{t=0}^{n} a_t^2\}$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) = P(a_0, \underset{\sim}{Z}) \Rightarrow P(\underset{\sim}{Z}) = \int P(a_0, \underset{\sim}{Z}) da_0$$

$$P(\underset{\sim}{Z}) = \sigma_a^{-n} |\Omega|^{1/2} \exp\{-\frac{1}{2\sigma_a^2} \dot{\underset{\sim}{Z}}^\top \Omega^{-1} \dot{\underset{\sim}{Z}}\}$$

$$P(a_0, \underset{\sim}{a}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2} \sum_{t=0}^{n} a_t^2\}$$

What is it ?

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2} \sum_{t=0}^{n} a_t^2\}$$

What is it ?

$$\begin{pmatrix} a_0 \\ Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ -\Delta C_\theta & D_\theta \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=0}^{n} a_t^2\}$$

What is it ?

$$\begin{pmatrix} a_0 \\ Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \boxed{\begin{pmatrix} 1 & \cdot \\ -\Delta C_\theta & D_\theta \end{pmatrix}} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{a}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=0}^{n} a_t^2\}$$

$$\begin{pmatrix} 1 & \cdot \\ -\Delta C_\theta & D_\theta \end{pmatrix}^{-1} = ?$$

$$\begin{pmatrix} a_0 \\ Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \boxed{\begin{pmatrix} 1 & \cdot \\ -\Delta C_\theta & D_\theta \end{pmatrix}} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}$$

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ -\Delta C_\theta & D_\theta \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ D_\theta^{-1}\Delta C_\theta & D_\theta^{-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix} = Y - X a_0$$

其中：

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \qquad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix}$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X a_0)^\top (Y - X a_0)\}$$

其中：

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \qquad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix}$$

$$D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X a_0)^\top (Y - X a_0)\}$$

其中 :

$$P(a_0, \underset{\sim}{Z}) = P(\underset{\sim}{Z}|a_0)P(a_0)$$

$$P(\underset{\sim}{Z}) = \int P(\underset{\sim}{Z}|a_0)P(a_0)da_0$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X a_0)^\top (Y - X a_0)\}$$

$$P(a_0, \underset{\sim}{Z}) = P(\underset{\sim}{Z}|a_0)P(a_0)$$

$$P(\underset{\sim}{Z}) = \int P(\underset{\sim}{Z}|a_0)P(a_0)da_0$$

其中 :

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \qquad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix}$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X a_0)^\top (Y - X a_0)\}$$

其中 :

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \qquad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix}$$

$$\boxed{\begin{array}{l} Z_1 = a_1 - \theta a_0 \\ a_0 = 0 \end{array}}$$

$$D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}$$

Likelihood Function for MA(1)

Conditional Likelihood :

$$P(a_0, \underset{\sim}{Z}) = P(\underset{\sim}{Z}|a_0)P(a_0)$$

Suppose $a_0 = 0$

$$P(\underset{\sim}{Z}|a_0 = 0) \to \ell(\theta|a_0 = 0)$$

$$\ell(\theta, \sigma_a^2|a_0 = 0, \underset{\sim}{Z}) \propto \sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}Y^\top Y\}$$

Note : $Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix}$

Likelihood Function for MA(1)

What is $D_\theta^{-1}\underset{\sim}{Z}$?

$$D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix} \Rightarrow D_\theta^{-1} = \begin{pmatrix} 1 & & & & \\ \theta & 1 & & & \\ \theta^2 & \theta & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \theta^{n-1} & \cdots & \theta^2 & \theta & 1 \end{pmatrix}$$

$$D_\theta^{-1}\underset{\sim}{Z} = \begin{pmatrix} 1 & & & & \\ \theta & 1 & & & \\ \theta^2 & \theta & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \\ \theta^{n-1} & \cdots & \theta^2 & \theta & 1 \end{pmatrix}\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{pmatrix}$$

$$D_\theta^{-1}\underset{\sim}{Z} = \underset{\sim}{a}^{(0)} = \begin{pmatrix} a_1^{(0)} \\ \vdots \\ a_n^{(0)} \end{pmatrix}$$

$$\underset{\sim}{Z} = D_\theta \underset{\sim}{a}^{(0)}$$

Likelihood Function for MA(1)

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}\begin{pmatrix} a_1^{(0)} \\ a_2^{(0)} \\ \vdots \\ a_n^{(0)} \end{pmatrix}$$

$$\begin{aligned} Z_1 &= a_1^{(0)} \\ Z_2 &= a_2^{(0)} - \theta a_1^{(0)} \Leftrightarrow a_2^{(0)} = Z_2 + \theta a_1^{(0)} \\ Z_3 &= a_3^{(0)} - \theta a_2^{(0)} \Leftrightarrow a_3^{(0)} = Z_3 + \theta a_2^{(0)} \end{aligned}$$

Recursive Calculation : $a_t^{(0)} = Z_t + \theta a_{t-1}^{(0)}$ with $a_0^{(0)} = 0$

Likelihood Function for MA(1)

$$\begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -\theta & 1 & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}\begin{pmatrix} a_1^{(0)} \\ a_2^{(0)} \\ \vdots \\ a_n^{(0)} \end{pmatrix} \Leftrightarrow D_\theta^{-1}\underset{\sim}{Z} = \underset{\sim}{a}^{(0)}$$

Two ways to get $\underset{\sim}{a}^{(0)}$:

- 用 $D_\theta^{-1}\underset{\sim}{Z} = \underset{\sim}{a}^{(0)}$ 硬算
- Recursive Calculation : $a_t^{(0)} = Z_t + \theta a_{t-1}^{(0)}$ with $a_0^{(0)} = 0$

Likelihood Function for MA(1)

Conditional Likelihood : Suppose $a_0 = 0$

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}, a_0 = 0) \propto \sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2\}$$

Likelihood Function for MA(1)

$Z \to Z_t - c$

To compute likelihood of θ, c, σ_a^2 :

Conditional on $a_0 = 0$,

$$\sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta,c))^2\}$$

其中： $a_t^{(0)} = Z_t - c + \theta a_{t-1}^{(0)}$ (with $a_0^{(0)} = 0$)

Likelihood Function for MA(1)

$Z \to Z_t - c$

To compute likelihood of θ, c, σ_a^2 :

Conditional on $a_0 = 0$

$$\sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta,c))^2\}$$

Data

其中： $a_t^{(0)} = Z_t - c + \theta a_{t-1}^{(0)}$ (with $a_0^{(0)} = 0$)

Likelihood Function for MA(1)

其中：

$$\sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta,c))^2\}$$

Suppose : $\theta = .5$ and $c = 2.5$

t	Z_t	$Z_t - c$	$a_t^{(0)}$	$(a_t^{(0)})^2$
1	2.3	-.2	-.2	.04
2	3.2	.7	.6	.36
3	3.5	1.0	1.3	1.69
4	2.9	.4	1.05	1.10
5	2.2	-.3	.23	.05

$$= \underset{3.24}{\sum_{t=1}^{5}(a_t^{(0)}(.5, 2.5))^2}$$

Likelihood Function for MA(1)

$Z \to Z_t - c$

To compute likelihood of θ, c, σ_a^2 :

Conditional on $a_0 = 0$

$$\sigma_a^{-n} \exp\{-\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta,c))^2\}$$

其中： $a_t^{(0)} = Z_t - c + \theta a_{t-1}^{(0)}$ (with $a_0^{(0)} = 0$)

$$\boxed{\min_{\theta, c}}$$

$$\sum(a_t^{(0)}(\widehat{\theta},\widehat{c}))^2$$

$$\widehat{\sigma_a^2} = \frac{1}{n}\sum(a_t^{(0)}(\widehat{\theta},\widehat{c}))^2$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \quad \propto \quad \sigma^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}\boxed{(Y - Xa_0)^\top (Y - Xa_0)}\}$$

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \quad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix} \quad \Delta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C_\theta = \theta$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \quad \propto \quad \sigma^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}\boxed{(Y - Xa_0)^\top (Y - Xa_0)}\}$$

$$\begin{aligned} & (Y - Xa_0)^\top (Y - Xa_0) \\ = \; & (Y - X\widehat{a_0})^\top (Y - X\widehat{a_0}) + (a_0 - \widehat{a_0})^2 X^\top X \end{aligned}$$

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \quad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix} \quad \Delta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C_\theta = \theta$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \quad \propto \quad \sigma^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}\boxed{(Y - Xa_0)^\top (Y - Xa_0)}\}$$

$$\begin{aligned} & (Y - Xa_0)^\top (Y - Xa_0) \\ = \; & (Y - X\widehat{a_0})^\top (Y - X\widehat{a_0}) + (a_0 - \widehat{a_0})^2 X^\top X \end{aligned}$$

$$\widehat{a_0} = (X^\top X)^{-1} X^\top Y$$

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \quad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix} \quad \Delta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C_\theta = \theta$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \quad \propto \quad \sigma^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}\boxed{(Y - Xa_0)^\top (Y - Xa_0)}\}$$

$$\begin{aligned} & (Y - Xa_0)^\top (Y - Xa_0) \\ = \; & (Y - X\widehat{a_0})^\top (Y - X\widehat{a_0}) + (a_0 - \widehat{a_0})^2 X^\top X \end{aligned}$$

$$\widehat{a_0} = (X^\top X)^{-1} X^\top Y$$

$$y = \beta x + \epsilon \Rightarrow \widehat{\beta} = (x^\top x)^{-1} x^\top y$$

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \quad X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix} \quad \Delta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C_\theta = \theta$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \quad \propto \quad \sigma^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2} \boxed{(Y - X a_0)^\top (Y - X a_0)}\}$$

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1} \underset{\sim}{Z} \end{pmatrix} \quad X = -\begin{pmatrix} 1 \\ D_\theta^{-1} \Delta C_\theta \end{pmatrix} \quad \Delta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C_\theta = \theta$$

Likelihood Function for MA(1)

$$\boxed{\begin{aligned} &P(a_0, \underset{\sim}{Z}) \\ \propto \quad & \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}[(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (X^\top X)(a_0 - \widehat{a}_0)^2]\} \end{aligned}}$$

Likelihood Function for MA(1)

$$\boxed{\begin{aligned} &P(a_0, \underset{\sim}{Z}) \\ \propto \quad & \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}[(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (X^\top X)(a_0 - \widehat{a}_0)^2]\} \end{aligned}}$$

Likelihood Function for MA(1)

$$\boxed{\begin{aligned} &P(a_0, \underset{\sim}{Z}) \\ \propto \quad & \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}[(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (X^\top X)(a_0 - \widehat{a}_0)^2]\} \end{aligned}}$$

Integrating out a_0 ,we finally obtain :

$$\boxed{\begin{aligned} &P(\underset{\sim}{Z}) \\ \propto \quad & \sigma_a^{-n} (X^\top X)^{-1/2} \exp\{-\frac{1}{2\sigma_a^2}(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0)\} \end{aligned}}$$

What are $\widehat{a}_0, \; Y - X\widehat{a}_0$?

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1} \underset{\sim}{Z} \end{pmatrix} \quad X = -\begin{pmatrix} 1 \\ D_\theta^{-1} \Delta C_\theta \end{pmatrix} \quad \Delta = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C_\theta = \theta$$

- See Page 160 of Notes

Likelihood Function for MA(1)

$$
\begin{aligned}
& P(a_0, \underset{\sim}{Z}) \\
\propto \quad & \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}[(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (X^\top X)(a_0 - \widehat{a}_0)^2]\}
\end{aligned}
$$

$$
\begin{aligned}
& (Y - Xa_0)^\top (Y - Xa_0) \\
= \quad & (Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (a_0 - \widehat{a}_0)^2 X^\top X
\end{aligned}
$$

Likelihood Function for MA(1)

What are $\widehat{a}_0,\ Y - X\widehat{a}_0$?

$$\widehat{a}_0 = (X^\top X)^{-1} X^\top Y$$

$$X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix} = -\begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \vdots \\ \theta^n \end{pmatrix}$$

$$X^\top X = \begin{pmatrix} 1 + \theta^2 + \cdots + \theta^{2n} \end{pmatrix} = \frac{1 - \theta^{2(n+1)}}{1 - \theta^2}$$

$$D_\theta^{-1}\Delta\theta = \begin{pmatrix} \theta \\ \theta^2 \\ \vdots \\ \theta^n \end{pmatrix}$$

$$D_\theta^{-1}\Delta = \begin{pmatrix} 1 & & & \\ \theta & \ddots & & \\ \vdots & \ddots & \ddots & \\ \theta^{n-1} & \cdots & \theta & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^{n-1} \end{pmatrix}$$

Likelihood Function for MA(1)

What are $\widehat{a}_0,\ Y - X\widehat{a}_0$?

$$\widehat{a}_0 = (X^\top X)^{-1} X^\top Y$$

$$X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\Delta C_\theta \end{pmatrix} = -\begin{pmatrix} 1 \\ \theta \\ \theta^2 \\ \vdots \\ \theta^n \end{pmatrix}$$

$$
\begin{aligned}
X^\top Y &= -\begin{pmatrix} 1 & \theta & \theta^2 & \cdots & \theta^n \end{pmatrix} \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix} \\
&= -\begin{pmatrix} 1 & \theta & \theta^2 & \cdots & \theta^n \end{pmatrix} \begin{pmatrix} \cdot \\ a_1^{(0)} \\ a_2^{(0)} \\ \vdots \\ a_n^{(0)} \end{pmatrix} \\
&= -\sum_{t=1}^{n} \theta^t a_t^{(0)}
\end{aligned}
$$

Likelihood Function for MA(1)

What are $\widehat{a}_0,\ Y - X\widehat{a}_0$?

$$X^\top X = \frac{1 - \theta^{2(n+1)}}{1 - \theta^2}$$

$$X^\top Y = \sum_{t=1}^{n} \theta^t a_t^{(0)}$$

$$\widehat{a}_0 = (X^\top X)^{-1} X^\top Y$$

$$\widehat{a}_0 = -\frac{1 - \theta^2}{1 - \theta^{2(n+1)}} \sum_{t=1}^{n} \theta^t a_t^{(0)}$$

Likelihood Function for MA(1)

$$\begin{aligned} P(\underset{\sim}{Z}) &\propto \text{likelihood function} \\ &\propto \sigma_a^{-n}(X^\top X)^{-1/2}\exp\{-\frac{1}{2\sigma_a^2}(Y - X\widehat{a_0})^\top(Y - X\widehat{a_0}) \end{aligned}$$

$\widehat{a_0}$ calculable

What is $Y - X\widehat{a_0}$?

时间序列分析

第八讲(上)

Today's Topic

1. Estimation (L.S. linear + nonlinear)
   - MA(1) conditional likelihood
     exact Likelihood
     Iterative Scheme
   - MA(q)
   - ARMA(p,q)
2. Diagnostic Checks
   - Box & Pierce Box & Ljung
3. Examples (BJ F, A, C)
4. Seasonal Models

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ -\triangle C_\theta & D_\theta \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}$$

$$Z_1 = a_1 - \theta a_0$$

其中：

$$\underset{\sim}{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \quad \underset{\sim}{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & \ddots & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}$$

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ -\triangle C_\theta & D_\theta \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}$$

$$P(a_0, \underset{\sim}{a}) \rightarrow P(a_0, \underset{\sim}{Z}) = \int P(\underset{\sim}{Z}|a_0)P(a_0)da_0 = P(\underset{\sim}{Z})$$

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ -\triangle C_\theta & D_\theta \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix}$$

$$P(a_0, \underset{\sim}{a}) \rightarrow P(a_0, \underset{\sim}{Z}) = \int P(\underset{\sim}{Z}|a_0)P(a_0)da_0 = P(\underset{\sim}{Z})$$

$$\begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ D_\theta^{-1}\triangle C_\theta & D_\theta^{-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}$$

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ D_\theta^{-1}\triangle C_\theta & D_\theta^{-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix} = Y - Xa_0$$

其中: $X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\triangle C_\theta \end{pmatrix}$

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix}$$

Likelihood Function for MA(1)

$$\begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix} = \begin{pmatrix} 1 & \cdot \\ D_\theta^{-1}\triangle C_\theta & D_\theta^{-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \underset{\sim}{Z} \end{pmatrix}$$

$$\begin{pmatrix} a_0 \\ \underset{\sim}{a} \end{pmatrix} = Y - Xa_0$$

其中: $X = -\begin{pmatrix} 1 \\ D_\theta^{-1}\triangle C_\theta \end{pmatrix}$

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1}\underset{\sim}{Z} \end{pmatrix}$$

$$\boxed{P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - Xa_0)^\top (Y - Xa_0)\}}$$

Likelihood Function for MA(1)

conditional likelihood on $a_0 = 0$

Likelihood Function for MA(1)

conditional likelihood on $a_0 = 0$

MA(1) :

$$Z_t = a_t - \theta a_{t-1} \qquad a_1 = Z_1 + \theta a_0$$
$$Z_2 = a_2 - \theta a_1 \qquad a_2 = Z_2 + \theta a_1$$
$$Z_3 = a_3 - \theta a_2 \qquad a_3 = Z_3 + \theta a_2$$

Likelihood Function for MA(1)

conditional likelihood on $a_0 = 0$

MA(1) :

$$Z_t = a_t - \theta a_{t-1} \qquad a_1 = Z_1 + \theta a_0$$
$$Z_2 = a_2 - \theta a_1 \qquad a_2 = Z_2 + \theta a_1$$
$$Z_3 = a_3 - \theta a_2 \qquad a_3 = Z_3 + \theta a_2$$

Likelihood Function for MA(1)

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - Xa_0)^\top (Y - Xa_0)\}$$

$\boxed{\text{conditional likelihood on } a_0 = 0}$

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-n} \exp\{\frac{1}{2\sigma_a^2} Y^\top Y\}$$

$$Y = D_\theta^{-1} \underset{\sim}{Z} = \begin{pmatrix} 1 & & & \\ -\theta & \ddots & & \\ \vdots & \ddots & \ddots & \\ -\theta^{n-1} & \cdots & -\theta & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix}$$

Likelihood Function for MA(1)

To calculate $D_\theta^{-1} \underset{\sim}{Z}$

Given θ

$$a^{(0)}(\theta, c) = D_\theta^{-1}(\underset{\sim}{Z} - c)$$
$$\Leftrightarrow \quad D_\theta a^{(0)}(\theta, c) = \underset{\sim}{Z} - c$$
$$\Rightarrow \quad \underset{\sim}{Z_1} = a_1^{(0)}(\theta, c) - \theta a_0^{(0)}(\theta, c)$$
$$\Rightarrow \quad a_t^{(0)}(\theta, c) = Z_t + a_{t-1}^{(0)}(\theta, c) \qquad \text{with } a_0^{(0)}(\theta, c) = 0$$

Note :

$$D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & \ddots & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}$$

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | a_0 = 0, \underset{\sim}{Z}) \propto \sigma_a^{-n} \exp\{\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2\}$$

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | a_0 = 0, \underset{\sim}{Z}) \propto \sigma_a^{-n} \exp\{\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2\}$$

$$\boxed{\min_{\theta, c} S^{(0)}(\theta, c) = S^{(0)}(\widehat{\theta}, \widehat{c})}$$

$$\boxed{\widehat{\sigma_a^2} = \frac{1}{n}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2}$$

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | a_0 = 0, \underset{\sim}{Z}) \propto \sigma_a^{-n} \exp\{\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2\}$$

NON-linear : $\boxed{\min_{\theta, c} S^{(0)}(\theta, c) = S^{(0)}(\widehat{\theta}, \widehat{c})}$

$$\boxed{\widehat{\sigma_a^2} = \frac{1}{n}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2}$$

Linear :

AR(1)	$a_t = Z_t - c - \phi Z_{t-1}$
AR(2)	$a_t = Z_t - c - \phi_1 Z_{t-1} - \phi_2 Z_{t-2}$

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | a_0 = 0, \underset{\sim}{Z}) \propto \sigma_a^{-n} \exp\{\frac{1}{2\sigma_a^2}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2\}$$

NON-linear : $\boxed{\min_{\theta, c} S^{(0)}(\theta, c) = S^{(0)}(\widehat{\theta}, \widehat{c})}$

$$\boxed{\widehat{\sigma_a^2} = \frac{1}{n}\sum_{t=1}^{n}(a_t^{(0)}(\theta, c))^2}$$

MA(1) : $\boxed{a^{(0)} = D_\theta^{-1}\underset{\sim}{Z} \Leftrightarrow D_\theta a^{(0)} = \underset{\sim}{Z}}$

Likelihood Function for MA(1)

Exact Likelihood :

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \underbrace{\sigma_a^{-(n+1)} \exp\{\frac{1}{2\sigma_a^2}(Y - Xa_0)^\top (Y - Xa_0)\}}_{P(a_0, \underset{\sim}{Z}) \to P(\underset{\sim}{Z})}$$

其中 :

$$Y = \begin{pmatrix} \cdot \\ D_\theta^{-1} \underset{\sim}{Z} \end{pmatrix} \quad X = -\begin{pmatrix} 1 \\ D_\theta^{-1} \triangle C_\theta \end{pmatrix}$$

Likelihood Function for MA(1)

Exact Likelihood :

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{\frac{1}{2\sigma_a^2}(Y - Xa_0)^\top (Y - Xa_0)\}$$

$$\begin{aligned} & (Y - Xa_0)^\top (Y - Xa_0) \\ = \; & (Y - X\hat{a}_0)^\top (Y - X\hat{a}_0) + (a_0 - \hat{a}_0)^2 X^\top X \end{aligned}$$

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{\frac{1}{2\sigma_a^2}(Y - Xa_0)^\top (Y - Xa_0)\}$$

$$\begin{aligned} & (Y - Xa_0)^\top (Y - Xa_0) \\ = \; & (Y - X\hat{a}_0)^\top (Y - X\hat{a}_0) + (a_0 - \hat{a}_0)^2 X^\top X \end{aligned}$$

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X\hat{a}_0)^\top (Y - X\hat{a}_0) + (a_0 - \hat{a}_0)^2 X^\top X\}$$

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{\frac{1}{2\sigma_a^2}(Y - Xa_0)^\top (Y - Xa_0)\}$$

$$\begin{aligned} & (Y - Xa_0)^\top (Y - Xa_0) \\ = \; & (Y - X\hat{a}_0)^\top (Y - X\hat{a}_0) + (a_0 - \hat{a}_0)^2 X^\top X \end{aligned}$$

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X\hat{a}_0)^\top (Y - X\hat{a}_0) + (a_0 - \hat{a}_0)^2 X^\top X\}$$

此部分与 a_0 无关

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{\frac{1}{2\sigma_a^2}(Y - X a_0)^\top (Y - X a_0)\}$$

$$\begin{aligned} & (Y - X a_0)^\top (Y - X a_0) \\ = \; & (Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (a_0 - \widehat{a}_0)^2 X^\top X \end{aligned}$$

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (a_0 - \widehat{a}_0)^2 X^\top X\}$$

Exact Likelihood :

$$\ell \propto P(\underset{\sim}{Z}) \propto \sigma_a^{-n} (X^\top X)^{1/2} \exp\{-\frac{1}{2\sigma_a^2}(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0)\}$$

Likelihood Function for MA(1)

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{\frac{1}{2\sigma_a^2}(Y - X a_0)^\top (Y - X a_0)\}$$

$$\begin{aligned} & (Y - X a_0)^\top (Y - X a_0) \\ = \; & (Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (a_0 - \widehat{a}_0)^2 X^\top X \end{aligned}$$

$$P(a_0, \underset{\sim}{Z}) \propto \sigma_a^{-(n+1)} \exp\{-\frac{1}{2\sigma_a^2}(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0) + (a_0 - \widehat{a}_0)^2 X^\top X\}$$

Exact Likelihood :

怎么算?

$$\ell \propto P(\underset{\sim}{Z}) \propto \sigma_a^{-n} (X^\top X)^{1/2} \exp\{-\frac{1}{2\sigma_a^2}(Y - X\widehat{a}_0)^\top (Y - X\widehat{a}_0)\}$$

Likelihood Function for MA(1)

$$\widehat{a}_0 = (X^\top X)^{-1} X^\top Y = -\frac{1 - \theta^{2(n+1)}}{1 - \theta^2} \sum_{t=1}^{n} \theta_t a_t^{(0)}(\theta, c)$$

其中 :

$$X = -\begin{pmatrix} 1 \\ \theta \\ \vdots \\ \theta^n \end{pmatrix}$$

$$a_t^{(0)}(\theta, c) = Z_t - c + \theta a_{t-1}^{(0)}(\theta, c)$$

with $a_0^{(0)}(\theta, c) = 0$

Likelihood Function for MA(1)

$$\widehat{a}_0 = (X^\top X)^{-1} X^\top Y = -\frac{1 - \theta^{2(n+1)}}{1 - \theta^2} \sum_{t=1}^{n} \theta_t a_t^{(0)}(\theta, c)$$

$$\begin{aligned} Y - X\widehat{a}_0 &= \begin{pmatrix} \cdot \\ D_\theta^{-1} \underset{\sim}{Z} \end{pmatrix} + \begin{pmatrix} 1 \\ D_\theta^{-1} \triangle C_\theta \end{pmatrix} \widehat{a}_0 \\ &= \begin{pmatrix} \widehat{a}_0 \\ D_\theta^{-1} (\underset{\sim}{Z} + \triangle C_\theta \widehat{a}_0) \end{pmatrix} \end{aligned}$$

Likelihood Function for MA(1)

$$\begin{pmatrix} Z_1 - c \\ Z_2 - c \\ \vdots \\ Z_n - c \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \theta \widehat{a_0} = \begin{pmatrix} Z_1 - c + \theta \widehat{a_0} \\ Z_2 - c \\ \vdots \\ Z_n - c \end{pmatrix}$$

$$\sum^{n}_{1} \theta_t a_t^{(0)}(\theta, c)$$

$$\Delta C_\theta \Big) \widehat{a_0}$$

$$= \Big(D_\theta^{-1}(\underset{\sim}{Z} + \Delta C_\theta \widehat{a_0}) \Big)$$

Likelihood Function for MA(1)

$$\tilde{\underset{\sim}{a}} = D_\theta^{-1}(\underset{\sim}{Z} + \Delta C_\theta \widehat{a_0})$$

$$\Leftrightarrow \quad D_\theta \tilde{\underset{\sim}{a}} = \underset{\sim}{Z} + \Delta C_\theta \widehat{a_0}$$

$$\Rightarrow \quad \tilde{a}_t(\theta, c) = Z_t + \theta \widehat{a_{t-1}}(\theta, c)$$

其中：

$$D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & \ddots & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix} \qquad \widehat{\underset{\sim}{a}} = \begin{pmatrix} \widehat{a_1} \\ \widehat{a_2} \\ \vdots \\ \widehat{a_n} \end{pmatrix}$$

Likelihood Function for MA(1)

Exact Likelihood Function:

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-n} \{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-1/2} \exp\{-\frac{1}{2\sigma_a^2} \sum_{t=0}^{n} (\tilde{a}_t(\theta, c))^2\}$$

• See Page 161-162 of Notes

Likelihood Function for MA(1)

Exact Likelihood Function:

$$\ell(\theta, c, \sigma_a^2 | \underset{\sim}{Z}) \propto \sigma_a^{-n} \{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-1/2} \exp\{-\frac{1}{2\sigma_a^2} \sum_{t=0}^{n} (\tilde{a}_t(\theta, c))^2\}$$

Given (θ, c)

$$\widehat{\sigma_a^2} = \frac{1}{n} \sum_{t=0}^{n} (\tilde{a}_t(\theta, c))^2$$

Thus, there are 2 ways to get the likelihood

• See Page 161-162 of Notes

Likelihood Function for MA(1)

Exact Likelihood Function:

$$\ell(\theta, c, \sigma_a^2|\underset{\sim}{Z}) \propto \sigma_a^{-n}\{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-1/2}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2\}$$

max out σ_a^2 for given (θ, c)

$$\begin{aligned}\ell(\theta, c|\underset{\sim}{Z}) &\propto \{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-n/2n}\{\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2\}^{-n/2} \\ &\propto \{\sum_{t=0}^{n}(\tilde{b}_t(\theta, c))^2\}^{-n/2} \\ \text{where} \quad \tilde{b}_t(\theta, c) &= \tilde{a}_t(\theta, c)\{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{1/2n}\end{aligned}$$

• See Page 161-162 of Notes

Likelihood Function for MA(1)

$$\ell(\theta, c|\underset{\sim}{Z}) \propto \{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-n/2n}\{\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2\}^{-n/2}$$

$$\{(\frac{1-\theta^{2(n+1)}}{1-\theta^2})^{1/n}\}^{-n/2}$$

Way 1

$$\begin{aligned}\ell(\theta, c|\underset{\sim}{Z}) &\propto \{(\frac{1-\theta^{2(n+1)}}{1-\theta^2})^{1/n}\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2\}^{-n/2} \\ &\propto \{\sum_{t=0}^{n}(\tilde{b}_t(\theta, c))^2\}^{-n/2}\end{aligned}$$

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Likelihood Function for MA(1)

Way 2

$$\sigma_a^{-n/2}\{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-1/2}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2\}$$

$$\boxed{x = e^{\log x}}$$

$$\exp\{-\frac{\sigma_a^2}{2\sigma_a^2}\log(\frac{1-\theta^{2(n+1)}}{1-\theta^2})\}$$

$$\sigma_a^{-n/2}\exp\{-\frac{1}{2\sigma_a^2}[\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2 + \sigma_a^2\log(\frac{1-\theta^{2(n+1)}}{1-\theta^2})]\}$$

• See Page 162 of Notes

Likelihood Function for MA(1)

Way 2

$$\sigma_a^{-n/2}\{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-1/2}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2\}$$

> 0

$$\sigma_a^{-n/2}\exp\{-\frac{1}{2\sigma_a^2}[\sum_{t=0}^{n}(\tilde{a}_t(\theta, c))^2 + \sigma_a^2\log(\frac{1-\theta^{2(n+1)}}{1-\theta^2})]\}$$

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Likelihood Function for MA(1)

Way 2

$$\sigma_a^{-n/2}\{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-1/2}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=0}^{n}(\tilde{a}_t(\theta,c))^2\}$$

没有此项

$$\sigma_a^{-n/2}\exp\{-\frac{1}{2\sigma_a^2}[\sum_{t=0}^{n}(\tilde{a}_t(\theta,c))^2+\sigma_a^2\log(\frac{1-\theta^{2(n+1)}}{1-\theta^2})]\}$$

• See Page 162 of Notes

Likelihood Function for MA(1)

Way 2

$$\sigma_a^{-n/2}\{\frac{1-\theta^{2(n+1)}}{1-\theta^2}\}^{-1/2}\exp\{-\frac{1}{2\sigma_a^2}\sum_{t=0}^{n}(\tilde{a}_t(\theta,c))^2\}$$

1979 Hillman/Tiao *JASA*

iterate between σ_a^2 and (θ, c)

没有此项

$$\sigma_a^{-n/2}\exp\{-\frac{1}{2\sigma_a^2}[\sum_{t=0}^{n}(\tilde{a}_t(\theta,c))^2+\sigma_a^2\log(\frac{1-\theta^{2(n+1)}}{1-\theta^2})]\}$$

• See Page 162 of Notes

Likelihood Function for MA(1)

Conditional Likelihood

- One recursions with $a_0 = 0$
- For given parameter values (θ, c, σ_a^2)
- nonlinear L.S.

Exact Likelihood

Two recursion

- $a_t^{(0)}(\theta, c)$ with $a_0 = 0$
- $\widehat{a_0}$
- $\tilde{a}_t(\theta, c)$ with $\widehat{a_0}$

Likelihood Function for MA(1)

MA(1) $\underset{\sim}{Z} = D_\theta \underset{\sim}{a} - \triangle c_\theta a_0$

其中:

$c_\theta = \theta$ $\quad \triangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\quad D_\theta = \begin{pmatrix} 1 & & & \\ -\theta & \ddots & & \\ & \ddots & \ddots & \\ & & -\theta & 1 \end{pmatrix}$

Likelihood Function for MA(2)

MA(2) $Z_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$

If known $a_0 = a_{-1} = 0$

$$\begin{aligned} Z_1 &= a_1 - \theta_1 a_0 - \theta_2 a_{-1} \\ Z_2 &= a_2 - \theta_1 a_1 - \theta_2 a_0 \\ Z_3 &= a_3 - \theta_1 a_2 - \theta_2 a_1 \\ Z_4 &= a_4 - \theta_1 a_3 - \theta_2 a_2 \end{aligned}$$

Likelihood Function for MA(2)

$$\begin{aligned} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_n \end{pmatrix} &= \begin{pmatrix} 1 & & & & \\ -\theta_1 & \ddots & & & \\ -\theta_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & -\theta_2 & -\theta_1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} - \begin{pmatrix} \theta_2 & \theta_1 \\ & \theta_2 \\ & \\ & \\ & \end{pmatrix} \begin{pmatrix} a_{-1} \\ a_0 \end{pmatrix} \\ &= D_\theta \underset{\sim}{a} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_2 & \theta_1 \\ & \theta_2 \end{pmatrix} \begin{pmatrix} a_{-1} \\ a_0 \end{pmatrix} \\ &= D_\theta \underset{\sim}{a} - \triangle C_\theta \begin{pmatrix} a_{-1} \\ a_0 \end{pmatrix} \end{aligned}$$

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时间序列分析

第八讲（下）

Likelihood Function for MA(2)

$$\begin{pmatrix} \underset{\sim}{a}_* \\ \underset{\sim}{Z} \end{pmatrix} = \begin{pmatrix} I & \cdot \\ -\triangle C_\theta & D_\theta \end{pmatrix} \begin{pmatrix} \underset{\sim}{a}_* \\ \underset{\sim}{a} \end{pmatrix}$$

其中：

$$\triangle = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix};\quad C_\theta = \begin{pmatrix} \theta_2 & \theta_1 \\ & \theta_2 \end{pmatrix} \quad a_* = \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

或

$$C_\theta = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \end{pmatrix} \quad a_* = \begin{pmatrix} a_0 \\ a_{-1} \end{pmatrix}$$

• See Page 164 of Notes

ARIMA(p,d,q) - Model Building

1. Tentative Model Specification
   - SACF
   - SPACF
   - SEACF
2. Model Fitting or Model Estimation
   - MLE of unknown parameter
3. Diagnostic Checking

Diagnostic Check – AR(1)

$$Z_t = c + \phi Z_{t-1} + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$$

$$\Downarrow$$

Estimate $\widehat{c}$ and $\widehat{\phi}$

$$\Downarrow$$

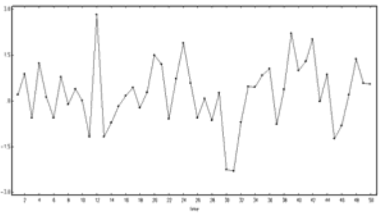
Residuals: $\widehat{a}_t = Z_t - \widehat{c} - \widehat{\phi} Z_{t-1}$

Diagnostic Check – AR(1)

$Z_t = c + \phi Z_{t-1} + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)$

Estimate $\widehat{c}$ and $\widehat{\phi}$

Residuals: $\widehat{a}_t = Z_t - \widehat{c}$

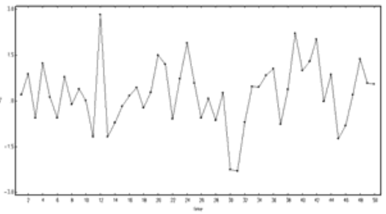


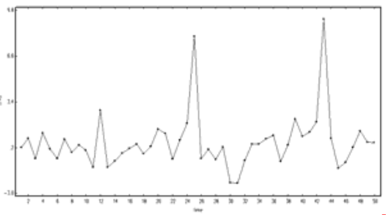
Diagnostic Check – AR(1)

$Z_t = c + \phi Z_{t-1} + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)$

Estimate $\widehat{c}$ and $\widehat{\phi}$

Residuals: $\widehat{a}_t = Z_t - \widehat{c}$



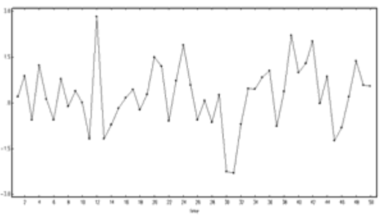


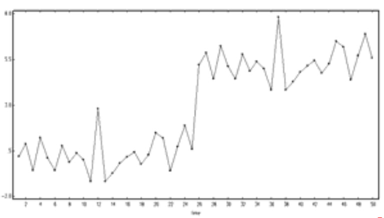
Diagnostic Check – AR(1)

$Z_t = c + \phi Z_{t-1} + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)$

Estimate $\widehat{c}$ and $\widehat{\phi}$

Residuals: $\widehat{a}_t = Z_t - \widehat{c}$



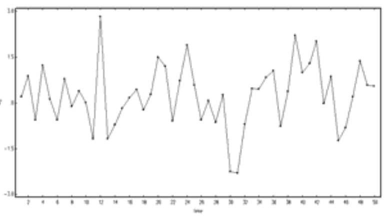


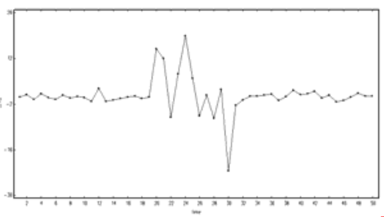
Diagnostic Check – AR(1)

$Z_t = c + \phi Z_{t-1} + a_t \qquad a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2)$

Estimate $\widehat{c}$ and $\widehat{\phi}$

Residuals: $\widehat{a}_t = Z_t - \widehat{c}$



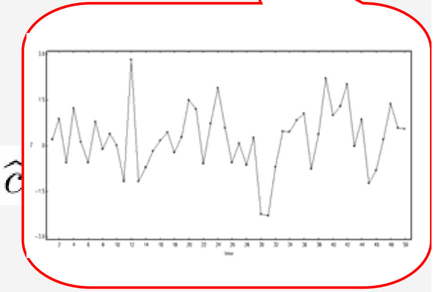


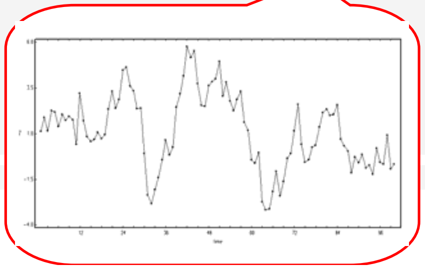
Diagnostic Check – AR(1)

$Z_t = c + \phi Z_{t-1} + a_t$ $\quad a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$

Estimate $\hat{c}$ and $\hat{\phi}$

Residuals: $\widehat{a_t} = Z_t - \hat{c}$





Diagnostic Check – AR(1)

1. Plot Residuals
2. Check if residual still have lag correlation

Diagnostic Check – AR(1)

1. Plot Residuals
2. Check if residual still have lag correlation

For Time Series Model

Most Popular Checking Technique:

Check to see if residuals are auto-correlated

Diagnostic Check – ARMA(1,1)

$$Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}$$

$$\Downarrow$$

$$\widehat{a_t} = Z_t - \widehat{\phi} Z_{t-1} + \widehat{\theta}\, \widehat{a_{t-1}}$$

Diagnostic Check – ARMA(1,1)

$$Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}$$

$$\Downarrow$$

$$\widehat{a}_t = Z_t - \widehat{\phi} Z_{t-1} + \widehat{\theta} \widehat{a_{t-1}}$$

If model adequate, $a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$

study to see if $\widehat{a}_t \to a_t$

Diagnostic Check – ARMA(1,1)

Theoretically if model is correct

for large sample: $\gamma_k(\widehat{a}_t) \to 0$

Diagnostic Check – ARMA(1,1)

Theoretically if model is correct

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SACF of $\widehat{a}_t$: $\gamma_k^2 \to \chi_1^2$

Diagnostic Check – ARMA(1,1)

Theoretically if model is correct

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SACF of $\widehat{a}_t$: $\gamma_k^2 \to \chi_1^2$

$$n \sum_{\ell=1}^{m} \gamma_\ell^2(\widehat{a}_t) \sim \chi_m^2$$

Diagnostic Check – ARMA(1,1)

Theoretically if model is correct

For large sample: $\gamma_k(\widehat{a}_t) \to 0$

SACF of $\widehat{a}_t$: $\gamma_k^2 \to \chi_1^2$

$$\cancel{n\sum_{\ell=1}^{m}\gamma_\ell^2(\widehat{a}_t) \sim \chi_m^2}$$

Diagnostic Check – ARMA(1,1)

Theoretically if model is correct

For large sample: $\gamma_k(\widehat{a}_t) \to 0$

SACF of $\widehat{a}_t$: $\gamma_k^2 \to \chi_1^2$

$$\cancel{n\sum_{\ell=1}^{m}\gamma_\ell^2(\widehat{a}_t) \sim \chi_m^2}$$

JASA (1970)

Box-Pierce

$$Q = \sum_{\ell=1}^{m}\gamma_\ell^2(\widehat{a}_t) \sim \chi_{m-(p+q)}^2$$

Diagnostic Check – ARMA(1,1)

Biometrika (1978)

Box–Ljung modified Q Stat：

Box-Pierce $\quad n\sum_{\ell=1}^{m}\gamma_\ell^2(\widehat{a}_t)$

Box-Ljung $\quad n(n+2)\sum_{\ell=1}^{m}\frac{1}{n-\ell}\gamma_\ell^2(\widehat{a}_t) \sim \chi_{m-(p+q)}^2$

Diagnostic Check

Examples：

- Box-Jenkins Series F (Page 252)
- Box-Jenkins Series A (Page 274)
- Generated ARIMA(0,1,1) (Page 260)

Diagnostic Check

Examples：

- Box-Jenkins Series F (Page 252)
- Box-Jenkins Series A (Page 274)
- Generated ARIMA(0,1,1) (Page 260)

model z(1) = (1)noise

对Z差分 MA(1)

$$\boxed{(1-B)Z_t = (1-\theta B)a_t}$$

Diagnostic Check

Examples：

- Box-Jenkins Series F (Page 252)
- Box-Jenkins Series A (Page 274)
- Generated ARIMA(0,1,1) (Page 260)

model z(1) = (1)noise 估计时加no check选项

对Z差分 MA(1)

$$\boxed{(1-B)Z_t = (1-\theta B)a_t}$$

Diagnostic Check

Examples：

- Box-Jenkins Series F (Page 252)
- Box-Jenkins Series A (Page 274)
- Generated ARIMA(0,1,1) (Page 260)

ARMA(1,1)

model z(1) = (1)noise $Z_t - Z_{t-1} = a_t - \theta a_{t-1}$

model (1)z = (1)noise $Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}$

Seasonal Model

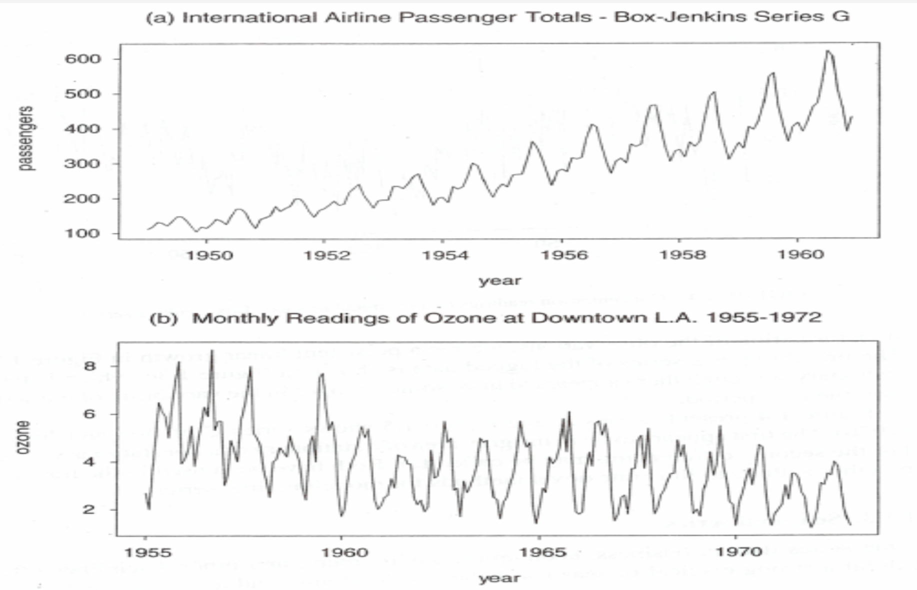


FIGURE 1.6 Two examples of seasonal series.

Seasonal Model

$$
\begin{aligned}
Z_t &= \eta \sin(\frac{2\pi}{12}t + \phi) + e_t \\
&= \eta \sin \frac{2\pi}{12} t \cos\phi + \eta \cos \frac{2\pi}{12} t \sin\phi + e_t \\
&= x_{1t}\beta_1 + x_{2t}\beta_2 + e_t
\end{aligned}
$$

其中：

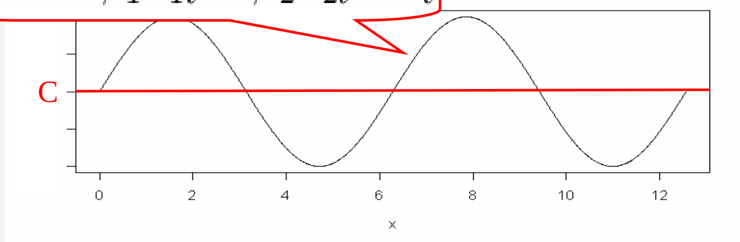
$$x_{1t} = \sin\frac{2\pi}{12}t \qquad x_{2t} = \cos\frac{2\pi}{12}t$$

$$\beta_1 = \eta\cos\phi \qquad \beta_2 = \eta\sin\phi$$

- See Page 167 of Notes

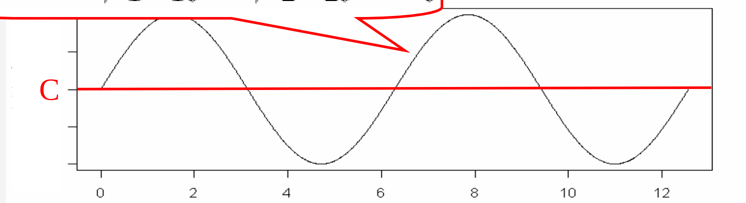
Seasonal Model

$$y_t = c + \beta_1 x_{1t} + \beta_2 x_{2t} + \epsilon_t$$

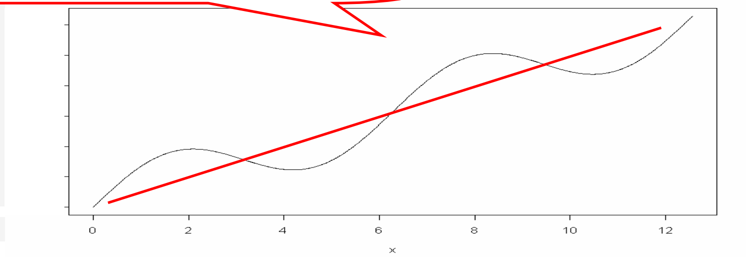


Seasonal Model

$$y_t = c + \beta_1 x_{1t} + \beta_2 x_{2t} + \epsilon_t$$

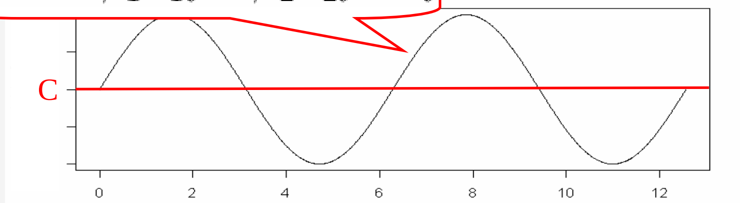


$$y_t = c + \alpha t + \beta_1 x_{1t} + \beta_2 x_{2t} + \epsilon_t$$

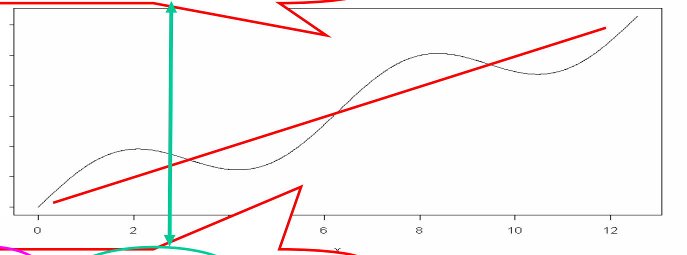


Seasonal Model

$$y_t = c + \beta_1 x_{1t} + \beta_2 x_{2t} + \epsilon_t$$



$$y_t = c + \alpha t + \beta_1 x_{1t} + \beta_2 x_{2t} + \epsilon_t$$



$$y_t = trend + seasonal + noise$$

时间序列分析

第九讲(上)

Topics

1. Review
   - ARIMA(p,d,q) models
   - Model Building
2. Seasonal Models
   - BJ Approach
   - Airline Model (various details)
   - Multiplicative models
   - Ozone example
3. Linear Dynamic Model
   - Intervention Analysis

Seasonal Data - Examples

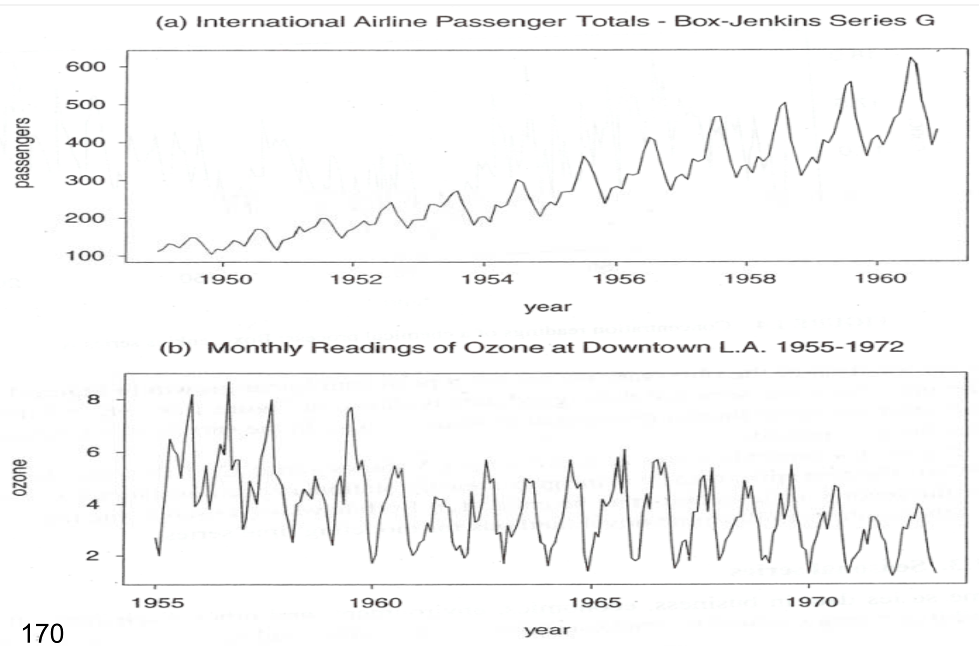
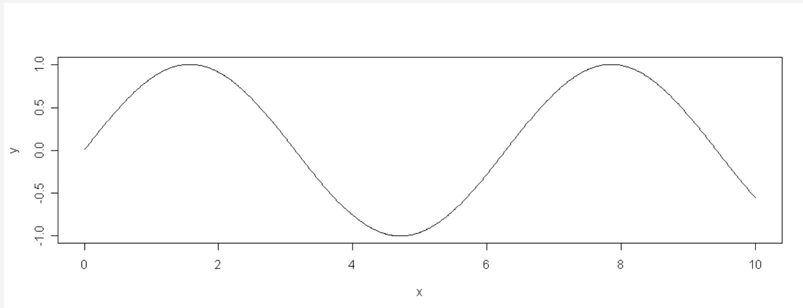


FIGURE 1.6 Two examples of seasonal series.

Seasonal Data

Deterministic Function :

$$Y_t = d\sin(\frac{2\pi}{12}t + \omega) + \epsilon_t$$



Seasonal Data

Deterministic Function :

$$Y_t = \boxed{d\sin(\frac{2\pi}{12}t+\omega)} + \epsilon_t$$

Linear Model :

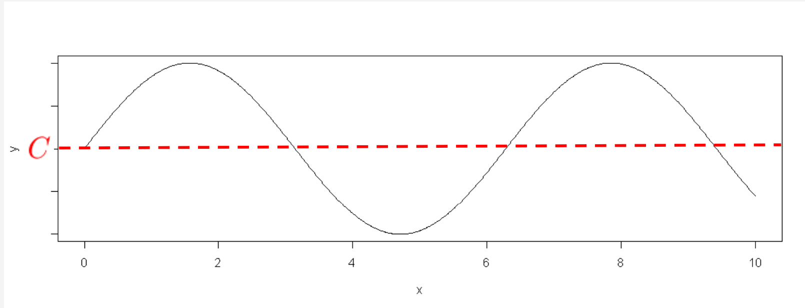
$$d\sin(\frac{2\pi}{12}t+\omega) = \beta_1 x_{1t} + \beta_2 x_{2t}$$

其中 : $\begin{cases} \beta_1 = d\cos\omega \\ \beta_2 = d\sin\omega \end{cases}$ $\begin{cases} x_{1t} = \sin\frac{2\pi}{12}t \\ x_{2t} = \cos\frac{2\pi}{12}t \end{cases}$

Seasonal Data

均值不为0的季节数据处理 :

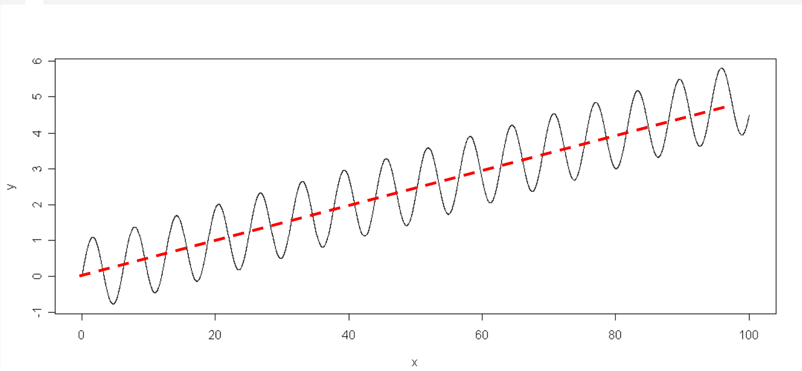
$$Y_t = C + \beta_1 x_{1t} + \beta_2 x_{2t} + e_t$$



Seasonal Data

有趋势性的季节数据处理 :

$$\begin{aligned} Y_t &= C + \alpha t + \beta_1 x_{1t} + \beta_2 x_{2t} + e_t \\ &= trend + seasonal + e_t \end{aligned}$$



Seasonal Data

Bacon – Two Way Table :

year \ month	J	F	M	A	M	J	J	A	S	O	N	D
1	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}
2	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}	Z_{18}	Z_{19}	Z_{20}	Z_{21}	Z_{22}	Z_{23}	Z_{24}
3	Z_{25}	Z_{26}	Z_{27}	Z_{28}	Z_{29}	Z_{30}	Z_{31}	Z_{32}	Z_{33}	Z_{34}	Z_{35}	Z_{36}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Seasonal Data

Bacon – Two Way Table :

year \ month	J	F	M	A	M	J	J	A	S	O	N	D
1	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}
2	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}	Z_{18}	Z_{19}	Z_{20}	Z_{21}	Z_{22}	Z_{23}	Z_{24}
3	Z_{25}	Z_{26}	Z_{27}	Z_{28}	Z_{29}	Z_{30}	Z_{31}	Z_{32}	Z_{33}	Z_{34}	Z_{35}	Z_{36}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

What's the relationship between Z_t and Z_{t-12} ?

Seasonal MA

MA(1) : $Z_t = a_t - \theta a_{t-1}$

Seasonal MA(1) : $Z_t = e_t - \Theta e_{t-12}$

Seasonal MA

MA(1) : $Z_t = a_t - \theta a_{t-1}$

Seasonal MA(1) : $Z_t = e_t - \Theta e_{t-12}$

If $e_t \overset{i.i.d.}{\sim} N(0, \sigma_e^2)$

then $Z_t = (1 - \Theta B^{12}) e_t$

$$\rho_\ell \begin{cases} \neq 0, & \ell = 12; \\ = 0, & \text{otherwise.} \end{cases}$$

Seasonal MA

MA(1) : $Z_t = a_t - \theta a_{t-1}$

Seasonal MA(1) : $Z_t = e_t - \Theta e_{t-12}$

If $e_t \overset{i.i.d.}{\sim} N(0, \sigma_e^2)$

then $Z_t = (1 - \Theta B^{12}) e_t$

$$\rho_\ell \begin{cases} \neq 0, & \ell = 12; \\ = 0, & \text{otherwise.} \end{cases}$$

Seasonal MA(2) : $Z_t = (1 - \Theta_1 B^s - \Theta_2 B^{2s}) e_t$

Seasonal MA(q)

Seasonal AR

$$Y_t = \varphi_1 Y_{t-12}$$

AR(1) : $(1-\phi B)Z_t = a_t$

Seasonal AR(1) : $(1-\varphi B^s)Z_t = e_t$

Seasonal AR(p)$_s$:

• See Page 167-168 of Notes

Multiplicative Model

Bacon – Two Way Table :

year \ month	J	F	M	A	M	J	J	A	S	O	N	D
1	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}
2	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}	Z_{18}	Z_{19}	Z_{20}	Z_{21}	Z_{22}	Z_{23}	Z_{24}
3	Z_{25}	Z_{26}	Z_{27}	Z_{28}	Z_{29}	Z_{30}	Z_{31}	Z_{32}	Z_{33}	Z_{34}	Z_{35}	Z_{36}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Multiplicative Model

Bacon – Two Way Table :

year \ month	J	F	M	A	M	J	J	A	S	O	N	D
1	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}
2	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}	Z_{18}	Z_{19}	Z_{20}	Z_{21}	Z_{22}	Z_{23}	Z_{24}
3	Z_{25}	Z_{26}	Z_{27}	Z_{28}	Z_{29}	Z_{30}	Z_{31}	Z_{32}	Z_{33}	Z_{34}	Z_{35}	Z_{36}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Periodic Time Series Model

Multiplicative Model

Seasonal MA(1) :

$$\boxed{\begin{aligned} Z_t &= e_t - \Theta_{12} e_{t-12} \\ &= (1-\Theta_{12}B^{12})e_t \end{aligned}}$$

- If e_t i.i.d. , then we can only accommodate seasonal relationship
- If e_t NOT i.i.d. , but simply an MA(1) model : $\boxed{\begin{aligned} e_t &= a_t - \theta a_{t-1} \\ &= (1-\theta B)a_t \end{aligned}}$

$$\boxed{e_t = (1-\theta B)a_t}$$

$$\boxed{Z_t = (1-\Theta_{12}B^{12})e_t}$$

year \ month	J	F	M	A	M	J	J	A	S	O	N	D
1	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}
2	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}	Z_{18}	Z_{19}	Z_{20}	Z_{21}	Z_{22}	Z_{23}	Z_{24}
3	Z_{25}	Z_{26}	Z_{27}	Z_{28}	Z_{29}	Z_{30}	Z_{31}	Z_{32}	Z_{33}	Z_{34}	Z_{35}	Z_{36}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Multiplicative Model

$$\begin{cases} Z_t = (1-\Theta_{12}B^{12})e_t \\ e_t = (1-\theta B)a_t \end{cases}$$

Multiplicative Model

$$\Rightarrow \quad Z_t = (1-\Theta_{12}B^{12})(1-\theta B)a_t$$

$$\Leftrightarrow \quad Z_t = (1-\theta B-\Theta_{12}B^{12}+\theta\Theta_{12}B^{13})a_t$$

Multiplicative Model

$$Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t$$

Correlation Function :

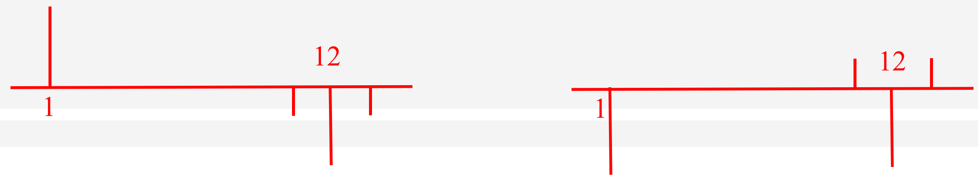
$\rho_1,\ \rho_{12},\ \rho_{13},\ \rho_{11} \neq 0,$

all others $= 0$

$$\rho_{11} = \rho_{13} = \rho_1 \times \rho_{12}$$

$$\rho_1 = \frac{-\theta}{1+\theta^2} \qquad \rho_{12} = \frac{-\Theta_{12}}{1+\Theta_{12}^2}$$

ACF Pattern :



Multiplicative Model

Not original data has this type of model

But : $(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t$

Multiplicative Model

Not original data has this type of model

But : $\boxed{(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t}$

Most useful member of the general class (p,d,q)

$$\phi(B)(1-B)^d Z_t = \theta(B)a_t$$

Seasonal $(P,D,Q)_s$

Multiplicative Model

$$\varphi(B^s)(1-B^s)^{D_s}Z_t = \Theta_q(B^s)e_t$$

If : $\phi(B)(1-B)^d e_t = \theta_q(B)a_t$

Then : $\phi(B)(1-B)^d\varphi(B^s)(1-B^s)^{D_s}Z_t = \theta_q(B)\Theta_Q(B^s)a_t$

Non Seasonal Part/Factors

Seasonal Factors

Multiplicative Model

Most Useful/Common :

$$\boxed{(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t}$$

Multiplicative Model

Most Useful/Common :

$$\boxed{(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t}$$

Airline Example (Page 315):

Multiplicative Model

Most Useful/Common :

$$\boxed{(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t}$$

Airline Example (Page 315):

For Airline data, after $(1\text{-}B)(1\text{-}B^{12})Z_t$

it roughly consistent with correlation pattern of

$(1\text{-}\theta B)(1\text{-}\Theta_{12}B^{12})a_t$

Or we can use

$(1\text{-}B)(1\text{-}B^{12})Z_t = (1\text{-}\theta B)(1\text{-}\Theta_{12}B^{12})a_t$

时间序列分析

第九讲(下)

Multiplicative Model

Most Useful/Common :

$$\boxed{(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t}$$

Airline Example (Page 315):

For Airline data, after (1-B)(1-B^{12})Z

it roug … on pattern of

(1-

Airline Model

Or we ca

(1-B) … B^{12})Z_t = (1-θB)(1-$\Theta_{12}B^{12}$)a_t

Airline Model

$$\boxed{(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta_{12}B^{12})a_t}$$

Forecast : $Z_1, \cdots, Z_T$ known, predict $Z_{T+\ell}, \ell = 1, 2, 3, \cdots$

其中T: forecast origin

For Z_{T+l} :

$$\begin{aligned} Z_{T+\ell} &= Z_{T+\ell-1} + Z_{T+\ell-12} - Z_{T+\ell-13} \\ &\quad + a_{T+\ell} - \theta a_{T+\ell-1} - \Theta a_{T+\ell-12} + \theta\Theta a_{T+\ell-13} \end{aligned}$$

When $l > 13$:

$$\widehat{Z_T(\ell)} = \widehat{Z_T(\ell-1)} + \widehat{Z_T(\ell-12)} - \widehat{Z_T(\ell-13)}$$

$$(1 - B - B^{12} + B^{13})\widehat{Z_T(\ell)} = 0$$

• See Page 171 of Notes

Airline Model

Data :

y \ m	J	F	M	A	M	J	J	A	S	O	N	D
1	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	Z_{12}
2	Z_{13}	Z_{14}	Z_{15}	Z_{16}	Z_{17}	Z_{18}	Z_{19}	Z_{20}	Z_{21}	Z_{22}	Z_{23}	Z_{24}
3	Z_{25}	Z_{26}	Z_{27}	Z_{28}	Z_{29}	Z_{30}	Z_{31}	Z_{32}	Z_{33}	Z_{34}	Z_{35}	Z_{36}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Forecast :

y \ m	J	F	M	A	M	J	J	A	S	O	N	D
0	$\widehat{Z_T(1)}$	$\widehat{Z_T(2)}$	$\widehat{Z_T(3)}$	$\widehat{Z_T(4)}$	$\widehat{Z_T(5)}$	$\widehat{Z_T(6)}$	$\widehat{Z_T(7)}$	$\widehat{Z_T(8)}$	$\widehat{Z_T(9)}$	$\widehat{Z_T(10)}$	$\widehat{Z_T(11)}$	$\widehat{Z_T(12)}$
1	$\widehat{Z_T(13)}$	$\widehat{Z_T(14)}$	$\widehat{Z_T(15)}$	$\widehat{Z_T(16)}$	$\widehat{Z_T(17)}$	$\widehat{Z_T(18)}$	$\widehat{Z_T(19)}$	$\widehat{Z_T(20)}$	$\widehat{Z_T(21)}$	$\widehat{Z_T(22)}$	$\widehat{Z_T(23)}$	$\widehat{Z_T(24)}$
2	$\widehat{Z_T(25)}$	$\widehat{Z_T(26)}$	$\widehat{Z_T(27)}$	$\widehat{Z_T(28)}$	$\widehat{Z_T(29)}$	$\widehat{Z_T(30)}$	$\widehat{Z_T(31)}$	$\widehat{Z_T(32)}$	$\widehat{Z_T(33)}$	$\widehat{Z_T(34)}$	$\widehat{Z_T(35)}$	$\widehat{Z_T(36)}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

• See Page 171 of Notes

Airline Model

$$\begin{aligned} Z_{T+\ell} &= Z_{T+\ell-1} + Z_{T+\ell-12} - Z_{T+\ell-13} \\ &\quad + a_{T+\ell} - \theta a_{T+\ell-1} - \Theta a_{T+\ell-12} + \theta\Theta a_{T+\ell-13} \end{aligned}$$

Forecast :

$$\ell = m + 12n \begin{cases} n = 0, 1, 2, \cdots, & \text{year;} \\ m = 1, 2, \cdots, 12, & \text{month of the year.} \end{cases}$$

• See Page 171 of Notes

Airline Model

$$\begin{aligned} Z_{T+\ell} &= Z_{T+\ell-1} + Z_{T+\ell-12} - Z_{T+\ell-13} \\ &\quad + a_{T+\ell} - \theta a_{T+\ell-1} - \Theta a_{T+\ell-12} + \theta\Theta a_{T+\ell-13} \end{aligned}$$

Forecast :

$$\ell = m + 12n \begin{cases} n = 0, 1, 2, \cdots, & \text{year;} \\ m = 1, 2, \cdots, 12, & \text{month of the year.} \end{cases}$$

$$\widehat{Z_T(\ell)} = \widehat{Z_T(m)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n$$

• See Page 171 of Notes

Airline Model

y \ m	J	F	M	A	M	J	J	A	S	O	N	D
0	$\widehat{Z_T(1)}$	$\widehat{Z_T(2)}$	$\widehat{Z_T(3)}$	$\widehat{Z_T(4)}$	$\widehat{Z_T(5)}$	$\widehat{Z_T(6)}$	$\widehat{Z_T(7)}$	$\widehat{Z_T(8)}$	$\widehat{Z_T(9)}$	$\widehat{Z_T(10)}$	$\widehat{Z_T(11)}$	$\widehat{Z_T(12)}$
1	$\widehat{Z_T(13)}$	$\widehat{Z_T(14)}$	$\widehat{Z_T(15)}$	$\widehat{Z_T(16)}$	$\widehat{Z_T(17)}$	$\widehat{Z_T(18)}$	$\widehat{Z_T(19)}$	$\widehat{Z_T(20)}$	$\widehat{Z_T(21)}$	$\widehat{Z_T(22)}$	$\widehat{Z_T(23)}$	$\widehat{Z_T(24)}$
2	$\widehat{Z_T(25)}$	$\widehat{Z_T(26)}$	$\widehat{Z_T(27)}$	$\widehat{Z_T(28)}$	$\widehat{Z_T(29)}$	$\widehat{Z_T(30)}$	$\widehat{Z_T(31)}$	$\widehat{Z_T(32)}$	$\widehat{Z_T(33)}$	$\widehat{Z_T(34)}$	$\widehat{Z_T(35)}$	$\widehat{Z_T(36)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Forecast :

$$\ell = m + 12n \begin{cases} n = 0, 1, 2, \cdots, & \text{year;} \\ m = 1, 2, \cdots, 12, & \text{month of the year.} \end{cases}$$

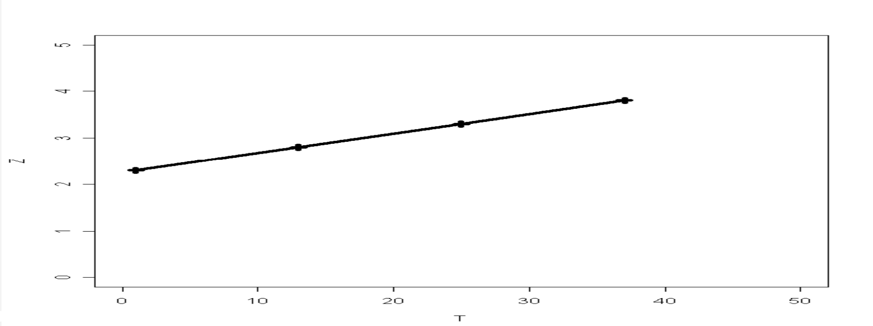
$$\begin{aligned} \widehat{Z_T(\ell)} &= \widehat{Z_T(m)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n \\ &= \widehat{Z_T(1)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n \end{aligned}$$

• See Page 171 of Notes

Airline Model

Forecast : $$\begin{aligned} \widehat{Z_T(\ell)} &= \widehat{Z_T(m)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n \\ &= \underbrace{\widehat{Z_T(1)}}_{2.3} + \underbrace{\{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}}_{.5}n \end{aligned}$$

m=1 : 2.3, 2.8, 3.3, 3.8



• See Page 171 of Notes

Airline Model

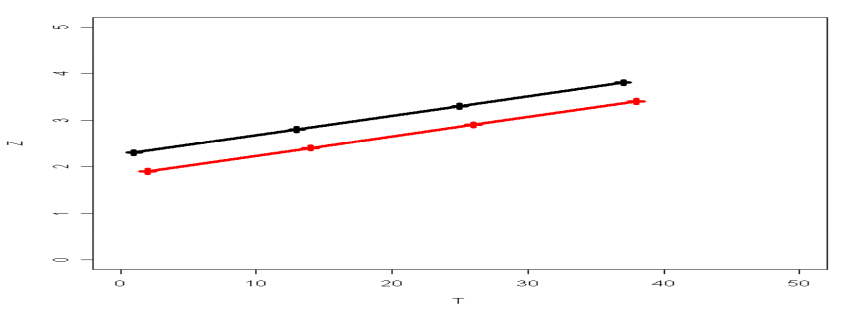
Forecast :

$$\widehat{Z_T(\ell)} = \widehat{Z_T(m)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n$$
$$= \widehat{Z_T(1)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n$$

Same slope

m=1 : 2.3, 2.8, 3.3, 3.8

m=2 : $\widehat{Z_T(\ell)} = \widehat{Z_T(2)} + .5n$ 1.9, 2.4, 2.9, 3.4



s

Airline Model

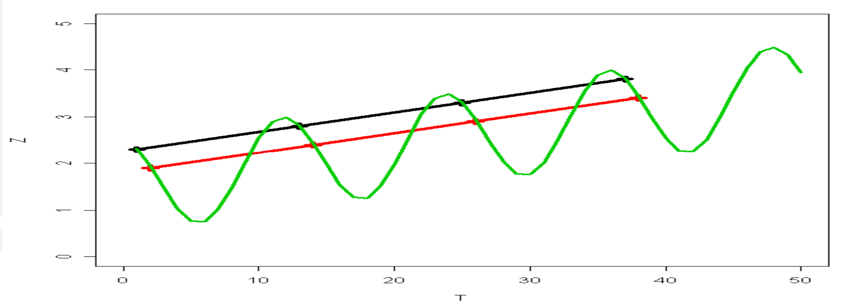
Forecast :

$$\widehat{Z_T(\ell)} = \widehat{Z_T(m)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n$$
$$= \widehat{Z_T(1)} + \{\widehat{Z_T(13)} - \widehat{Z_T(1)}\}n$$

Same slope

m=1 : 2.3, 2.8, 3.3, 3.8

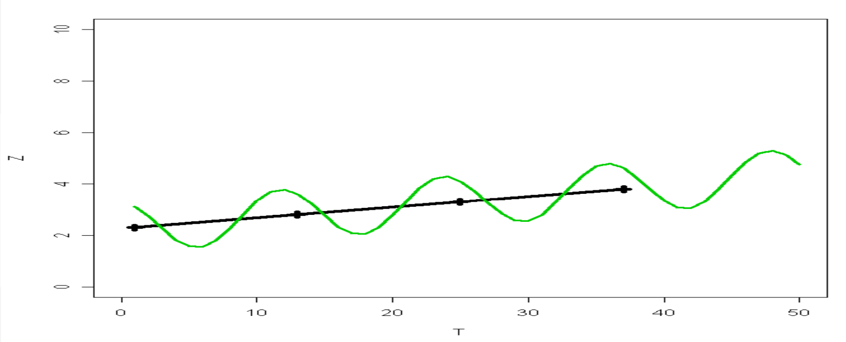
m=2 : $\widehat{Z_T(\ell)} = \widehat{Z_T(2)} + .5n$ 1.9, 2.4, 2.9, 3.4



s

Airline Model

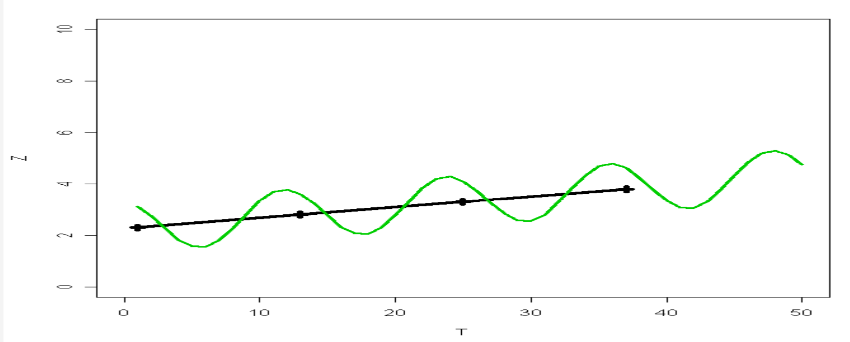
Forecast :



• See Page 171 of Notes

Airline Model

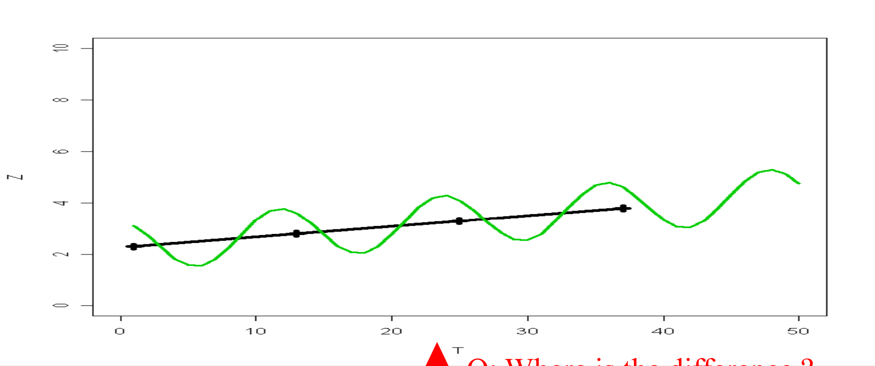
Forecast :



$$Y_t = c_0 + c_1 t + d\sin(\frac{2\pi}{12}t + \omega) + \epsilon_t$$

Airline Model

Forecast :

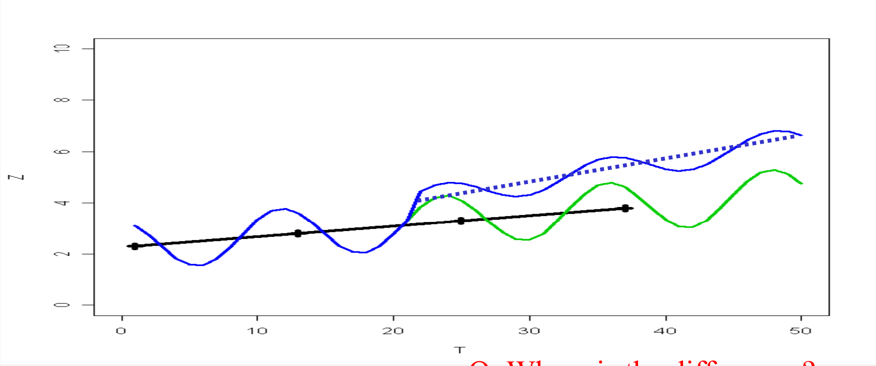


Q: Where is the difference ?

$$Y_t = c_0 + c_1 t + d\sin(\frac{2\pi}{12}t + \omega) + \epsilon_t$$

Airline Model

Forecast :



Q: Where is the difference ?

A: trend and seasonality can change over time

$$Y_t = c_0 + c_1 t + d\sin(\frac{2\pi}{12}t + \omega) + \epsilon_t$$

Airline Model

model :

$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

- See Page 172 of Notes

Airline Model

model :

$$(1-B)^2(1+B+\cdots+B^{11})$$

$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

$$\begin{aligned} 1-x^2 &= (1-x)(1+x) \\ 1-x^3 &= (1-x)(1+x+x^2) \\ 1-x^{12} &= (1-x)(1+x+\cdots+x^{11}) \end{aligned}$$

- See Page 172 of Notes

Airline Model

model :

$$(1-B)(1-B^{12})Z_t=(1-\theta B)(1-\Theta B^{12})a_t$$

$$(1-B)(1-B^{12})Z_t = (1-B)^2(1+B+\cdots+B^{11})Z_t$$

• See Page 172 of Notes

Airline Model

model :

$$(1-B)(1-B^{12})Z_t=(1-\theta B)(1-\Theta B^{12})a_t$$

$$(1-B)(1-B^{12})Z_t = (1-B)^2(1+B+\cdots+B^{11})Z_t$$

• See Page 172 of Notes

Airline Model

model :

$$(1-B)(1-B^{12})Z_t=(1-\theta B)(1-\Theta B^{12})a_t$$

$$\begin{aligned}(1-B)(1-B^{12})Z_t &= (1-B)^2\underbrace{(1+B+\cdots+B^{11})Z_t}_{W_t}\\ (1+B+\cdots+B^{11})Z_t &= Z_t+Z_{t-1}+\cdots+Z_{t-11}\\ \frac{(1+B+\cdots+B^{11})Z_t}{12} &= \cdots\end{aligned}$$

• See Page 172 of Notes

Airline Model

$$(1-B)(1-B^{12})Z_t=(1-\theta B)(1-\Theta B^{12})a_t$$

$$\begin{aligned}(1-B)(1-B^{12})Z_t &= (1-B)^2\underbrace{(1+B+\cdots+B^{11})Z_t}_{W_t}\\ (1+B+\cdots+B^{11})Z_t &= Z_t+Z_{t-1}+\cdots+Z_{t-11}\\ \frac{(1+B+\cdots+B^{11})Z_t}{12} &= \cdots\end{aligned}$$

$$(1-B)^2W_t = (\quad)(\quad)a_t$$

其中 W_t : 12月移动平均

• See Page 172 of Notes

Airline Model

$$(1-B)\boxed{(1-B^{12})}Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

$$\boxed{\begin{aligned} 1-B^{12} &= (1-B^6)(1+B^6) \\ (1-B^6) &= (1-B^3)(1+B^3) \\ (1+B^6) &= (1+B^2)(1-B^2+B^4) \end{aligned}}$$

• See Page 166-175 of Notes

Airline Model

$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

Non Seasonal Part/Factors

Seasonal Factors

• See Page 166-175 of Notes

Airline Model

$$(1-B) \quad Z_t = (1-\theta B) \quad a_t$$

Non Seasonal Part/Factors

Seasonal Factors

• See Page 166-175 of Notes

Airline Model

$$(1-B^{12})Z_t = (1-\Theta B^{12})a_t$$

Non Seasonal Part/Factors

Seasonal Factors

• See Page 166-175 of Notes

Airline Model

$$(1-B)(1-B^{12})Z_t \quad = \quad (1-\theta B)(1-\Theta B^{12})a_t$$

Non Seasonal Part/Factors

Seasonal Factors

$$\left(\frac{1-B}{1-\theta B}\right)\left(\frac{1-B^{12}}{1-\Theta B^{12}}\right)Z_t \quad = \quad a_t$$

• See Page 166-175 of Notes

Airline Model

$$\left(\frac{1-B}{1-\theta B}\right)\left(\frac{1-B^{12}}{1-\Theta B^{12}}\right)Z_t = a_t$$

$$\Leftrightarrow \left\{1-\frac{(1-\theta)B}{1-\theta B}\right\}\left\{1-\frac{(1-\Theta)B^{12}}{1-\Theta B^{12}}\right\}Z_t = a_t$$

$$\Rightarrow \widehat{Z_T(1)} = \frac{1-\theta}{1-\theta B}Z_T + \frac{1-\Theta}{1-\Theta B^{12}}\left[Z_{T+1-12} - \frac{1-\theta}{1-\theta B}Z_{T-12}\right]$$

• See Page 166-175 of Notes

Airline Model

$$\left(\frac{1-B}{1-\theta B}\right)\left(\frac{1-B^{12}}{1-\Theta B^{12}}\right)Z_t = a_t$$

$$\Leftrightarrow \left\{1-\frac{(1-\theta)B}{1-\theta B}\right\}\left\{1-\frac{(1-\Theta)B^{12}}{1-\Theta B^{12}}\right\}Z_t = a_t$$

$$\Rightarrow \widehat{Z_T(1)} = \frac{1-\theta}{1-\theta B}Z_T + \frac{1-\Theta}{1-\Theta B^{12}}\left[Z_{T+1-12} - \frac{1-\theta}{1-\theta B}Z_{T-12}\right]$$

2006.12

2006.11

2005.12

2005.11

• See Page 166-175 of Notes

Airline Model

$$\left(\frac{1-B}{1-\theta B}\right)\left(\frac{1-B^{12}}{1-\Theta B^{12}}\right)Z_t = a_t$$

$$\Leftrightarrow \left\{1-\frac{(1-\theta)B}{1-\theta B}\right\}\left\{1-\frac{(1-\Theta)B^{12}}{1-\Theta B^{12}}\right\}Z_t = a_t$$

$$\Rightarrow \widehat{Z_T(1)} = \frac{1-\theta}{1-\theta B}Z_T + \frac{1-\Theta}{1-\Theta B^{12}}\left[Z_{T+1-12} - \frac{1-\theta}{1-\theta B}Z_{T-12}\right]$$

2006.12

2006.11

2005.12

2005.11

$$-(1-\theta)[Z_T + \theta Z_{T-1} + \theta^2 Z_{T-2} + \theta^3 Z_{T-3} + \cdots]$$

• See Page 166-175 of Notes

Airline Model

$$\left(\frac{1-B}{1-\theta B}\right)\left(\frac{1-B^{12}}{1-\Theta B^{12}}\right)Z_t = a_t$$

$$\Rightarrow \widehat{Z_T(1)} = \frac{1-\theta}{1-\theta B}Z_T + \frac{1-\Theta}{1-\Theta B^{12}}\left[Z_{T+1-12} - \frac{1-\theta}{1-\theta B}Z_{T-12}\right]$$

2006.12

2006.11

2005.12

2005.11

$-(1-\theta)[Z_T+\theta Z_{T-1}+\theta^2 Z_{T-2}+\theta^3 Z_{T-3}$

去年的误差，if we use exponential smoothing

- See Page 166-175 of Notes

Airline Model

$$\left(\frac{1-B}{1-\theta B}\right)\left(\frac{1-B^{12}}{1-\Theta B^{12}}\right)Z_t = a_t$$

$$\Rightarrow \widehat{Z_T(1)} = \frac{1-\theta}{1-\theta B}Z_T + \frac{1-\Theta}{1-\Theta B^{12}}\left[Z_{T+1-12} - \frac{1-\theta}{1-\theta B}Z_{T-12}\right]$$

2006.12

2006.11

2005.12

2005.11

$-(1-\theta)[Z_T+\theta Z_{T-1}+\theta^2 Z_{T-2}+$

去年的误差，if we use exponential smoothing

$$\frac{1-\Theta}{1-\Theta B^{12}} = (1-\Theta)(1+\Theta B^{12}+\Theta B^{24}+\cdots)$$

- See Page 166-175 of Notes

LA OZONE DATA

$$(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

- See Page 328 of Notes

LA OZONE DATA

$$Z_t = S_t + \epsilon_t$$

- 月资料：$S_t = S_{t-12}$ (deterministic seasonality)

- See Page 328 of Notes

LA OZONE DATA

$$\begin{aligned} Z_t &= S_t + \epsilon_t \\ &\Downarrow \quad (S_t = S_{t-12}) \\ (1-B^{12})Z_t &= (1-B^{12})\epsilon_t \end{aligned}$$

• See Page 328 of Notes

LA OZONE DATA

$$\begin{aligned} Z_t &= S_t + \epsilon_t \\ &\Downarrow \quad (S_t = S_{t-12}) \\ (1-B^{12})Z_t &= (1-B^{12})\epsilon_t \end{aligned}$$

$$(1-B^{12})Z_t = (1-\Theta B^{12})a_t$$

1. $\widehat{\Theta} = 1$时应该如何处理？
2. Conditional likelihood or Exact Likelihood?

• See Page 328 of Notes

时间序列分析

第十讲(上)

Topics

1. Dynamic Models
2. Intervention Analysis
   - Ozone example
   - Crest example
   - WEPCO example
3. Outlier Detection
   - Effect on obs + residuals
   - Various type of outliers (IO, AO, LS, TC)
   - Iterative detection procedure

   Examples: 1. Crest
   2. Production
   3. Food Data

Linear Dynamic System

$$y_t = \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + \nu_j x_{t-j} + \cdots$$

- ν_j: effect of x_{t-j} on y_t
- ν_j: impulse response

• See Page 176 of Notes

Linear Dynamic System

$$y_t = \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + \nu_j x_{t-j} + \cdots$$

- ν_j: effect of x_{t-j} on y_t
- ν_j: impulse response

transfer function

$$y_t = \nu(B) x_t$$

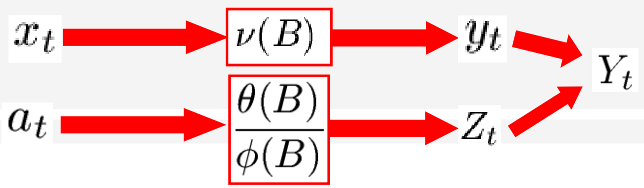
$$\begin{aligned}
\nu(B) &= \nu_0 + \nu_1 B + \nu_2 B^2 + \cdots = \frac{\omega(B)}{\delta(B)} B^b \qquad b = \text{delay} \\
\omega(B) &= \omega_0 - \omega_1 B - \cdots - \omega_r B^r \\
\delta(B) &= \delta_0 - \delta_1 B - \cdots - \delta_r B^r
\end{aligned}$$

Linear Dynamic System

$$y_t = \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + \nu_j x_{t-j} + \cdots$$

Observe :

$$\begin{aligned} Y_t &= y_t + Z_t \\ y_t &= \nu(B)x_t \qquad \{x_t\}\perp\{Z_t\} \\ Z_t &= \frac{\theta(B)}{\varphi(B)}a_t \qquad a_t \overset{i.i.d.}{\sim} N(0,\sigma_a^2) \end{aligned}$$



Linear Dynamic System

1. x_t deterministic :

$$\begin{aligned} Y_t &= \nu(B)x_t + Z_t \\ &= \underbrace{\nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + \nu_k x_{t-k}}_{\text{Regression model}} + \underbrace{Z_t}_{\text{with time series error term}} \end{aligned}$$

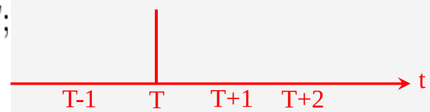
2. x_t stochastic :

Linear Dynamic System

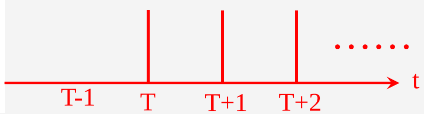
$$y_t = \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + \nu_j x_{t-j} + \cdots$$

deterministic
- indicator type model
- general

(a) $$x_t = P_t^{(T)} = \begin{cases} 1, & t = T; \\ 0, & ow. \end{cases}$$



(b) $$x_t = S_t^{(T)} = \begin{cases} 1, & t \geq T; \\ 0, & ow. \end{cases}$$



• See Page 178 of Notes

Linear Dynamic System

Input : $$x_t = S_t^{(T)} = \begin{cases} 1, & t \geq T; \\ 0, & ow. \end{cases}$$

Response : $$\begin{aligned} y_t &= \nu(B)x_t \\ &= \frac{\omega(B)}{\delta(B)}B^b \end{aligned}$$

$$y_t = \frac{\omega B}{1-\delta B}S_t^{(T)}$$

ω : constant

$\frac{\omega}{1-\delta}$: gain

• See Page 178 of Notes

Linear Dynamic System

Example : $\boxed{y_t = \frac{\omega B}{1-\delta B} S_t^{(T)}}$

First order dynamic model
very very useful!

ω : constant

$\frac{\omega}{1-\delta}$: gain

Linear Dynamic System

Example : $\boxed{y_t = \frac{\omega B}{1-\delta B} S_t^{(T)}}$

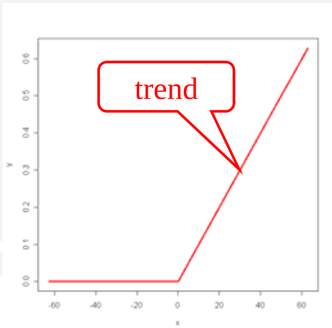
First order dynamic model
very very useful!

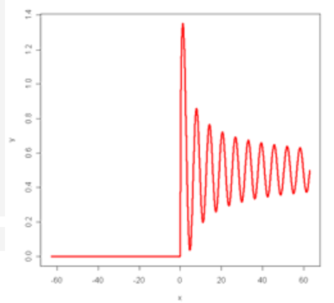
ω : constant

$\frac{\omega}{1-\delta}$: gain

If **δ**= 1 : $y_t = \frac{\omega B}{1-B} S_t^{(T)}$

$y_t = \frac{\omega B}{1-\delta_1 B - \delta_2 B^2} S_t^{(T)}$



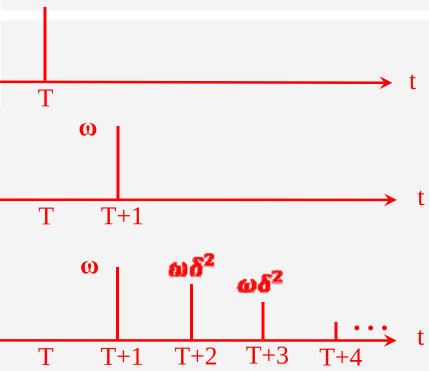


Linear Dynamic System

Input : $x_t = P_t^{(T)} = \begin{cases} 1, & t = T; \\ 0, & ow. \end{cases}$

Response : $\omega B P_t^{(T)}$

$\frac{\omega B P_t^{(T)}}{1-\delta B}$

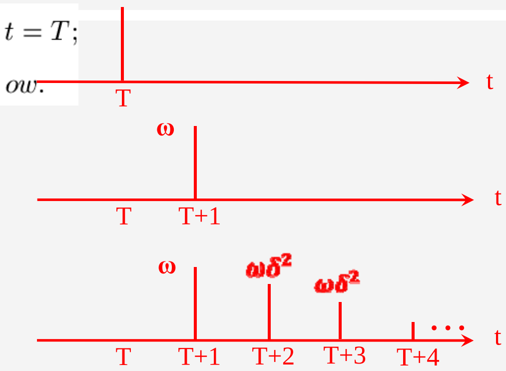


Linear Dynamic System

Input : $x_t = P_t^{(T)} = \begin{cases} 1, & t = T; \\ 0, & ow. \end{cases}$

Response : $\omega B P_t^{(T)}$

$\frac{\omega B P_t^{(T)}}{1-\delta B}$



$\left(\frac{\omega_1}{1-B} + \frac{\omega_2}{1-\delta B}\right) B P_t^{(T)}$

永久性效果　　　渐衰性效果

LA Ozone

• See Page 328 of Notes

LA Ozone

• MOZ

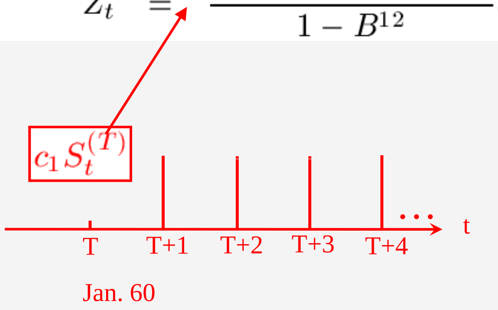
$$
\begin{aligned}
(1 - B^{12})Z_t &= (1 - \theta B)(1 - \Theta B^{12})a_t \\
Z_t &= \frac{(1 - \theta B)(1 - \Theta B^{12})}{1 - B^{12}} a_t
\end{aligned}
$$

• See Page 333 of Notes

LA Ozone

• MOZ

$$
\begin{aligned}
(1 - B^{12})Z_t &= (1 - \theta B)(1 - \Theta B^{12})a_t \\
Z_t &= \frac{(1 - \theta B)(1 - \Theta B^{12})}{1 - B^{12}} a_t
\end{aligned}
$$

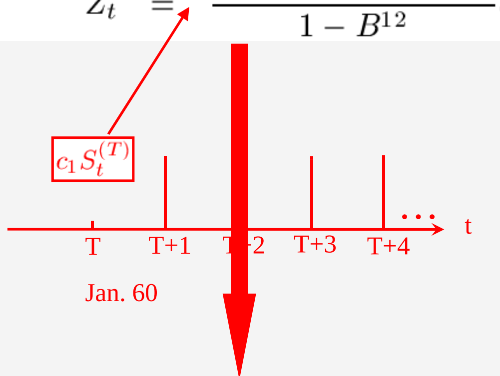


• See Page 336 of Notes

LA Ozone

• MOZ

$$
\begin{aligned}
(1 - B^{12})Z_t &= (1 - \theta B)(1 - \Theta B^{12})a_t \\
Z_t &= \frac{(1 - \theta B)(1 - \Theta B^{12})}{1 - B^{12}} a_t
\end{aligned}
$$



• MOZZ

$$
(1 - B^{12})Z_t = c_1(1 - B^{12})S_t^{(T)} + (1 - \theta B)(1 - \Theta B^{12})a_t
$$

• See Page 337 of Notes

LA Ozone

t=60时有
level shift

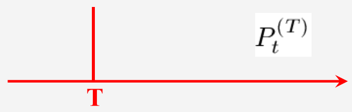
$$
\begin{aligned}
Y_t &= c_1 X_{1t} + \frac{c_2}{1-B^{12}} X_{2t} + \frac{c_3}{1-B^{12}} X_{3t} + \frac{(1-\theta B)(1-\Theta B^{12})}{1-B^{12}} a_t \\
&\Downarrow \\
(1-B^{12}) Y_t &= c_1 X_{1t}(1-B^{12}) + c_2 X_{2t} + c_3 X_{3t} + (1-\theta B)(1-\Theta B^{12}) a_t
\end{aligned}
$$

- See Page 338 of Notes

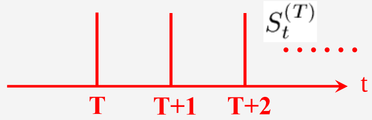
Crest

$$
\begin{aligned}
& \frac{\omega}{1-\delta B} \frac{1}{1-B} P_t^{(T)} + \frac{1-\theta B}{1-B} a_t \\
= & \frac{\omega}{1-\delta B} S_t^{(T)} + \frac{1-\theta B}{1-B} a_t
\end{aligned}
$$

$$
(a) \qquad x_t = P_t^{(T)} = \begin{cases} 1, & t = T; \\ 0, & ow. \end{cases}
$$



$$
(b) \qquad x_t = S_t^{(T)} = \begin{cases} 1, & t \geq T; \\ 0, & ow. \end{cases}
$$



$$
\boxed{(1-B) S_t^{(T)} = P_t^{(T)}}
$$

- See Page 344 of Notes

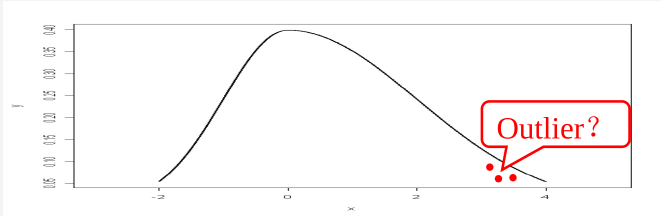
时间序列分析

第十讲（下）

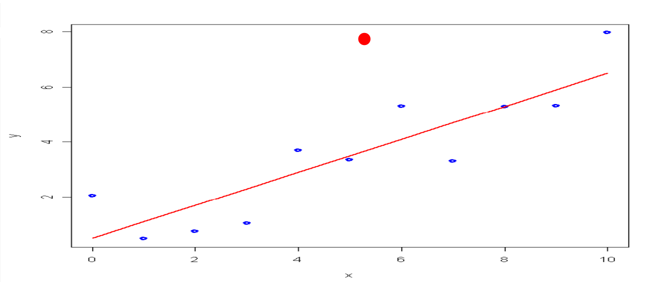
Outlier Detection

Outlier detection :

- What is outlier ? (异常值，特异值)
- outlier is relative to a model
- outlier is quite different between regression and time series



Outlier Detection



$$y_t = x_i^\top \beta + e_i + \omega P_t^{(T)}$$

In time series, AR(1) model $y_t = \phi y_{t-1} + a_t$

Outlier Detection

	Y_t特大 at $t = T$
line	$Y_T = \phi Y_{T-1} + a_T$
	Y_T $\quad T = 51 \quad$ 特大
line	$Y_{T+1} = \phi Y_T + a_{T+1}$

Outlier in Time Series is more complex !

-- Fox (1972) JRSS

• See Page 184 of Notes

Outlier Detection

4 common types of outlier : $x_t = P_t^{(T)} = \begin{cases} 1, & t = T; \\ 0, & ow. \end{cases}$

$\mathbf{Y_t}$'s obs : $y_t = \psi(B) a_t$

- Additive Outlier — **AO** $y_t = \psi(B) a_t + \omega P_t^{(b)}$
  gross error model
- Innovational Outlier — **IO** $y_t = \psi(B)[a_t + \omega P_t^{(b)}]$
  special shock at $T = b$.
  work through the dynamic system
- Level Shift — **LS** $y_t = \psi(B) a_t + \frac{\omega}{1-B} P_t^{(b)}$
  level shift at $t = b$
- Temporary Change — **TC** $y_t = \psi(B) a_t + \frac{\omega}{1-\delta B} P_t^{(b)}$
  transient change beginning at $t = b$

Outlier Detection

IO : $y_t = \psi(B)(a_t + \omega P_t^{(b)})$

Ignoring outlier, residual :

$$\begin{aligned} y_t &= \psi(B) a_t \\ \pi(B) y_t &= e_t \\ y_t &= \pi_1 y_{t-1} + \pi_2 y_{t-2} + \cdots + e_t \end{aligned}$$

Considering outlier, residual :

$$\begin{aligned} e_t &= a_t + \omega P_t^{(b)} \\ \widehat{\omega_{I,b}} &= e_b \end{aligned}$$

Standardize : $\lambda_{I,t} = e_t / \hat{\sigma}_t$

Outlier Detection

AO : $y_t = \psi(B) a_t + \omega P_t^{(b)}$

Ignoring outlier, residual :

$$\begin{aligned} e_t &= \pi(B) y_t \\ &= \psi^{-1}(B) y_t \end{aligned}$$

Considering outlier, residual :

$$\begin{aligned} e_t &= a_t + \omega \pi(B) P_t^{(b)} \\ &= a_t + \omega x_t \end{aligned}$$

• See Page 185 of Notes

Outlier Detection

$$\begin{aligned} e_t &= a_t + \omega \pi(B) P_t^{(b)} \\ &= a_t + \boxed{\omega}\,\boxed{x_t} \end{aligned}$$

$$x_t = (1 - \pi_1 B - \pi_2 B^2 - \cdots) P_t^{(b)}$$

$$-\pi B P_t^{(b)} = -\pi P_{t-1}^{(b)}$$

t	$P_t^{(b)}$	x_t
1	0	0
2	⋮	⋮
⋮	⋮	⋮
b-1	0	0
b	1	1
b+1	0	$-\pi_1$
⋮	0	$-\pi_2$
⋮	⋮	$-\pi_3$
⋮	⋮	⋮

$$\begin{aligned} \widehat{\omega_{A,b}} &= \frac{\sum x_t e_t}{\sum x_t^2} \\ &= \frac{e_b - \pi_1 e_{b+1} - \pi_2 e_{b+2} + \cdots}{1 + \pi_1^2 + \pi_2^2 + \cdots} \\ &= \frac{\pi(F) e_b}{1 + \pi_1^2 + \pi_2^2 + \cdots} \end{aligned}$$

Outlier Detection

4 common types of outlier : $x_t = P_t^{(T)} = \begin{cases} 1, & t = T; \\ 0, & ow. \end{cases}$

$\mathbf{Y_t}$'s obs : $y_t = \psi(B)a_t$

Additive Outlier — **AO** $y_t = \psi(B)a_t + \omega P_t^{(b)}$
gross error model

Innovational Outlier — **IO** $y_t = \psi(B)[a_t + \omega P_t^{(b)}]$
special shock at $T = b$.
work through the dynamic system

Level Shift — **LS** $y_t = \psi(B)a_t + \frac{\omega}{1-B}P_t^{(b)}$
level shift at $t = b$

Temporary Change — **TC** $y_t = \psi(B)a_t + \frac{\omega}{1-\delta B}P_t^{(b)}$
transient change beginning at $t = b$

Crest

IO at t=b : $y_t = \psi(B)[a_t + \omega P_t^{(b)}]$

If model is (0,1,1) :

$$\begin{aligned} \psi(B) &= \frac{1-\theta B}{1-B} \\ y_t &= \frac{1-\theta B}{1-B}[a_t + \omega P_t^{(b)}] \\ &= \frac{1-\theta B}{1-B}a_t + \frac{\omega}{1-B}(1-\theta B)P_t^{(b)} \end{aligned}$$

Note : for nonstationary series, IO and LS tends to get mixed up

• See Page 344 of Notes

Production Data

Model : $(1-B)Z_t = (1-\theta B)a_t$

Estimation and outlier :
1. Level Shift at $T = 47$
2. $\widehat{\theta} \doteq 1$

差分后 : $Z = c + a_t$

• See Page 352 of Notes

时间序列分析

第十一讲(上)

Topics

1. Food Example (regression + time series)
2. MTS Examples
3. Transfer function model (unidirectional)
4. Vector ARMA
   - Vector MA(1) properties
     - individual
     - transfer function
     - moments
5. Cross Correlations

   $\Gamma(\ell) = E(\underset{\sim}{Z_{t-\ell}}\underset{\sim}{Z_t}^{\top})$
6. Identification of MA model

Food Example

• See Page 360 of Notes

Food Example

Regression Model :

$$\begin{aligned} y_t &= \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \beta_3 x_{3t} + N_t \\ \widehat{N_t} &= y_t - \widehat{\beta_0} - \widehat{\beta_1} x_{1t} - \widehat{\beta_2} x_{2t} - \widehat{\beta_3} x_{3t} \end{aligned}$$

Analysis $\widehat{N_t}$
- individual
- transfer function
- moments

• See Page 360 of Notes

Multivariate Time Series

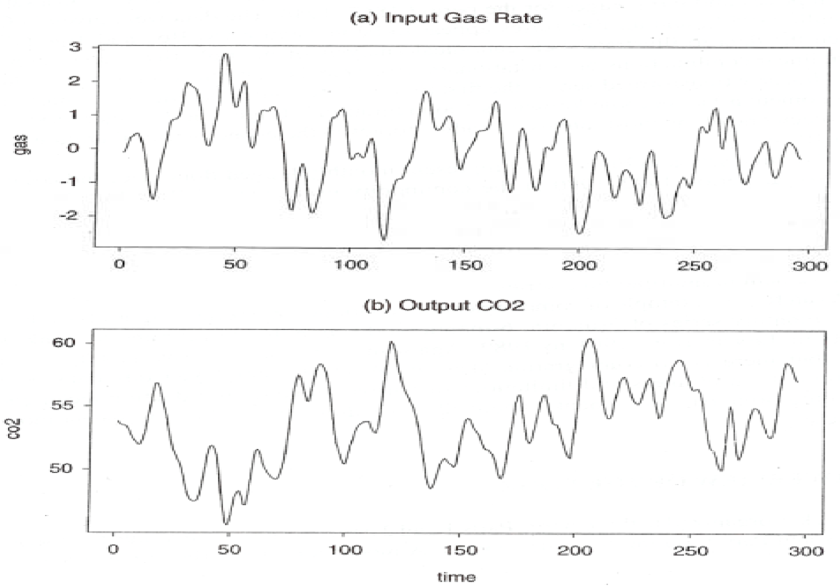


FIGURE 1.10 The gas furnace data: Box–Jenkins series J.

Multivariate Time Series

1.2. OVERVIEW OF THE BOOK 11

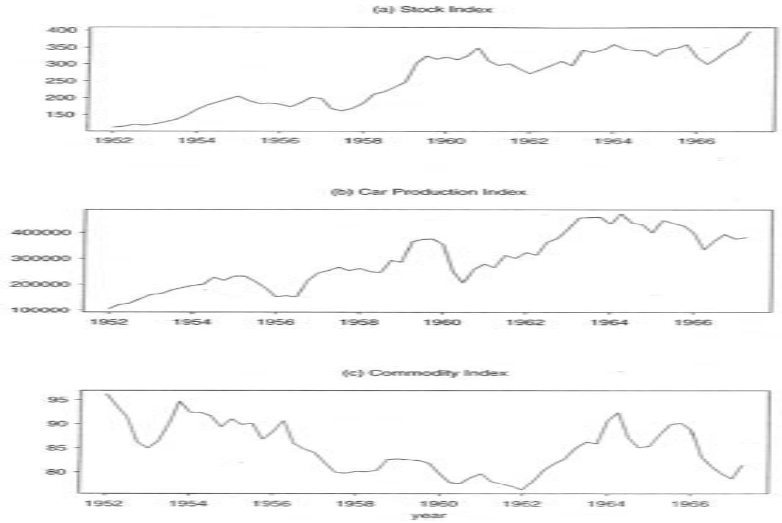


FIGURE 1.11 Quarterly *Financial Times* data 1952–1967.

Multivariate Time Series

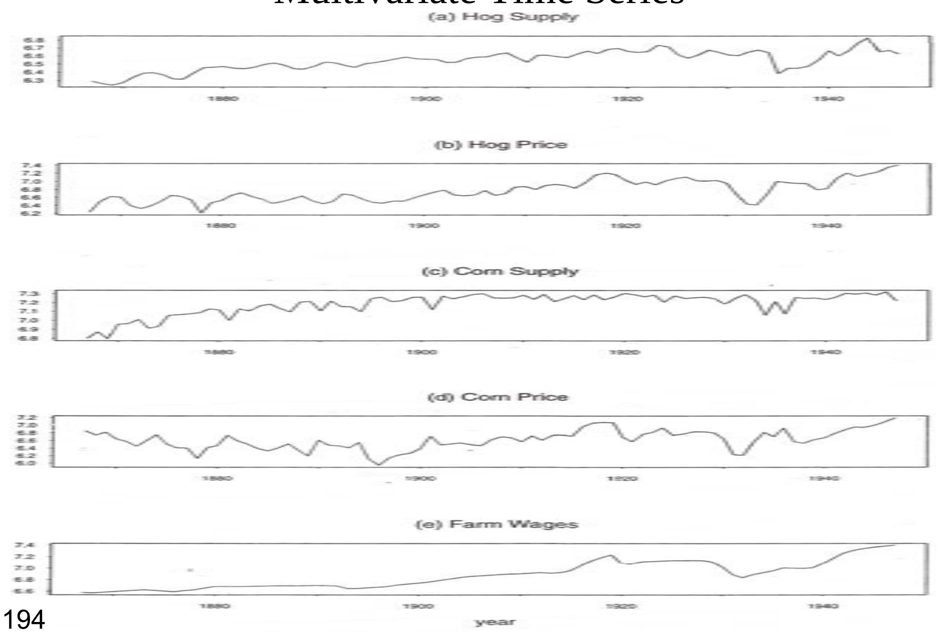


FIGURE 1.12 Quenouille's hog data 1867–1948.

Multivariate Time Series

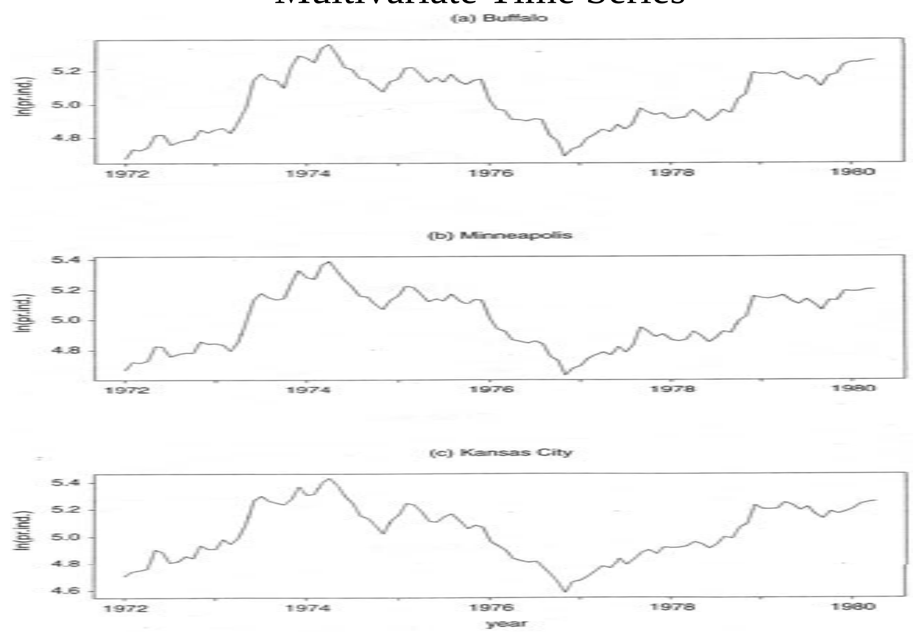


FIGURE 1.14 Logarithms of monthly flour price indices 1972–1980.

Multivariate Time Series

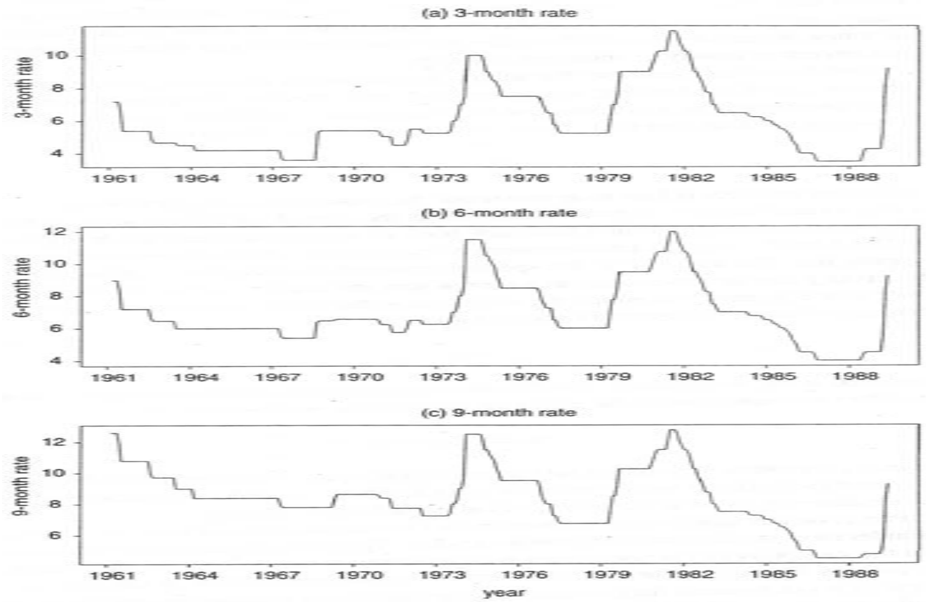


FIGURE 1.15 Taiwan's interest rate data 1961–1989.

Multivariate Time Series

Analysis several time series together

分析目的 :

- 分析序列间的关系
- 包括领先（落后）关系

如何对这些序列建模?

- Dynamic Model
- Transfer Function Modeling
- ……

时间序列分析

第十一讲(下)

Topics

1. Food Example (regression + time series)
2. MTS Examples
3. Transfer function model (unidirectional)
4. Vector ARMA
   - Vector MA(1) properties: • individual • transfer function • moments
5. Cross Correlations

   $\Gamma(\ell) = E(\underset{\sim}{Z_{t-\ell}}\underset{\sim}{Z_t}^\top)$
6. Identification of MA model

Transfer Function Model

Chapter 11 of ***BJR*** : Dynamic model

Transfer function modelling

单向 unidirectional relationship :

$$Z_{1t} \underset{\not\leftarrow}{\rightarrow} Z_{2t} \underset{\not\leftarrow}{\rightarrow} Z_{3t}$$

优点 :

- logically simple
- easy to be understood

Transfer Function Model

$$Z_{1t} = L_1(B)\alpha_{1t}$$

$$L_1(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

$$Z_{2t} = \nu_{12}(B) Z_{1t} + L_2(B)\alpha_{2t}$$

$$\nu_{12}(B) = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots) Z_{1t}$$

$$= \nu_0 Z_{1t} + \nu_1 Z_{1t-1} + \nu_2 Z_{1t-2} + \cdots$$

$$L_2(B) = (\psi_{20} + \psi_{21} B + \psi_{22} B^2 + \cdots)\alpha_{2t}$$

其中 : $\{\alpha_{1t}\} \perp \{\alpha_{2t}\}$

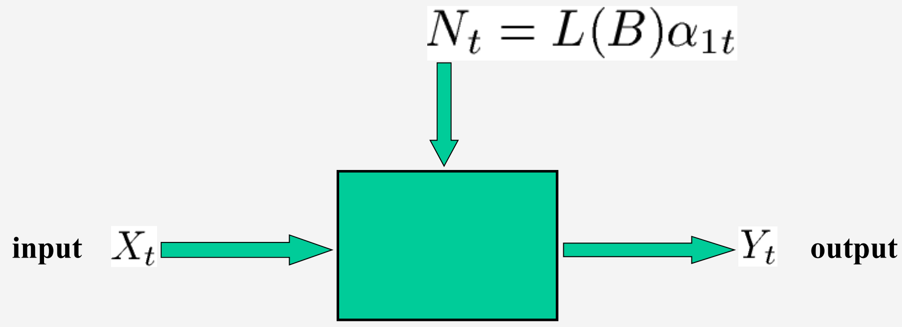
$i.i.d.$ $\quad$ $i.i.d.$

$N(0, \sigma_1^2)$ $\quad$ $N(0, \sigma_2^2)$

Transfer Function Model

$$
\begin{tikzcd}
\alpha_{1t} \arrow[d, "L_1(B)"'] & \alpha_{2t} \arrow[d, "L_2(B)"'] \\
Z_{1t} \arrow[r, "\nu_{12}(B)"] & Z_{2t}
\end{tikzcd}
$$

Transfer Function Model



$$Y_t = \nu(B)X_t + N_t$$

α_{1t}: i.i.d. series

Transfer Function Model

$$
\begin{aligned}
Y_t &= \nu(B)X_t + N_t \\
N_t &\sim L_1(B)\alpha_{1t} \\
X_t &\sim L_2(B)\alpha_{2t}
\end{aligned}
$$

$$\{\alpha_{1t}\}\perp\{\alpha_{2t}\} \Leftrightarrow \{X_t\}\perp\{N_t\}$$

Univariate ARIMA Model

Generalize to Vector ARMA Model

Vector ARMA Model

Univaraite : ARMA(p,q)

$$Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

$$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$$

Bivariate ARMA :

$$\underset{\sim}{Z_t} - \phi_1 \underset{\sim}{Z_{t-1}} - \cdots - \phi_p \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \theta_1 \underset{\sim}{a_{t-1}} - \cdots - \theta_q \underset{\sim}{a_{t-q}}$$

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad \underset{\sim}{a_t} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \overset{i.i.d.}{\sim} N_2(\underset{\sim}{0}, \Sigma)$$

$$\phi's = \begin{pmatrix} \phi_{11}^{(s)} & \phi_{12}^{(s)} \\ \phi_{21}^{(s)} & \phi_{22}^{(s)} \end{pmatrix}_{2\times 2} \qquad \theta's = \begin{pmatrix} \theta_{11}^{(s)} & \theta_{12}^{(s)} \\ \theta_{21}^{(s)} & \theta_{22}^{(s)} \end{pmatrix}_{2\times 2} \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$s = 1, \cdots, p$$

Vector ARMA Model

Z_{1t} income at time t

Z_{2t} spending at time t

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{cases} \text{Income Eqn.} & Z_{1t} = \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ \text{Spending Eqn.} & Z_{2t} = \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{cases}$$

Vector ARMA Model

Z_{1t} income at time t

Z_{2t} spending at time t

此项为0时，即为一个单向的AR(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{cases} \text{Income Eqn.} & Z_{1t} = \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ \text{Spending Eqn.} & Z_{2t} = \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{cases}$$

Vector ARMA Model

Z_{1t} income at time t

Z_{2t} spending at time t

此两项为0时，即为两个单独的AR(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{cases} \text{Income Eqn.} & Z_{1t} = \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ \text{Spending Eqn.} & Z_{2t} = \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{cases}$$

Vector ARMA Model

Z_{1t} income at time t

Z_{2t} spending at time t

$$\boxed{\underset{\sim}{Z}_t - \phi \underset{\sim}{Z}_{t-1} = \underset{\sim}{a}_t}$$

$$\Leftrightarrow \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{cases} \text{Income Eqn.} & Z_{1t} = \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ \text{Spending Eqn.} & Z_{2t} = \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{cases}$$

Vector ARMA Model

Vector ARMA :

$$\underset{\sim}{Z}_t - \phi_1 \underset{\sim}{Z}_{t-1} - \cdots - \phi_p \underset{\sim}{Z}_{t-p} = \underset{\sim}{a}_t - \theta_1 \underset{\sim}{a}_{t-1} - \cdots - \theta_q \underset{\sim}{a}_{t-q}$$

$$\underset{\sim}{Z}_t = \begin{pmatrix} Z_{1t} \\ \vdots \\ Z_{Kt} \end{pmatrix} \qquad \underset{\sim}{a}_t = \begin{pmatrix} a_{1t} \\ \vdots \\ a_{Kt} \end{pmatrix} \overset{i.i.d.}{\sim} N_K(\underset{\sim}{0}, \Sigma)$$

$$\phi' s = \begin{pmatrix} \phi_{11}^{(s)} & \cdots & \phi_{1K}^{(s)} \\ \vdots & \ddots & \vdots \\ \phi_{K1}^{(s)} & \cdots & \phi_{KK}^{(s)} \end{pmatrix}_{K\times K} \qquad \theta' s = \begin{pmatrix} \theta_{11}^{(s)} & \cdots & \theta_{1K}^{(s)} \\ \vdots & \ddots & \vdots \\ \theta_{K1}^{(s)} & \cdots & \theta_{KK}^{(s)} \end{pmatrix}_{K\times K} \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1K} \\ \vdots & \ddots & \vdots \\ \sigma_{K1} & \cdots & \sigma_{KK} \end{pmatrix}$$

$$s = 1, \cdots, p$$

Topics

1. Food Example (regression + time series)
2. MTS Examples
3. Transfer function model (unidirectional)
4. Vector ARMA
   - Vector MA(1) properties $\left\{ \begin{array}{l} \bullet \text{ individual} \\ \bullet \text{ transfer function} \\ \bullet \text{ moments} \end{array} \right.$
5. Cross Correlations

   $\Gamma(\ell) = E(\underset{\sim}{Z}_{t-\ell} \underset{\sim}{Z}_t^{\top})$
6. Identification of MA model

Vector MA(1) Model

Vector MA(1) Dimension K=2

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{a}_t \overset{i.i.d.}{\sim} N_2(\underset{\sim}{0}, \Sigma)$$

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Example :

$$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \qquad \theta = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix}$$

Vector MA(1) Model

$$Z_{1,t} = a_{1,t} - \theta_{11} a_{1,t-1} - \theta_{12} a_{2,t-1}$$

Vector MA(1) Model

$$
\begin{aligned}
Z_{1,t} &= a_{1,t} - \theta_{11} a_{1,t-1} - \theta_{12} a_{2,t-1} \\
& \quad Z_{1t} \text{ individually MA}(1), \text{ NOT MA}(2) \\
Z_{1,t-1} &= a_{1,t-1} - \theta_{11} a_{1,t-2} - \theta_{12} a_{2,t-2} \\
Z_{1,t-2} &= a_{1,t-2} - \theta_{11} a_{1,t-3} - \theta_{12} a_{2,t-3}
\end{aligned}
$$

Vector MA(1) Model

$$
\begin{aligned}
\underset{\sim}{Z_t} &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\
&= (I - \theta B) \underset{\sim}{a_t}
\end{aligned}
$$

Vector MA(1) Model

$$
\begin{aligned}
\underset{\sim}{Z_t} &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\
&= (I - \theta B) \underset{\sim}{a_t} \\
I - \theta B &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B \\
&= \begin{pmatrix} 1 - \theta_{11} B & -\theta_{12} B \\ -\theta_{21} B & 1 - \theta_{22} B \end{pmatrix}
\end{aligned}
$$

Vector MA(1) Model

$$
\begin{aligned}
\underset{\sim}{Z_t} &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\
&= (I - \theta B) \underset{\sim}{a_t} \\
I - \theta B &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B \\
&= \begin{pmatrix} 1 - \theta_{11} B & -\theta_{12} B \\ -\theta_{21} B & 1 - \theta_{22} B \end{pmatrix}
\end{aligned}
$$

Invert :

$$
\begin{aligned}
\underset{\sim}{a_t} &= (I - \theta B)^{-1} \underset{\sim}{Z_t} \\
(I - \theta B)^{-1} &= I + \theta B + \theta^2 B^2 + \cdots
\end{aligned}
$$

Vector MA(1) Model

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$
$$= (I - \theta B)\underset{\sim}{a_t}$$

$$I - \theta B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B$$

$$= \begin{pmatrix} 1 - \theta_{11} B & \\ -\theta_{21} B & 1 - \theta_{22} B \end{pmatrix}$$

All eigenvalue of **θ** lying inside unit circle

zeros of |I- **θ** B| all lying outside unit circle

Invert : $\underset{\sim}{a_t} = (I - \theta B)^{-1} \underset{\sim}{Z_t}$

$$(I - \theta B)^{-1} = I + \theta B + \theta^2 B^2 + \cdots$$

Vector MA(1) Model

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

if $\theta_{12} = 0$, then

$$Z_{1t} = (1 - \theta_{11} B) a_{1t}$$
$$Z_{2t} = (1 - \theta_{22} B) a_{2t-1} - \theta_{21} a_{1t-1}$$
$$= (1 - \theta_{22} B) a_{2t-1} - \theta_{21} \frac{Z_{1t-1}}{1 - \theta_{11} B}$$

Vector MA(1) Model

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

if $\theta_{12} = 0$, then

$$Z_{1t} = (1 - \theta_{11} B) a_{1t}$$
$$\boxed{Z_{2t}} = (1 - \theta_{22} B) a_{2t-1} - \theta_{21} a_{1t-1}$$
$$= (1 - \theta_{22} B) a_{2t-1} - \theta_{21} \frac{\boxed{Z_{1t-1}}}{1 - \theta_{11} B}$$

Unidirectional Relationship ?

Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

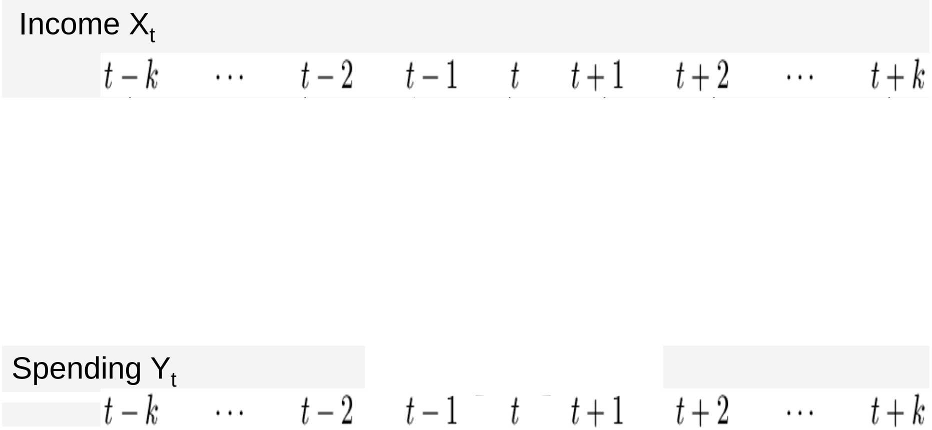
More than one series, correlations:

Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z}_t = \underset{\sim}{a}_t - \theta \underset{\sim}{a}_{t-1}$$

More than one series, correlations:

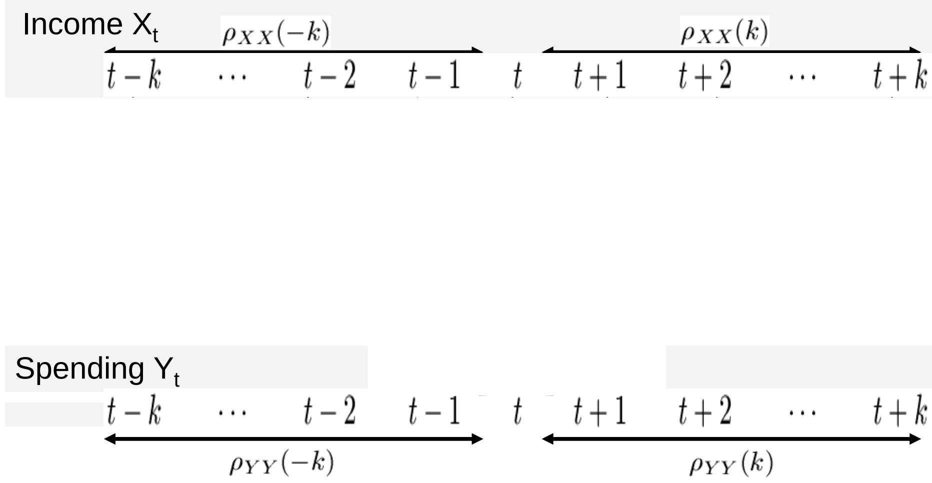


Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z}_t = \underset{\sim}{a}_t - \theta \underset{\sim}{a}_{t-1}$$

More than one series, correlations:

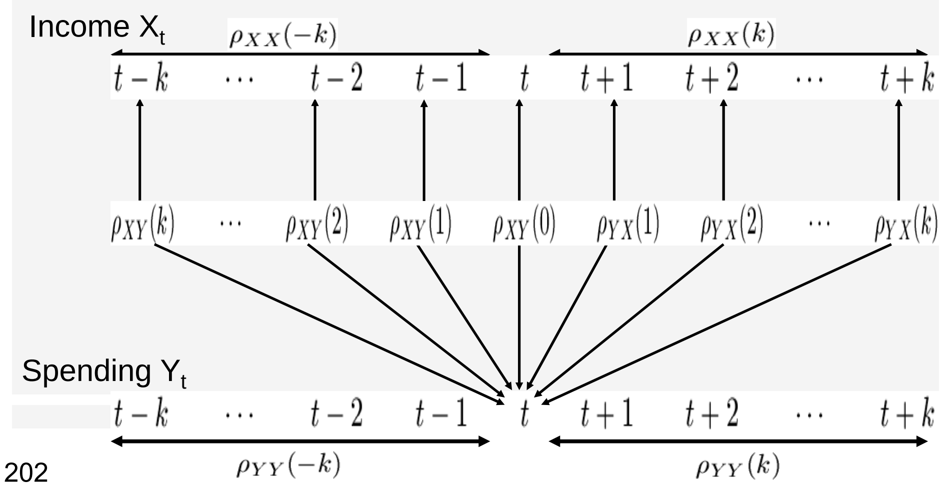


Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z}_t = \underset{\sim}{a}_t - \theta \underset{\sim}{a}_{t-1}$$

More than one series, correlations:

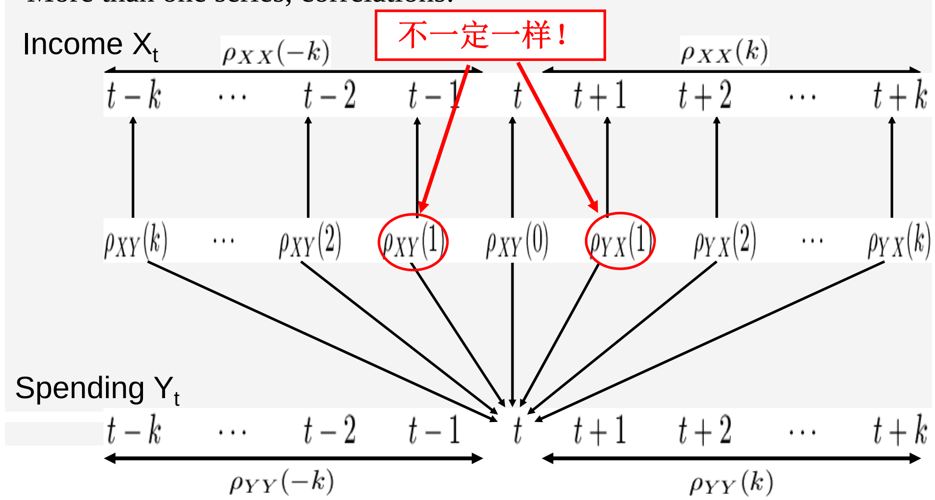


Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z}_t = \underset{\sim}{a}_t - \theta \underset{\sim}{a}_{t-1}$$

More than one series, correlations:



Vector MA(1) Model

$\underset{\sim}{\mu} = 0$

$\rho_{XY}(K) = \rho_{YX}(-K)$

Cross covariance matrix :

$$E\begin{pmatrix} Z_{1t-\ell} \\ Z_{2t-\ell} \end{pmatrix}\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} \qquad \gamma_{12}(\ell) \neq \gamma_{21}(\ell)$$

$$\begin{aligned} \Gamma(\ell) &= E(\underset{\sim}{Z_{t-\ell}}\underset{\sim}{Z_t}^\top) \\ &= \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix} \end{aligned}$$

Vector MA(1) Model

Cross Correlation Matrix of $\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$

$$\begin{aligned} \rho(\ell) &= \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \Gamma(\ell) \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{11}(\ell) & \rho_{12}(\ell) \\ \rho_{21}(\ell) & \rho_{22}(\ell) \end{pmatrix} \end{aligned}$$

$$\Gamma(0) = \begin{pmatrix} \gamma_{11}(0) & \gamma_{12}(0) \\ \gamma_{21}(0) & \gamma_{22}(0) \end{pmatrix}$$

Vector MA(1) Model

MA(1) :

$$\begin{aligned} \underset{\sim}{Z_t} &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\ \Gamma(0) &= E(\underset{\sim}{Z_t}\underset{\sim}{Z_t}^\top) \\ &= E(\underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}})(\underset{\sim}{a_t}^\top - \underset{\sim}{a_{t-1}}^\top \theta^\top) \\ &= \Sigma + \theta\Sigma\theta^\top \\ \Gamma(1) &= E(\underset{\sim}{Z_{t-1}}\underset{\sim}{Z_t}^\top) \\ &= E(\underset{\sim}{a_{t-1}} - \theta \underset{\sim}{a_{t-2}})(\underset{\sim}{a_t}^\top - \underset{\sim}{a_{t-1}}^\top \theta^\top) \\ &= -\Sigma\theta^\top \\ &\neq 0 \\ \Gamma(2) &= \Gamma(3) = \cdots = 0 \end{aligned}$$

Vector MA(1) Model

For MA(1) :

Cross Correlation Matrices :

$$\rho(1) \neq 0$$

$$\rho(2) = \rho(3) = \cdots = 0$$

时间序列分析

第十二讲(上)

Topics

1. Review of CCF

   Vector MA(1)

2. Vector AR(1)
3. Generalization of PACF
4. Identification of Vector AR + Vector MA
5. SCC Example
6. Cross Correlation *v.s.* Dynamic Relationship
7. Gas Furnace Example

Review

Vector ARMA :

$$\underset{\sim}{Z_t} - \phi_1 \underset{\sim}{Z_{t-1}} - \cdots - \phi_p \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \theta_1 \underset{\sim}{a_{t-1}} - \cdots - \theta_q \underset{\sim}{a_{t-q}}$$

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ \vdots \\ Z_{Kt} \end{pmatrix} \qquad \underset{\sim}{a_t} = \begin{pmatrix} a_{1t} \\ \vdots \\ a_{Kt} \end{pmatrix} \overset{i.i.d.}{\sim} N_K(\underset{\sim}{0}, \Sigma)$$

$$\phi's = \begin{pmatrix} \phi_{11}^{(s)} & \cdots & \phi_{1K}^{(s)} \\ \vdots & \ddots & \vdots \\ \phi_{K1}^{(s)} & \cdots & \phi_{KK}^{(s)} \end{pmatrix}_{K\times K} \qquad \theta's = \begin{pmatrix} \theta_{11}^{(s)} & \cdots & \theta_{1K}^{(s)} \\ \vdots & \ddots & \vdots \\ \theta_{K1}^{(s)} & \cdots & \theta_{KK}^{(s)} \end{pmatrix}_{K\times K} \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1K} \\ \vdots & \ddots & \vdots \\ \sigma_{K1} & \cdots & \sigma_{KK} \end{pmatrix}$$

系数太多！

Review

Simple Example :

MA(1), K=2

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{a_t} \overset{i.i.d.}{\sim} N_2(\underset{\sim}{0}, \Sigma)$$

Example：

$$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \qquad \theta = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix}$$

Review

$$
\begin{aligned}
Z_t &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\
Z_{t-1} &= \underset{\sim}{a_{t-1}} - \theta \underset{\sim}{a_{t-2}} \\
Z_{t-2} &= \underset{\sim}{a_{t-2}} - \theta \underset{\sim}{a_{t-3}}
\end{aligned}
$$

MA(1) : $\rho_{Z_1 Z_2}(\ell) = 0$

$$\underset{\sim}{\rho(\ell)} = \underset{\sim}{0} \qquad \ell \geq 2$$

$$
\begin{aligned}
\Gamma(\ell) &= E(\underset{\sim}{Z_{t-\ell}} \underset{\sim}{Z_t}^{\top}) \\
\rho(\ell) &: \text{normalized } \Gamma(\ell)
\end{aligned}
$$

Review

MA的两个特点：

(1) $\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$

$\{Z_{1t}\}$ model? MA(1) or MA(0)

$\{Z_{2t}\}$

(2) $\underset{\sim}{Z_t} = (I - \theta B)\underset{\sim}{a_t}$

Invertibility $(I - \theta B)^{-1} \underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$(I + \theta B + \theta^2 B^2 + \cdots)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$$

Review

MA的两个特点：

(1) $\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$

$\{Z_{1t}\}$ model? MA(1) or MA(0)

Invertibility : all the eigenvalue **|α|<1**

(2) $\underset{\sim}{Z_t} = (I - \theta B)\underset{\sim}{a_t}$

$$
\begin{aligned}
(I - \theta B)^{-1} \underset{\sim}{Z_t} &= \underset{\sim}{a_t} \\
(I + \theta B + \theta^2 B^2 + \cdots)\underset{\sim}{Z_t} &= \underset{\sim}{a_t}
\end{aligned}
$$

Review

$\underset{\sim}{Z_t} = (I - \theta B)\underset{\sim}{a_t} \qquad (I - \theta B)^{-1} \underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$
\begin{aligned}
I - \theta B &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B \\
&= \begin{pmatrix} 1 - \theta_{11} B & -\theta_{12} B \\ -\theta_{21} B & 1 - \theta_{22} B \end{pmatrix} \\
(I - \theta B)^{-1} &= \frac{1}{|I - \theta B|} \begin{pmatrix} 1 - \theta_{22} B & \theta_{12} B \\ \theta_{21} B & 1 - \theta_{11} B \end{pmatrix} \\
|I - \theta B| &= (1 - \theta_{11} B)(1 - \theta_{22} B) - \theta_{12}\theta_{21} B^2
\end{aligned}
$$

Review

$$(I-\theta B)^{-1}\underset{\sim}{Z} = \frac{1}{(1-\theta_{11}B)(1-\theta_{22}B)-\theta_{12}\theta_{21}B^2}\begin{pmatrix} 1-\theta_{22}B & \theta_{12}B \\ \theta_{21}B & 1-\theta_{11}B \end{pmatrix}\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$$

Review

$$(I-\theta B)^{-1}\underset{\sim}{Z} = \frac{1}{(1-\theta_{11}B)(1-\theta_{22}B)-\theta_{12}\theta_{21}B^2}\begin{pmatrix} 1-\theta_{22}B & \theta_{12}B \\ \theta_{21}B & 1-\theta_{11}B \end{pmatrix}\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$$

$$\boxed{(1-\alpha_1 B)(1-\alpha_2 B)}$$

Invertibility = zeros of $|I-\theta B|$ all lying outside unit cycle

Vector AR(1)

K=2 : $(I-\phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

Z_1: income

Z_2: consumption

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} B\right]\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}\begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$Z_{1t} = \phi_{11}Z_{1t-1} + \phi_{12}Z_{2t-1} + a_{1t}$$
$$Z_{2t} = \phi_{21}Z_{1t-1} + \phi_{22}Z_{2t-1} + a_{2t}$$

Vector AR(1)

K=2 : $(I-\phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

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$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}\begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

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$$Z_{2t} = \phi_{21}Z_{1t-1} + \phi_{22}Z_{2t-1} + a_{2t}$$

Vector AR(1)

K=2 :

$$(I-\phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$$

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

Z_1: income

Z_2: consumption

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} B\right] \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{aligned} Z_{1t} &= \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ Z_{2t} &= \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{aligned}$$

Vector AR(1)

Example :

$$\phi = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

Vector AR(1)

$$(I-\phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$$

$$\begin{aligned} \underset{\sim}{Z_t} &= (I-\phi B)^{-1}\underset{\sim}{a_t} \\ &= (I+\phi B+\phi^2B^2+\phi^3B^3+\cdots)\underset{\sim}{a_t} \\ &= \underset{\sim}{a_t} + \phi\underset{\sim}{a_{t-1}} + \phi^2\underset{\sim}{a_{t-2}} + \phi^3\underset{\sim}{a_{t-3}} + \cdots \end{aligned}$$

Vector AR(1)

$$(I-\phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$$

$$\begin{aligned} \underset{\sim}{Z_t} &= (I-\phi B)^{-1}\underset{\sim}{a_t} \\ &= (I+\phi B+\phi^2B^2+\phi^3B^3+\cdots)\underset{\sim}{a_t} \\ &= \underset{\sim}{a_t} + \phi\underset{\sim}{a_{t-1}} + \phi^2\underset{\sim}{a_{t-2}} + \phi^3\underset{\sim}{a_{t-3}} + \cdots \end{aligned}$$

eigen values of ϕ, $|\alpha_i| < 1$, then it will converge

Vector AR(1)

$(I - \phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$\begin{aligned} \underset{\sim}{Z_t} &= (I - \phi B)^{-1}\underset{\sim}{a_t} \\ &= (I + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)\underset{\sim}{a_t} \\ &= \underset{\sim}{a_t} + \phi \underset{\sim}{a_{t-1}} + \phi^2 \underset{\sim}{a_{t-2}} + \phi^3 \underset{\sim}{a_{t-3}} + \cdots \end{aligned}$$

eigen values of ϕ, $|\alpha_i| < 1$, then it will converge

$$\begin{aligned} \Gamma(0) &= \mathrm{Cov}(\underset{\sim}{Z_t}\underset{\sim}{Z_t}^\top) \\ &= \Sigma + \phi\Sigma\phi^\top + \phi^2\Sigma(\phi^\top)^2 + \phi^3\Sigma(\phi^\top)^3 + \cdots \end{aligned}$$

Vector AR(1)

$\Gamma(0) = \Sigma + \phi\Gamma(0)\phi^\top$

If Σ, ϕ known, linear in elements of $\Gamma(0)$, we may get $\Gamma(0)$.

Vector AR(1)

$$\begin{aligned} (I - \phi B)^{-1} &= \frac{1}{|I - \phi B|}\begin{pmatrix} 1 - \phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1 - \phi_{11}B \end{pmatrix} \\ &= \Gamma(0)(I - \theta B)^{-1} \end{aligned}$$

Zero of lying outside u.c.

Vector AR(1)

Bivariate AR(1)

$$\begin{pmatrix} 1 - \phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1 - \phi_{22}B \end{pmatrix}\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

What is the model for $\{Z_{1t}\}$?

Vector AR(1)

Bivariate AR(1)

$$\begin{pmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{pmatrix} \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

What is the model for $\{Z_{1t}\}$?

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{pmatrix}^{-1} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

Vector AR(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \frac{1}{|I-\phi B|} \begin{pmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{pmatrix} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$(1-\phi_{11}B)(1-\phi_{22}B)-\phi_{12}\phi_{21}B^2 = \varphi(B)$$

$$\underbrace{\varphi(B) \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}}_{\text{AR(2)}} = \underbrace{\begin{pmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{pmatrix} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}}_{\text{MA(1)}}$$

Vector AR(1)

In principle, each can be ARMA(2,1)

How to generalize?

Review : SPACF for univariate case ?

Vector AR(1)

$$\left.\begin{array}{lcl} \text{AR}(1) & & \widehat{\phi} \\ \text{AR}(2) & \neq & \widehat{\phi}_2 \\ \text{AR}(3) & \neq & \widehat{\phi}_3 \\ \text{AR}(4) & & \widehat{\phi}_4 \\ \vdots & & \end{array}\right\}$$

SPACF

Last coefficient estimated!

Vector AR(1)

$$\begin{aligned} y &= \beta x + \epsilon \\ \widehat{\beta} &= (x^\top x)^{-1} x^\top y \end{aligned}$$

Vector AR(1)

$$\begin{aligned} y &= \beta x + \epsilon \\ \widehat{\beta} &= (x^\top x)^{-1} x^\top y \end{aligned}$$

AR(1) $Z_t = \phi Z_{t-1} + a_t \qquad t = 1, \cdots, T$

$$\widehat{\phi} = \frac{\sum_{t=2}^{T} Z_t Z_{t-1}}{\sum_{t=2}^{T} Z_t^2}$$

Vector AR(1)

Vector :

$$\begin{aligned} \underset{\sim}{Z_t} &= \phi \underset{\sim}{Z_{t-1}} + \underset{\sim}{a_t} \\ \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} &= \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \qquad t = 1, \cdots, T \\ \underset{\sim}{Z_t}^\top &= \underset{\sim}{Z_{t-1}}^\top \phi^\top + \underset{\sim}{a_t}^\top \\ \begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} &= \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix} \end{aligned}$$

$$\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array} \begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

Vector AR(1)

$$\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \underset{\sim}{y_1} & \underset{\sim}{y_2} \end{pmatrix} = X \begin{pmatrix} \underset{\sim}{\phi_1} & \underset{\sim}{\phi_2} \end{pmatrix} + \begin{pmatrix} \underset{\sim}{a_1} & \underset{\sim}{a_2} \end{pmatrix} \Rightarrow \begin{cases} \underset{\sim}{y_1} = X \underset{\sim}{\phi_1} + \underset{\sim}{a_1} \\ \underset{\sim}{y_2} = X \underset{\sim}{\phi_2} + \underset{\sim}{a_2} \end{cases}}$$

$$\begin{aligned} Y &= X\theta + \epsilon \\ \widehat{\theta} &= (X^\top X)^{-1} X^\top Y \end{aligned}$$

时间序列分析

第十二讲(下)

Vector AR(1)

$$\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \underset{\sim}{y_1} & \underset{\sim}{y_2} \end{pmatrix} = X \begin{pmatrix} \underset{\sim}{\phi_1} & \underset{\sim}{\phi_2} \end{pmatrix} + \begin{pmatrix} \underset{\sim}{a_1} & \underset{\sim}{a_2} \end{pmatrix} \Rightarrow \begin{cases} \underset{\sim}{y_1} = X\underset{\sim}{\phi_1} + \underset{\sim}{a_1} \\ \underset{\sim}{y_2} = X\underset{\sim}{\phi_2} + \underset{\sim}{a_2} \end{cases}}$$

$$\begin{aligned} Y &= X\theta + \epsilon \\ \widehat{\theta} &= (X^\top X)^{-1} X^\top Y \end{aligned}$$

Vector AR(1)

$$\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

$$\begin{aligned} Y &= X\theta + \epsilon \\ \widehat{\theta} &= (X^\top X)^{-1} X^\top Y \end{aligned}$$

fitted $\widehat{Y} = X\widehat{\theta}$

Residual $Y - \widehat{Y}$

Bartlett 1938

LR Test

Test $\underset{\sim}{\theta} = \underset{\sim}{0} \quad v.s. \quad \underset{\sim}{\theta} \neq \underset{\sim}{0}$

Bartlett Test

Regression : $S(0) = Y^\top Y$

residual $(Y - \widehat{Y})^\top (Y - \widehat{Y}) = S(1)$

$$U_1 = \frac{|S(1)|}{|S(0)|}$$

Bartlett Test : $M(1) = -(N - .5 - K)\log(U_1) \sim \chi^2(4)$

test: $\theta = 0 \ v.s. \ \theta \neq 0$

Ratio of Determinant $\quad k=2 \Leftrightarrow k^2=4$

M(1) : corresponds to 1st Partial in Univariate Case

Bartlett Test

AR(2)

$$\underset{\sim}{Z_t} = \phi_1 \underset{\sim}{Z_{t-1}} + \phi_2 \underset{\sim}{Z_{t-2}} + \underset{\sim}{a_t}$$

$$\underset{\sim}{Z_t}^\top = \underset{\sim}{Z_{t-1}}^\top \phi_1^\top + \underset{\sim}{Z_{t-2}}^\top \phi_2^\top + \underset{\sim}{a_t}^\top$$

$$\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} = \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \phi_1^\top + \begin{pmatrix} Z_{1t-2} & Z_{2t-2} \end{pmatrix} \phi_2^\top + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}$$

$t = 3, \cdots, T$

$$Y = X_1\theta_1 + X_2\theta_2 + \underset{\sim}{\epsilon}$$

Bartlett Test

Test $\phi_2^\top = \theta_2 = 0$

$$\widehat{Y_2} = X_1\widehat{\theta_1} + X_2\widehat{\theta_2}$$

$$X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$$

$$S(2) = (Y - \widehat{Y_2})^\top (Y - \widehat{Y_2})$$

$$\frac{|S(2)|}{|S(1)|} = U_2$$

Bartlett Test : $M(2) = -(N - .5 - 2K)\log(U_2) \sim \chi^2(4)$

Ratio of Determinant : k=2 ⇔ k^2=4

Example

Generate AR(1) $\underset{\sim}{Z_t} - \phi \underset{\sim}{Z_{t-1}} = \underset{\sim}{a_t}$

$$\phi = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

Generate MA(1) $\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$

$$\theta = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

Example

The Financial Times Data :

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{pmatrix} = \begin{pmatrix} \text{stock} \\ \text{car} \\ \text{commodity} \end{pmatrix}$$

T = 15, -8

R^2=88%

Example

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ Z_{3t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \\ a_{3t} \end{pmatrix}$$

Example

$$y_t = \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t$$

Assume $\{X_t\}$, $\{N_t\}$ independent

Cross Correlation 可以用么？

时间序列分析

第十三讲(上)

Today's Topic

Ⅰ. **Review**

ARIMA model building

Seasonal model multiplicative/non-multiplicative

Intervention Analysis + Outlier

$$\begin{aligned} Y_t &= y_t + N_t \\ y_t &= \nu(B)x_t = \frac{\omega(B)}{\delta(B)}B^b x_t \\ N_t &= ARIMA \\ \{N_t\} &\perp \{x_t\} \end{aligned}$$

Outlier: IO, AO, LS, TC

Iterative procedure

Ⅱ **Examples**

airline data, ozone data, crest, WEPCO, Food Example

Ⅲ **Vector Series**

Review - 1: ARIMA Model

ARIMA(p,d,q) $\Phi_p(B)(1-B)^d Z_t = c + \Theta_q(B)a_t$

3 stages

- **Tentative Specification**
  - **SACF**
  - **SPACF**
  - **SEACF**
- **Estimation** — **MLE**
- **Diagnostic checking**

Review - 2: Seasonal Models

$$\varphi_p(B^{12})(1-B^{12})^{d_1} Z_t = \Theta_q(B^{12})e_t$$

Review - 2: Seasonal Models

$$\varphi_p(B^{12})(1-B^{12})^{d_1}Z_t = \Theta_q(B^{12})\boxed{e_t}$$

$$\boxed{\phi_p(B)(1-B)^d e_t = \theta_q(B)a_t}$$

Review - 2: Seasonal Models

$$\boxed{\varphi_p(B^{12})(1-B^{12})^{d_1}Z_t = \Theta_q(B^{12})e_t}$$

+

$$\boxed{\phi_p(B)(1-B)^d e_t = \theta_q(B)a_t}$$

||

Multiplicative Model

Review - 3: Intervention Analysis

$$\begin{aligned} Y_t &= y_t + N_t \\ \text{Observed Time Series} &= \text{intervention} + \text{ARIMA} \end{aligned}$$

$$y_t = \nu(B)X_t = \frac{\omega(B)}{\delta(B)}B^b X_t$$

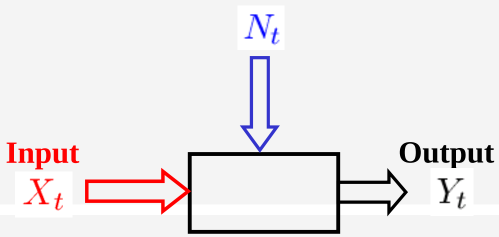
X_t: exogenous variable

$$X_t \text{ could be } \begin{cases} \text{deterministic indicator} \\ \quad * \ \textit{indicator type} \begin{cases} \textit{step change} & S_t^{(T)} \\ \textit{pulse change} & P_t^{(T)} \end{cases} \\ \\ \text{stochastic} \end{cases}$$

Review - 3: Intervention Analysis

$$\begin{aligned} Y_t &= y_t + N_t \\ \text{Observed Time Series} &= \text{intervention} + \text{ARIMA} \end{aligned}$$

$$y_t = \nu(B)X_t = \frac{\omega(B)}{\delta(B)}B^b X_t$$



Review - 4: Outlier

No-outlier situation: $Y_t = \psi(B)a_t$

Review - 4: Outlier

No-outlier situation: $Y_t = \psi(B)a_t$

Outlier Type:

IO: $Y_t = \psi(B)(a_t + \omega P_t^{(T)})$

Review - 4: Outlier

No-outlier situation: $Y_t = \psi(B)a_t$

Outlier Type:

IO: $Y_t = \psi(B)(a_t + \omega P_t^{(T)})$

AO: $Y_t = \psi(B)a_t + \omega P_t^{(T)}$

t = T时，Y_t 特大（小）

gross error model

Review - 4: Outlier

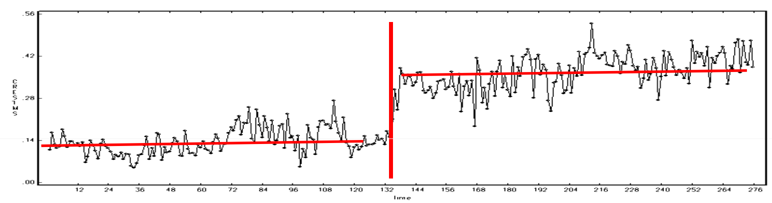
No-outlier situation: $Y_t = \psi(B)a_t$

Outlier Type:

IO: $Y_t = \psi(B)(a_t + \omega P_t^{(T)})$

AO: $Y_t = \psi(B)a_t + \omega P_t^{(T)}$

LS: $Y_t = \psi(B)a_t + \dfrac{\omega}{1-B} P_t^{(T)}$



Review - 4: Outlier

No-outlier situation: $Y_t = \psi(B)a_t$

Outlier Type:

$$\textbf{IO}: \quad Y_t = \psi(B)(a_t + \omega P_t^{(T)})$$

$$\textbf{AO}: \quad Y_t = \psi(B)a_t + \omega P_t^{(T)}$$

$$\textbf{LS}: \quad Y_t = \psi(B)a_t + \frac{\omega}{1-B}P_t^{(T)}$$

$$\textbf{TC}: \quad Y_t = \psi(B)a_t + \frac{\omega}{1-\delta B}P_t^{(T)} \qquad |\delta| < 1$$

Examples

Examples

- **Airline Model (from Page 315)**

Examples

- **Airline Model (from Page 315)**

$$\underbrace{(1-B)(1-B^{12})Z_t}_{w_t} = (1-\theta B)(1-\Theta_{12}B^{12})a_t$$

$$\rho_1 \neq 0$$

$$\rho_{12} \neq 0$$

$$\rho_{11} = \rho_{13} = \rho_1 \times \rho_{12}$$

Examples

- **Airline Model (from Page 315)**
- **Ozone Data (from Page 328)**

Examples

- **Airline Model (from Page 315)**
- **Ozone Data (from Page 328)**

$$\boxed{(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t}$$

Is it OK?

Examples

- **Airline Model (from Page 315)**
- **Ozone Data (from Page 328)**

$$\boxed{(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t}$$

Is it OK?

Intervention model:

$$\begin{aligned} Z_t &= c_1 x_{1t} + \frac{(1-\theta_1 B)(1-\Theta_{12}B^{12})}{1-B^{12}} a_t \\ (1-B^{12})Z_t &= c_1(1-B^{12})x_{1t} + (1-\theta_1 B)(1-\Theta_{12}B^{12})a_t \end{aligned}$$

Examples

- **Airline Model (from Page 315)**
- **Ozone Data (from Page 328)**

Summer trend and winter trend model:

$$\begin{aligned} Z_t &= c_1 X_{1t} + \frac{c_2}{1-B^{12}} X_{2t} + \frac{c_3}{1-B^{12}} X_{3t} \\ &+ \frac{(1-\theta_1 B)(1-\theta_{12}B^{12})}{1-B^{12}} a_t \end{aligned}$$

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**

Summer trend and winter trend model:

$$\begin{aligned} Z_t &= \boxed{c_1 X_{1t}} + \frac{c_2}{1-B^{12}} X_{2t} + \frac{c_3}{1-B^{12}} X_{3t} \\ &+ \boxed{\frac{(1-\theta_1 B)(1-\theta_{12} B^{12})}{1-B^{12}} a_t} \end{aligned}$$

as before

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**

Summer trend and winter trend model:

$$\begin{aligned} Z_t &= c_1 X_{1t} + \frac{c_2}{1-B^{12}} X_{2t} + \frac{c_3}{1-B^{12}} X_{3t} \\ &+ \frac{(1-\theta_1 B)(1-\theta_{12} B^{12})}{1-B^{12}} a_t \end{aligned}$$

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**

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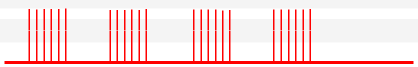
$$X_{2t} = \begin{cases} 1, & \text{Jun-Oct, Since 66;} \\ 0, & \text{otherwise.} \end{cases}$$



Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**

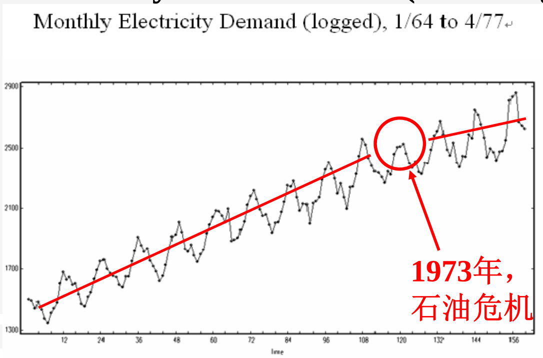
Summer trend and winter trend model:

$$\begin{aligned} Z_t &= c_1 X_{1t} + \overbrace{\boxed{\frac{c_2}{1-B^{12}} X_{2t}}}^{\textbf{summer trend}} + \overbrace{\boxed{\frac{c_3}{1-B^{12}} X_{3t}}}^{\textbf{non-summer trend}} \\ &+ \frac{(1-\theta_1 B)(1-\theta_{12} B^{12})}{1-B^{12}} a_t \end{aligned}$$

$$X_{2t} = \begin{cases} 1, & \text{Jun-Oct, Since 66;} \\ 0, & \text{otherwise.} \end{cases}$$

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**



Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**

$$Z_t = \frac{\omega_0}{1-B^{12}}\left(1-\frac{\omega_1}{1-\delta B^{12}}\right)X_t + \frac{(1-\theta B)(1-\Theta_{12}B^{12})}{1-B^{12}}a_t$$

$$X_t = \begin{cases} 1, & t \geq 13; \\ 0, & t < 13. \end{cases}$$

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**

$$Z_t = \frac{\omega_0}{1-B^{12}}\left(1-\frac{\omega_1}{1-\delta B^{12}}\right)X_t + \frac{(1-\theta B)(1-\Theta_{12}B^{12})}{1-B^{12}}a_t$$

$$X_t = \begin{cases} 1, & t \geq 13; \\ 0, & t < 13. \end{cases}$$

如此部分改为 $\frac{\omega_0}{1-B^{12}}X_t$，
则为一个 constant growth rate

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**

$$Z_t = \underbrace{\frac{\omega_0}{1-B^{12}}\overbrace{\left(1-\frac{\omega_1}{1-\delta B^{12}}\right)}^{\text{adjustment，最多变到 } 1-\frac{\omega_1}{1-\delta}}X_t}_{} + \frac{(1-\theta B)(1-\Theta_{12}B^{12})}{1-B^{12}}a_t$$

$$X_t = \begin{cases} 1, & t \geq 13; \\ 0, & t < 13. \end{cases}$$

如此部分改为 $\frac{\omega_0}{1-B^{12}}X_t$，
则为一个 constant growth rate

Examples

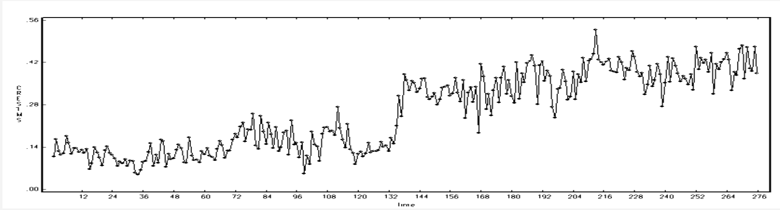
- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**

$$Z_t = \frac{\omega_0}{1-B^{12}}\left(1 + \frac{\omega_1}{1-\delta B^{12}}\right)X_t + \frac{(1-\theta B)(1-\Theta_{12}B^{12})}{1-B^{12}}a_t$$

$$X_t = \begin{cases} 1, & t \geq 13; \\ 0, & t < 13. \end{cases}$$

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**
- **Crest Market Share** **(from Page 344)**



Relationship between IO and LS

IO: $\quad Y_t = \psi(B)(a_t + \omega P_t^{(T)})$

LS: $\quad Y_t = \psi(B)a_t + \frac{\omega}{1-B}P_t^{(T)}$

if $\psi(B) = \frac{1-\theta B}{1-B}$, then

$$\begin{aligned} Y_t &= \left(\frac{1-\theta B}{1-B}\right)(a_t + \omega P_t^{(T)}) \\ &= \frac{1-\theta B}{1-B}a_t + \omega\frac{1-\theta B}{1-B}P_t^{(T)} \end{aligned}$$

Relationship between IO and LS

IO: $\quad Y_t = \psi(B)(a_t + \omega P_t^{(T)})$

LS: $\quad Y_t = \psi(B)a_t + \frac{\omega}{1-B}P_t^{(T)}$

if $\psi(B) = \frac{1-\theta B}{1-B}$, then

$$\begin{aligned} Y_t &= \left(\frac{1-\theta B}{1-B}\right)(a_t + \omega P_t^{(T)}) \\ &= \frac{1-\theta B}{1-B}a_t + \omega\frac{1-\theta B}{1-B}P_t^{(T)} \end{aligned}$$

ATTENETION: **IO and LS might be overlapped or mixed up**

时间序列分析

第十三讲(下)

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**
- **Crest Market Share** **(from Page 344)**
- **The Production Data** **(from Page 352)**

Examples

The Production Data **(from Page 352)**

$$(1-B)Z_t = \omega P_t^{(T)} + (1-\theta B)a_t$$

Examples

The Production Data **(from Page 352)**

$$\begin{aligned}(1-B)Z_t &= \omega P_t^{(T)} + (1-\theta B)a_t \\ &\Downarrow \qquad (\widehat{\theta} = 1.000) \\ (1-B)Z_t &= \omega P_t^{(T)} + (1-B)a_t\end{aligned}$$

Examples

The Production Data **(from Page 352)**

$$
\begin{aligned}
(1-B)Z_t &= \omega P_t^{(T)} + (1-\theta B)a_t \\
&\Downarrow \qquad (\widehat{\theta} = 1.000) \\
(1-B)Z_t &= \omega P_t^{(T)} + (1-B)a_t \\
&\Downarrow \qquad \text{两边除以 } (1-B) \\
Z_t &= \frac{\omega}{1-B}P_t^{(T)} + a_t
\end{aligned}
$$

But where is the constant ?

Examples

The Production Data **(from Page 352)**

$$
\begin{aligned}
(1-B)Z_t &= \omega P_t^{(T)} + (1-\theta B)a_t \\
&\Downarrow \qquad (\widehat{\theta} = 1.000) \\
(1-B)Z_t &= \omega P_t^{(T)} + (1-B)a_t \\
&\Downarrow \qquad \text{两边除以 } (1-B) \\
Z_t &= \cancel{\frac{\omega}{1-B}P_t^{(T)} + a_t}
\end{aligned}
$$

But where is the constant ?

Here is the constant

Actually: $Z_t = \mathbf{c} + \frac{\omega}{1-B}P_t^{(T)} + a_t$

Examples

The Production Data **(from Page 352)**

$$
\begin{aligned}
Z_t &= \mu + a_t \\
&\Downarrow \quad \text{差分} \\
(1-B)Z_t &= (1-B)\mu + (1-B)a_t \\
&= 0 + a_t - a_{t-1}
\end{aligned}
$$

Examples

- **Airline Model** **(from Page 315)**
- **Ozone Data** **(from Page 328)**
- **Monthly Electricity Demand** **(from Page 181)**
- **Crest Market Share** **(from Page 344)**
- **The Production Data** **(from Page 352)**
- **Weekly Sales of a Food Product** **(from Page 360)**

Review

1. 方法部分
2. 案例 (13个)
   - **3 generated : MA(1), AR(1), ARIMA(0,1,1)**
   - **BJR Series : F, C, A, Airline Data**
   - **2 Economic : Interest Rate, Treasury Bill, GNP**
   - **Ozone, Crest, Prod, Food**

Multivariate Time Series

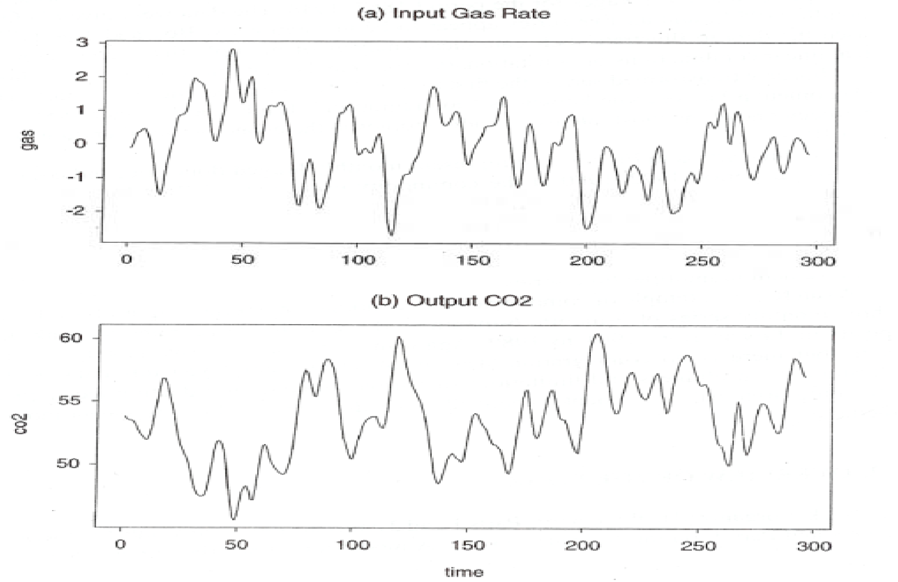
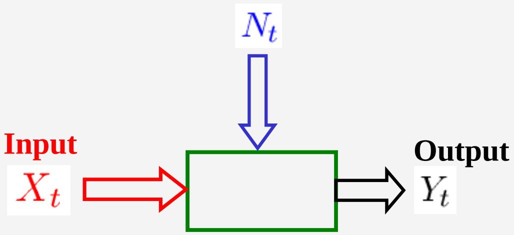


FIGURE 1.10 The gas furnace data: Box–Jenkins series J.

Multivariate Time Series

Gas Furnace Data



$$Y_t = \nu(B) X_t + N_t$$

Multivariate Time Series

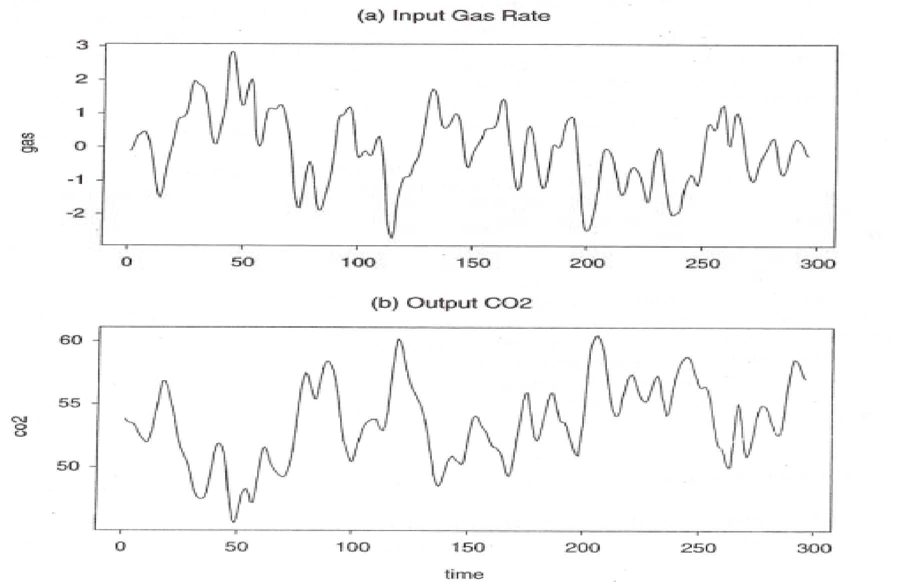


FIGURE 1.10 The gas furnace data: Box–Jenkins series J.

Multivariate Time Series

Quarterly data (SCC)

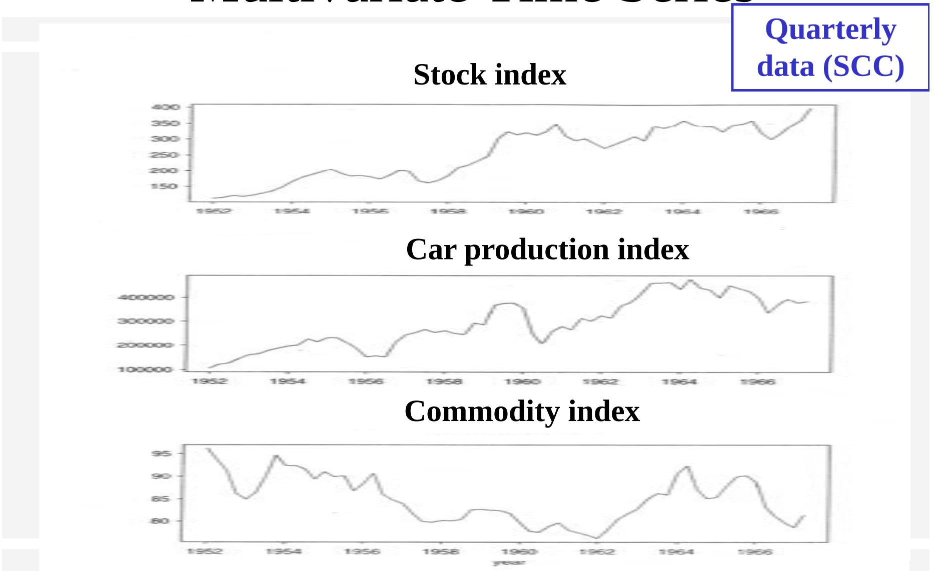


FIGURE 1.11 Quarterly Financial Times data 1952–1967.

Multivariate Time Series

Quenouille

Yearly data

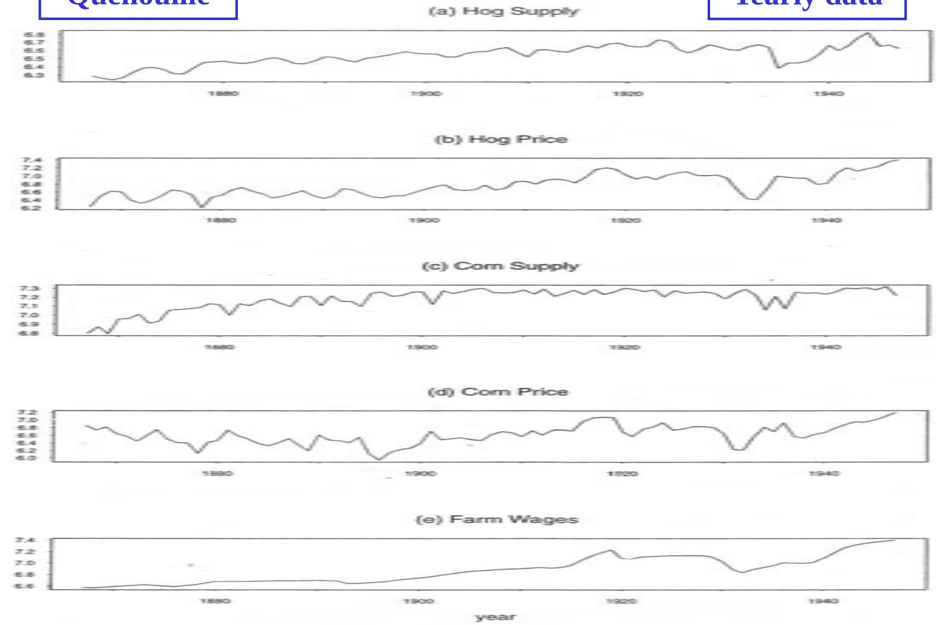


FIGURE 1.12 Quenouille's hog data 1867–1948.

Multivariate Time Series

Monthly flour data

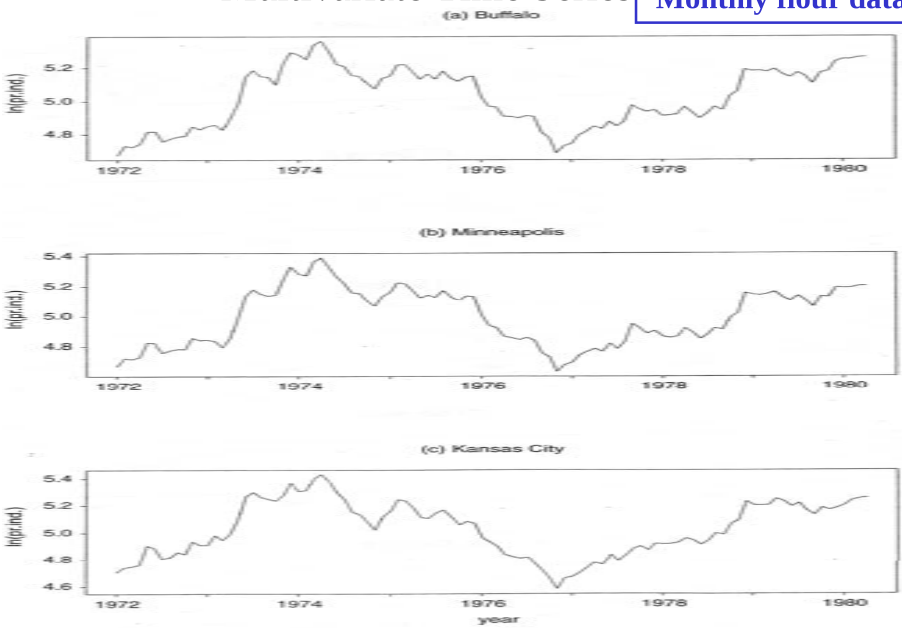


FIGURE 1.14 Logarithms of monthly flour price indices 1972–1980.

Multivariate Time Series

Monthly data

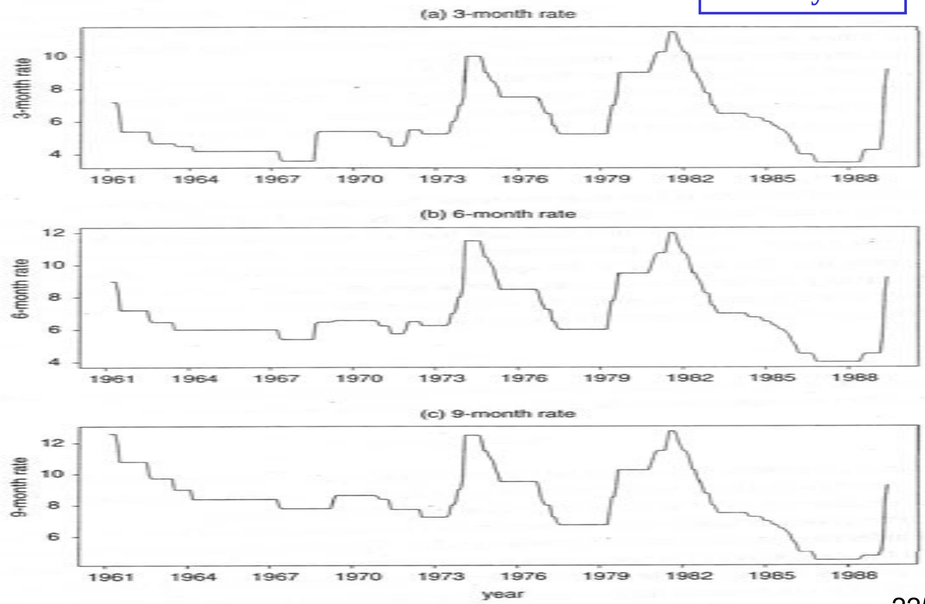


FIGURE 1.15 Taiwan's interest rate data 1961–1989.

Multivariate Time Series

Analysis several time series together

分析目的：

- 分析序列间的关系
- 包括领先（落后）关系

如何对这些序列建模？

- **Dynamic Model**
- **Transfer Function Modeling**
- ……

Topics

1. MTS Examples
2. Transfer function model (unidirectional)
3. Vector ARMA
   - Vector MA(1) properties $\begin{cases} \text{• individual} \\ \text{• transfer function} \\ \text{• moments} \end{cases}$
4. Cross Correlations

   $\Gamma(\ell) = E(\underset{\sim}{Z}_{t-\ell}\underset{\sim}{Z}_t^\top)$
5. Identification of MA model

Transfer Function Model

Chapter 11 of ***BJR*** : Dynamic model

Transfer function modeling

单向 unidirectional relationship :

$$Z_{1t} \underset{\not\leftarrow}{\rightarrow} Z_{2t} \underset{\not\leftarrow}{\rightarrow} Z_{3t}$$

优点 :

- logically simple
- easy to be understood

Transfer Function Model (unidirectional)

$$\begin{cases} Z_{1t} = L_1(B)\alpha_{1t} \\ \quad L_1(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots \\ Z_{2t} = \nu_{12}(B)Z_{1t} + L_2(B)\alpha_{2t} \\ \quad \nu_{12}(B) = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots)Z_{1t} \\ \quad\quad = \nu_0 Z_{1t} + \nu_1 Z_{1t-1} + \nu_2 Z_{1t-2} + \cdots \\ \quad L_2(B) = (\psi_{20} + \psi_{21} B + \psi_{22} B^2 + \cdots)\alpha_{2t} \end{cases}$$

其中 : $\{\alpha_{1t}\} \perp \{\alpha_{2t}\}$

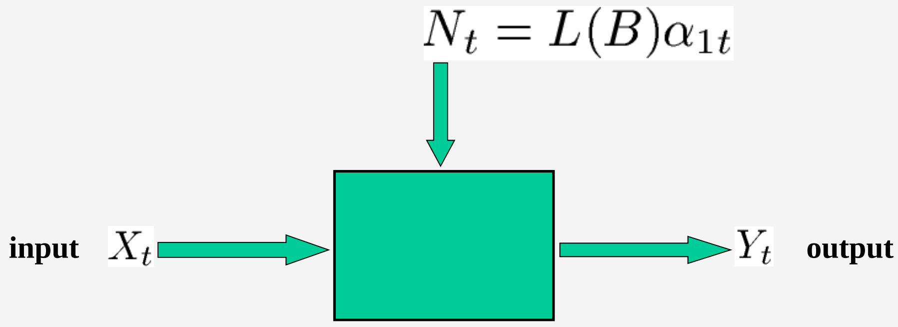
$i.i.d. \qquad i.i.d.$

$N(0, \sigma_1^2) \qquad N(0, \sigma_2^2)$

Transfer Function Model

$$\begin{tikzcd}
\alpha_{1t} \arrow[r, phantom, "\perp"] \arrow[d, "L_1(B)"'] & \alpha_{2t} \arrow[d, "L_2(B)"'] \\
Z_{1t} \arrow[r, "\nu_{12}(B)"] & Z_{2t}
\end{tikzcd}$$

Transfer Function Model



$$Y_t = \nu(B)X_t + N_t$$

α_{1t}: i.i.d. series

Transfer Function Model

$$\begin{aligned}
Y_t &= \nu(B)X_t + N_t \\
N_t &\sim L_1(B)\alpha_{1t} \\
X_t &\sim L_2(B)\alpha_{2t}
\end{aligned}$$

$$\{\alpha_{1t}\}\perp\{\alpha_{2t}\} \Leftrightarrow \{X_t\}\perp\{N_t\}$$

Univariate ARIMA Model

Generalize to Vector ARMA Model

时间序列分析

第十四讲(上)

Today's Topic

- Multivariate Time Series
- Transfer Function Model (unidirectional)
- Vector ARMA
- MA(1)

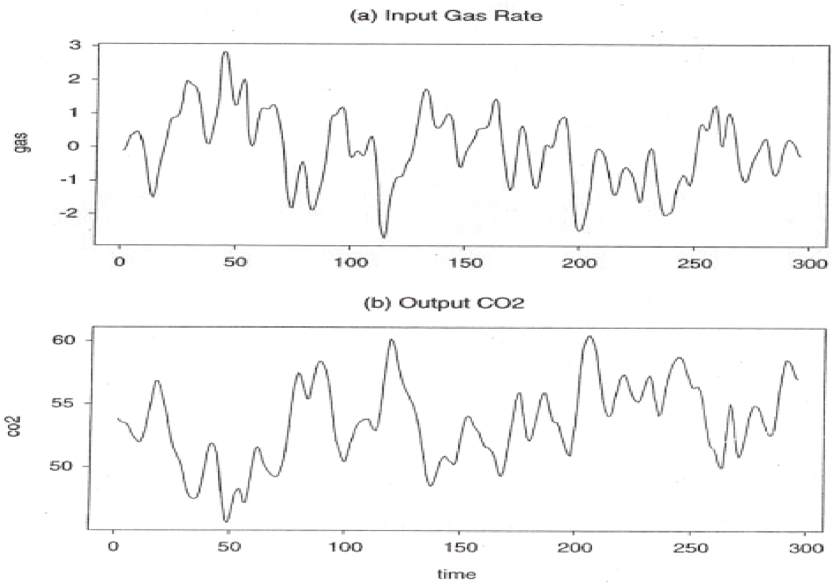
  Specification of MA cross corr

  Order tentative specification

  $$\begin{bmatrix} + & \cdot \\ \cdot & - \end{bmatrix}$$
- AR(1)

  Specification of AR Partial Autoregression
- SCC (Cross Correlation and Dynamic Relationship)
- Gas Furnace Data

Multivariate Time Series



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Multivariate Time Series

Quarterly data (SCC)

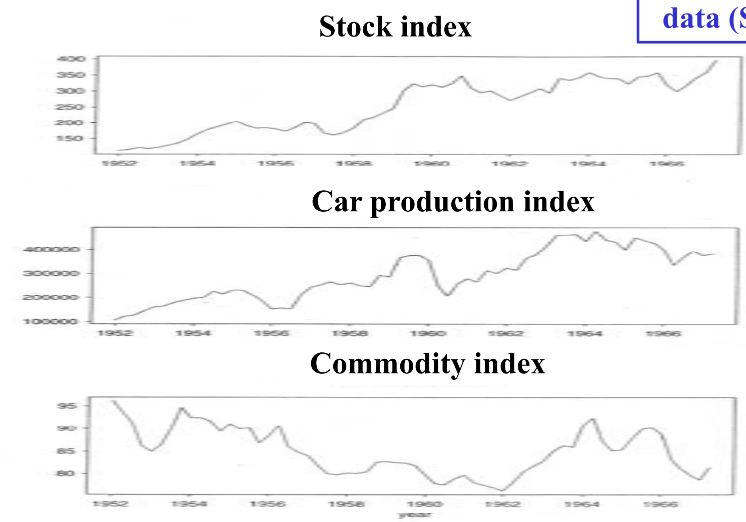


FIGURE 1.11 Quarterly Financial Times data 1952–1967.

Multivariate Time Series

Quenouille

Yearly data

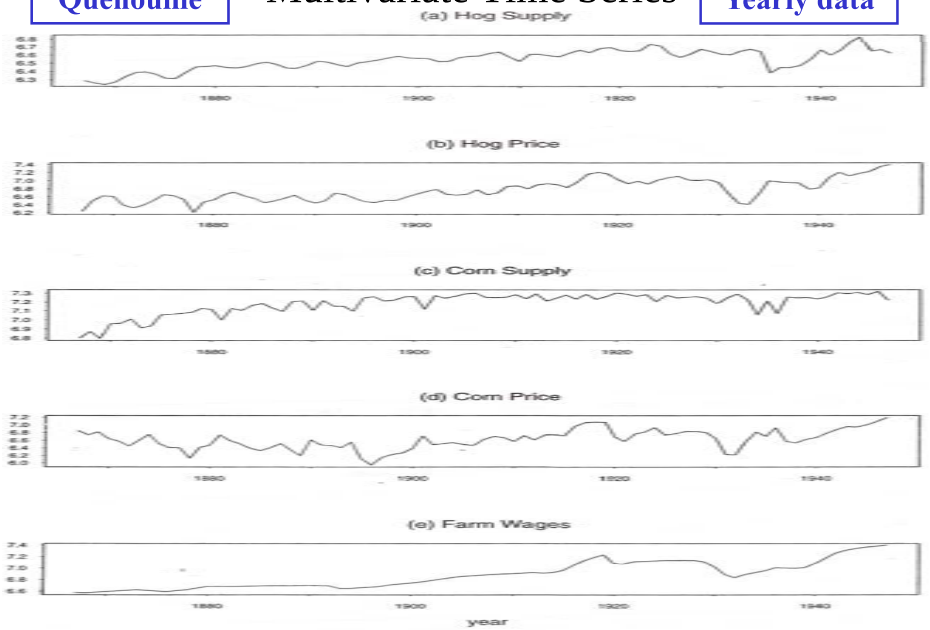


FIGURE 1.12 Quenouille's hog data 1867–1948.

Multivariate Time Series

Monthly flour data

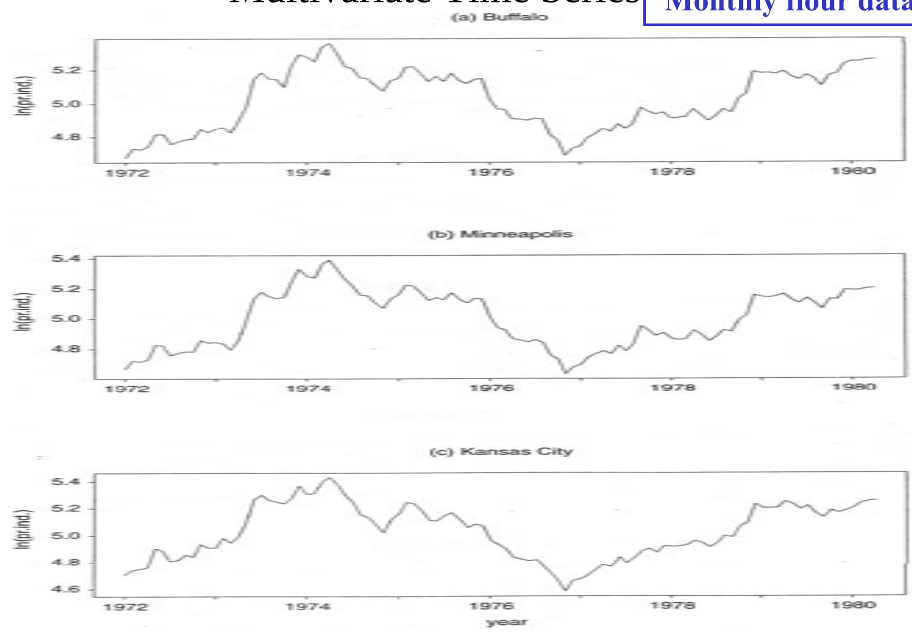
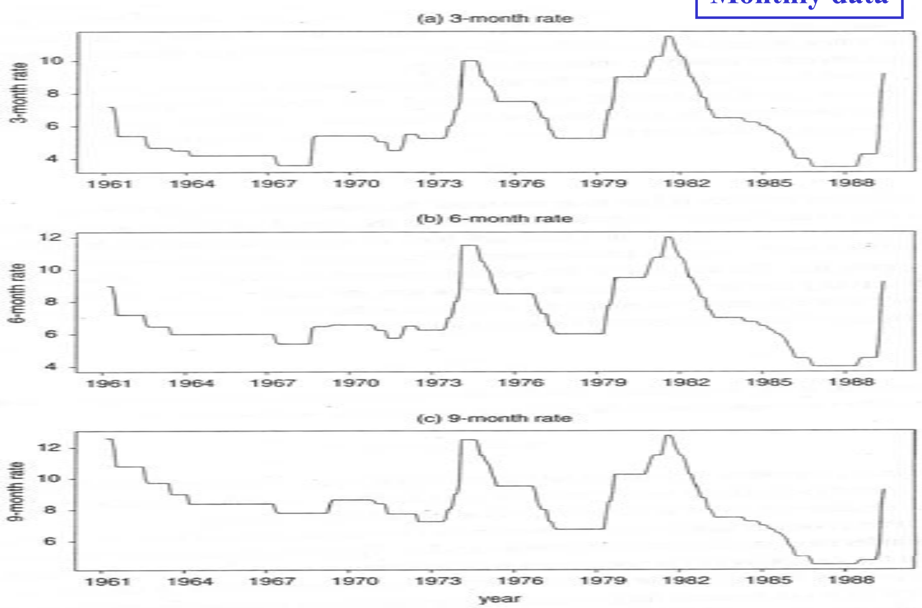


FIGURE 1.14 Logarithms of monthly flour price indices 1972–1980.

Multivariate Time Series

Monthly data



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- **Transfer Function Modeling**
- ……

Transfer Function Model

Chapter 11 of ***BJR*** : Dynamic model

Transfer function modeling

单向 unidirectional relationship :

$$Z_{1t} \underset{\not\leftarrow}{\overset{\rightarrow}{}} Z_{2t} \underset{\not\leftarrow}{\overset{\rightarrow}{}} Z_{3t}$$

优点 :

- logically simple
- easy to be understood

Transfer Function Model (unidirectional)

$$\begin{cases} Z_{1t} = L_1(B)\alpha_{1t} & \text{(input)} \\ \quad L_1(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots \\ Z_{2t} = \nu_{12}(B) Z_{1t} + L_2(B)\alpha_{2t} & \text{(output)} \end{cases}$$

$$\nu_{12}(B) = (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots) Z_{1t}$$

$$= \nu_0 Z_{1t} + \nu_1 Z_{1t-1} + \nu_2 Z_{1t-2} + \cdots$$

$$L_2(B) = (\psi_{20} + \psi_{21} B + \psi_{22} B^2 + \cdots)\alpha_{2t}$$

其中 : $\{\alpha_{1t}\} \perp \{\alpha_{2t}\}$

$i.i.d.$ $\quad i.i.d.$

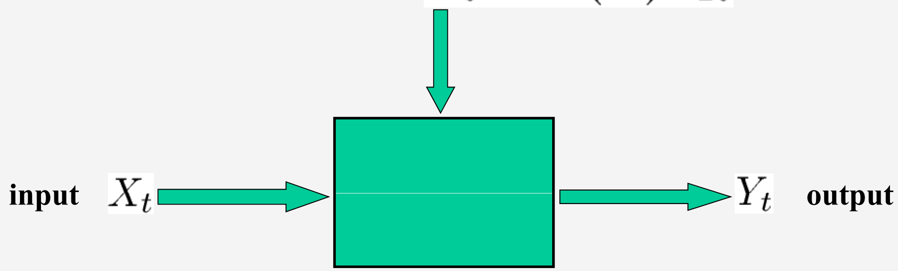
$N(0, \sigma_1^2)$ $\quad N(0, \sigma_2^2)$

Transfer Function Model

$$\begin{tikzcd} \alpha_{1t} \arrow[d, "L_1(B)"'] \arrow[r, phantom, "\perp"] & \alpha_{2t} \arrow[d, "L_2(B)"'] \\ Z_{1t} \arrow[r, "\nu_{12}(B)"] & Z_{2t} \arrow[l, bend left, "\times"] \end{tikzcd}$$

Transfer Function Model

$$N_t = L(B)\alpha_{1t}$$



$$Y_t = \nu(B) X_t + N_t$$

α_{1t}: i.i.d. series

Transfer Function Model

$$\begin{aligned} Y_t &= \nu(B)X_t + N_t \\ N_t &\sim L_1(B)\alpha_{1t} \\ X_t &\sim L_2(B)\alpha_{2t} \end{aligned}$$

$$\{\alpha_{1t}\}\perp\{\alpha_{2t}\} \Leftrightarrow \{X_t\}\perp\{N_t\}$$

Univariate ARIMA Model

Generalize to Vector ARMA Model

Transfer Function Model

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Transfer function modeling

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优点 :

- logically simple
- easy to be understood

Vector ARMA Model

Univaraite : ARMA(p,q)

$$Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

$$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$$

Bivariate ARMA :

$$\underset{\sim}{Z_t} - \phi_1 \underset{\sim}{Z_{t-1}} - \cdots - \phi_p \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \theta_1 \underset{\sim}{a_{t-1}} - \cdots - \theta_q \underset{\sim}{a_{t-q}}$$

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad \underset{\sim}{a_t} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \overset{i.i.d.}{\sim} N_2(\underset{\sim}{0}, \Sigma)$$

$$\phi' s = \begin{pmatrix} \phi_{11}^{(s)} & \phi_{12}^{(s)} \\ \phi_{21}^{(s)} & \phi_{22}^{(s)} \end{pmatrix}_{2\times 2} \qquad \theta' s^* = \begin{pmatrix} \theta_{11}^{(s^*)} & \theta_{12}^{(s^*)} \\ \theta_{21}^{(s^*)} & \theta_{22}^{(s^*)} \end{pmatrix}_{2\times 2} \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$s = 1, \cdots, p$ $\qquad$ $s^* = 1, \cdots, q$

Vector ARMA Model

Univaraite : ARMA(p,q)

$$Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

$$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$$

Bivariate ARMA :

$$\underset{\sim}{Z_t} - \phi_1 \underset{\sim}{Z_{t-1}} - \cdots - \phi_p \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \theta_1 \underset{\sim}{a_{t-1}} - \cdots - \theta_q \underset{\sim}{a_{t-q}}$$

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$s = 1, \cdots, p$ $\qquad$ $s^* = 1, \cdots, q$

Vector ARMA Model

Z_{1t} income at time t

Z_{2t} spending at time t

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{cases} \text{Income Eqn.} & Z_{1t} = \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ \text{Spending Eqn.} & Z_{2t} = \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{cases}$$

Vector ARMA Model

Univaraite : ARMA(p,q)

$$Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

$$a_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)$$

Bivariate ARMA :

$$\underset{\sim}{Z_t} - \phi_1 \underset{\sim}{Z_{t-1}} - \cdots - \phi_p \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \theta_1 \underset{\sim}{a_{t-1}} - \cdots - \theta_q \underset{\sim}{a_{t-q}}$$

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad \underset{\sim}{a_t} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \overset{i.i.d.}{\sim} N_2(\underset{\sim}{0}, \Sigma)$$

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$s = 1, \cdots, p$ $\qquad$ $s^* = 1, \cdots, q$

Vector ARMA Model

Z_{1t} income at time t

Z_{2t} spending at time t

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

此项为0时，即为一个单向的AR(1)

$$\begin{cases} \text{Income Eqn.} & Z_{1t} = \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ \text{Spending Eqn.} & Z_{2t} = \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{cases}$$

Vector ARMA Model

Z_{1t} income at time t

Z_{2t} spending at time t

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

此两项为0时，即为两个单独的AR(1)，但是不独立，因为noise项可能相关。

$$\begin{cases} \text{Income Eqn.} & Z_{1t} = \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ \text{Spending Eqn.} & Z_{2t} = \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{cases}$$

Vector ARMA Model

Vector ARMA :

$$\underset{\sim}{Z}_t - \phi_1 \underset{\sim}{Z}_{t-1} - \cdots - \phi_p \underset{\sim}{Z}_{t-p} = \underset{\sim}{a}_t - \theta_1 \underset{\sim}{a}_{t-1} - \cdots - \theta_q \underset{\sim}{a}_{t-q}$$

$$\underset{\sim}{Z}_t = \begin{pmatrix} Z_{1t} \\ \vdots \\ Z_{Kt} \end{pmatrix} \qquad \underset{\sim}{a}_t = \begin{pmatrix} a_{1t} \\ \vdots \\ a_{Kt} \end{pmatrix} \overset{i.i.d.}{\sim} N_K(\underset{\sim}{0}, \Sigma)$$

$$\phi' s = \begin{pmatrix} \phi_{11}^{(s)} & \cdots & \phi_{1K}^{(s)} \\ \vdots & \ddots & \vdots \\ \phi_{K1}^{(s)} & \cdots & \phi_{KK}^{(s)} \end{pmatrix}_{K\times K} \qquad \theta' s^* = \begin{pmatrix} \theta_{11}^{(s^*)} & \cdots & \theta_{1K}^{(s^*)} \\ \vdots & \ddots & \vdots \\ \theta_{K1}^{(s^*)} & \cdots & \theta_{2K}^{(s^*)} \end{pmatrix}_{K\times K} \qquad \Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1K} \\ \vdots & \ddots & \vdots \\ \sigma_{K1} & \cdots & \sigma_{KK} \end{pmatrix}$$

$$s = 1, \cdots, p \qquad s^* = 1, \cdots, q$$

Vector MA(1) Model

Vector MA(1) Dimension K=2

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{a}_t \overset{i.i.d.}{\sim} N_2(\underset{\sim}{0}, \Sigma)$$

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Example :

$$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \qquad \theta = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix}$$

Vector MA(1) Model

Vector MA(1) Dimension K=2

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

2x2 vector of shocks at time t

2x2 vector of shocks at time t-1

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Example :

$$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \qquad \theta = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix}$$

Vector MA(1) Model

Vector MA(1) $\boxed{Z_t = (1 - \theta B) a_t \Longrightarrow \underset{\sim}{Z}_t = (I - \Theta B) \underset{\sim}{a}_t}$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{a}_t \overset{i.i.d.}{\sim} N_2(\underset{\sim}{0}, \Sigma)$$

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Vector MA(1) Model

Vector MA(1) Dimension K=2

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

Example: See Page 5 of Chapter 14

$$\underset{\sim}{Z_t} = \underset{\sim}{\mu} + \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

$$\underset{\sim}{\mu} = \begin{pmatrix} 17 \\ 25 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \qquad \theta = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix}$$

Vector MA(1) Model

$$Z_{1,t} = a_{1,t} - \theta_{11} a_{1,t-1} - \theta_{12} a_{2,t-1}$$

Z_{1t} individually MA(1), NOT MA(2)

Vector MA(1) Model

$$Z_{1,t} = a_{1,t} - \theta_{11} a_{1,t-1} - \theta_{12} a_{2,t-1}$$

Z_{1t} individually MA(1), NOT MA(2)

$$Z_{1,t-1} = a_{1,t-1} - \theta_{11} a_{1,t-2} - \theta_{12} a_{2,t-2}$$

$$Z_{1,t-2} = a_{1,t-2} - \theta_{11} a_{1,t-3} - \theta_{12} a_{2,t-3}$$

Vector MA(1) Model

$$\begin{aligned} \underset{\sim}{Z_t} &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\ &= (I - \theta B) \underset{\sim}{a_t} \end{aligned}$$

Vector MA(1) Model

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$
$$= (I - \theta B)\underset{\sim}{a_t}$$
$$I - \theta B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B$$
$$= \begin{pmatrix} 1 - \theta_{11}B & -\theta_{12}B \\ -\theta_{21}B & 1 - \theta_{22}B \end{pmatrix}$$

Vector MA(1) Model

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$
$$= (I - \theta B)\underset{\sim}{a_t}$$
$$I - \theta B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B$$
$$= \begin{pmatrix} 1 - \theta_{11}B & -\theta_{12}B \\ -\theta_{21}B & 1 - \theta_{22}B \end{pmatrix}$$
$$\text{Invertibility : } \underset{\sim}{a_t} = (I - \theta B)^{-1}\underset{\sim}{Z_t}$$
$$(I - \theta B)^{-1} = I + \theta B + \theta^2 B^2 + \cdots$$

Vector MA(1) Model

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$
$$= (I - \theta B)\underset{\sim}{a_t}$$
$$I - \theta B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B$$
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$$(I - \theta B)^{-1} = I + \theta B + \theta^2 B^2 + \cdots$$

Vector MA(1) Model

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$
$$= (I - \theta B)\underset{\sim}{a_t}$$
$$(I - \theta B)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B$$
$$= \begin{pmatrix} 1 - \theta_{11}B & -\theta_{12}B \\ -\theta_{21}B & 1 - \theta_{22}B \end{pmatrix}^{-1}$$
$$\text{Invertibility : } \underset{\sim}{a_t} = (I - \theta B)^{-1}\underset{\sim}{Z_t}$$
$$(I - \theta B)^{-1} = I + \theta B + \theta^2 B^2 + \cdots$$

Vector MA(1) Model

$$\begin{pmatrix} 1-\theta_{11}B & -\theta_{12}B \\ -\theta_{12}B & 1-\theta_{22}B \end{pmatrix}^{-1} = \frac{1}{|I-\theta B|}\begin{pmatrix} 1-\theta_{22}B & \theta_{12}B \\ \theta_{21}B & 1-\theta_{11}B \end{pmatrix}$$

Vector MA(1) Model

$$\begin{pmatrix} 1-\theta_{11}B & -\theta_{12}B \\ -\theta_{12}B & 1-\theta_{22}B \end{pmatrix}^{-1} = \frac{1}{|I-\theta B|}\begin{pmatrix} 1-\theta_{22}B & \theta_{12}B \\ \theta_{21}B & 1-\theta_{11}B \end{pmatrix}$$

$$\begin{aligned} |I-\theta B| &= (1-\theta_{11}B)(1-\theta_{22}B)-\theta_{12}\theta_{21}B^2 \\ &= (1-\alpha_1 B)(1-\alpha_2 B) \end{aligned}$$

Vector MA(1) Model

$$\begin{pmatrix} 1-\theta_{11}B & -\theta_{12}B \\ -\theta_{12}B & 1-\theta_{22}B \end{pmatrix}^{-1} = \frac{1}{|I-\theta B|}\begin{pmatrix} 1-\theta_{22}B & \theta_{12}B \\ \theta_{21}B & 1-\theta_{11}B \end{pmatrix}$$

$$\begin{aligned} |I-\theta B| &= (1-\theta_{11}B)(1-\theta_{22}B)-\theta_{12}\theta_{21}B^2 \\ &= (1-\alpha_1 B)(1-\alpha_2 B) \\ & \quad |\alpha_i|<1 \quad i=1,2 \end{aligned}$$

Vector MA(1) Model

$$\begin{aligned} \underset{\sim}{Z_t} &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\ &= (I-\theta B)\underset{\sim}{a_t} \end{aligned}$$

$$I-\theta B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} B$$

All eigenvalue of **θ**lying inside unit circle

zeros of |I- **θ** B| all lying outside unit circle

Invertible : $\underset{\sim}{a_t} = (I-\theta B)^{-1}\underset{\sim}{Z_t}$

$$(I-\theta B)^{-1} = I + \theta B + \theta^2 B^2 + \cdots$$

Vector MA(1) Model

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

if $\theta_{12} = 0$, then

$$\begin{aligned} Z_{1t} &= (1 - \theta_{11}B)a_{1t} \\ Z_{2t} &= (1 - \theta_{22}B)a_{2t} - \theta_{21}a_{1t-1} \\ &= (1 - \theta_{22}B)a_{2t} - \theta_{21}\frac{Z_{1t-1}}{1 - \theta_{11}B} \end{aligned}$$

Vector MA(1) Model

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

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$$\begin{aligned} Z_{1t} &= (1 - \theta_{11}B)a_{1t} \\ \boxed{Z_{2t}} &= (1 - \theta_{22}B)a_{2t} - \theta_{21}a_{1t-1} \\ &= (1 - \theta_{22}B)a_{2t} - \theta_{21}\frac{\boxed{Z_{1t-1}}}{1 - \theta_{11}B} \end{aligned}$$

Unidirectional Relationship ?

Vector MA(1) Model

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

if $\theta_{12} = 0$, then

可以使得correlated的 a_{1t}, a_{2t}

将a_{2t}分解为ϵ_t和βa_{1t}，而$\epsilon_t \perp \beta a_{1t}$

$$\begin{aligned} Z_{1t} &= (1 - \theta_{11}B)a_{1t} \\ \boxed{Z_{2t}} &= (1 - \theta_{22}B)\boxed{a_{2t}} - \theta_{21}a_{1t-1} \\ &= (1 - \theta_{22}B)a_{2t} - \theta_{21}\frac{Z_{1t-1}}{1 - \theta_{11}B} \\ &= (1 - \theta_{22}B)\epsilon_t - (\omega_0 - \omega_1 B)\frac{\boxed{Z_{1t-1}}}{1 - \theta_{11}B} \end{aligned}$$

Unidirectional Relationship ?

Vector MA(1) Model

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix}$$

if $\theta_{12} = 0$, then

If θ is triangular

then unidirectional representation

$$\begin{aligned} Z_{1t} &= (1 - \theta_{11}B)a_{1t} \\ \boxed{Z_{2t}} &= (1 - \theta_{22}B)a_{2t} - \theta_{21}a_{1t-1} \\ &= (1 - \theta_{22}B)a_{2t} - \theta_{21}\frac{Z_{1t-1}}{1 - \theta_{11}B} \\ &= (1 - \theta_{22}B)\epsilon_t - (\omega_0 - \omega_1 B)\frac{\boxed{Z_{1t-1}}}{1 - \theta_{11}B} \end{aligned}$$

Unidirectional Relationship ?

Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

More than one series, correlations:

Income X_t

$t-k \quad \cdots \quad t-2 \quad t-1 \quad t \quad t+1 \quad t+2 \quad \cdots \quad t+k$

单变量时

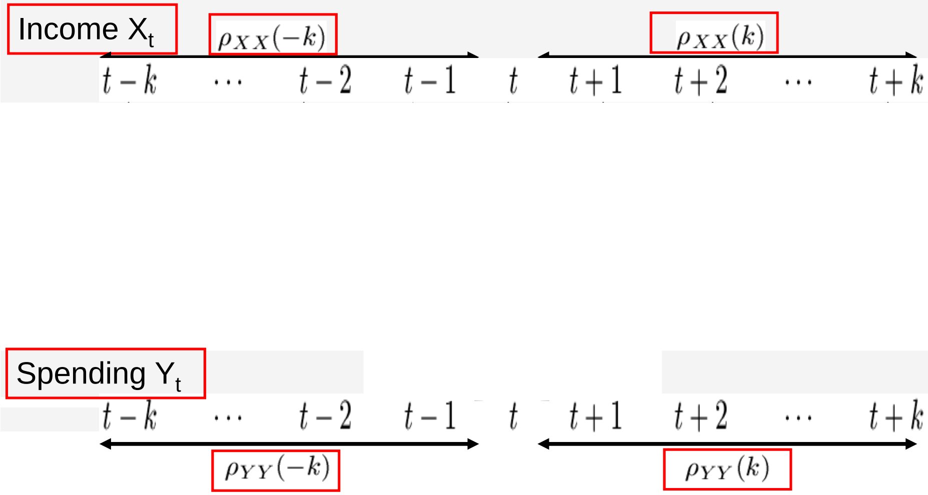
$$\rho_K = coor(Z_t, Z_{t-k})$$

Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

More than one series, correlations:



Vector MA(1) Model

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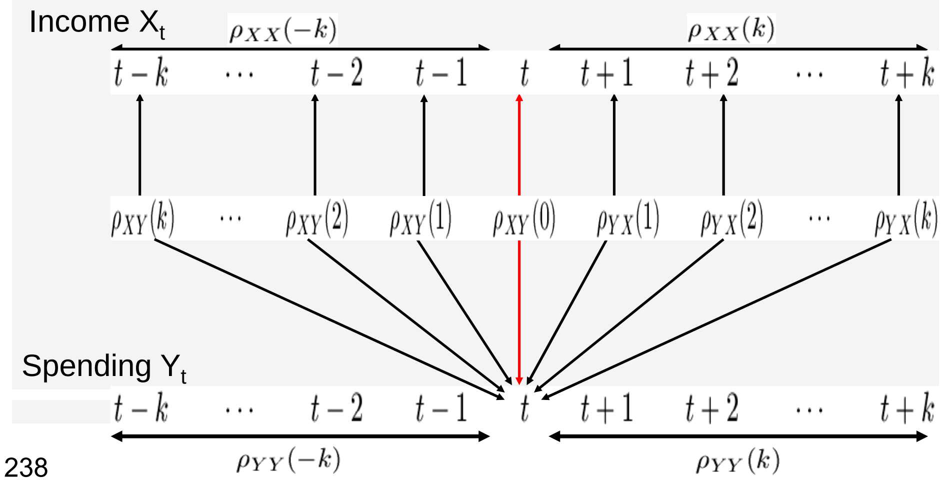

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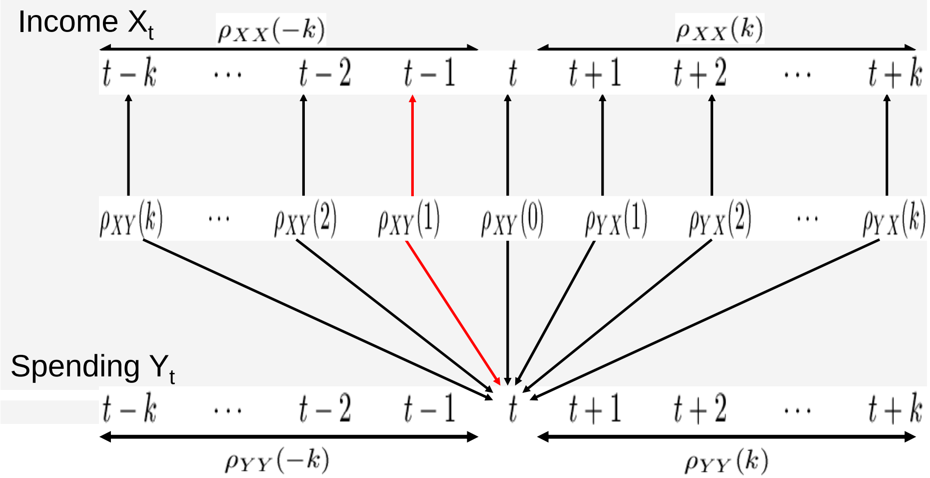
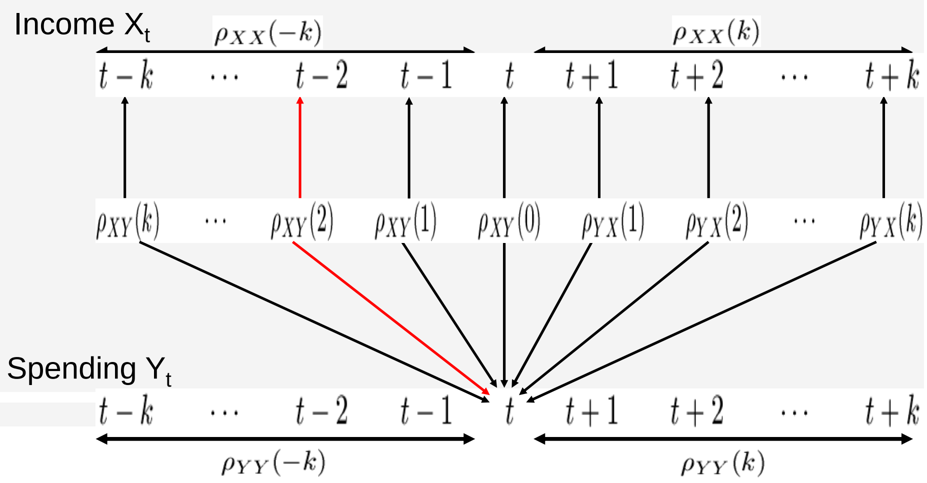


Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

More than one series, correlations:

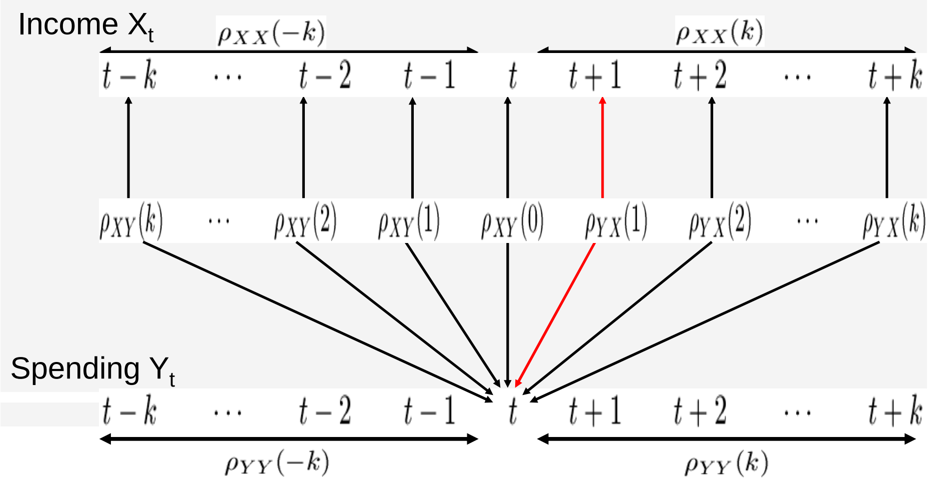


Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

More than one series, correlations:



Vector MA(1) Model

How to specify Vector MA(1) :

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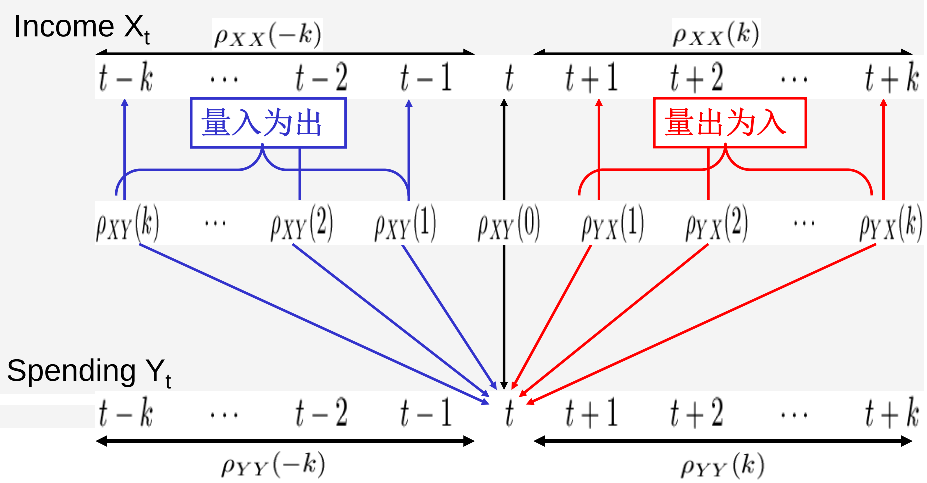
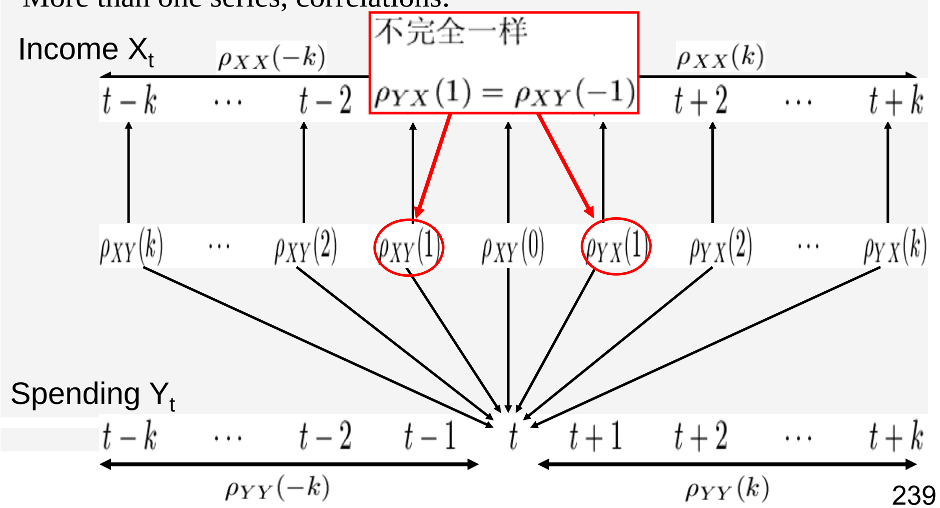


Vector MA(1) Model

How to specify Vector MA(1) :

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

More than one series, correlations:



Vector MA(1) Model

$\underset{\sim}{\mu} = 0$

$\rho_{XY}(K) = \rho_{YX}(-K)$

Cross covariance matrix :

$$E\begin{pmatrix} Z_{1t-\ell} \\ Z_{2t-\ell} \end{pmatrix}\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} \qquad \gamma_{12}(\ell) \neq \gamma_{21}(\ell)$$

$$\begin{aligned} \Gamma(\ell) &= E(\underset{\sim}{Z_{t-\ell}}\underset{\sim}{Z_t}^\top) \\ &= \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix} \end{aligned}$$

$\ell = 0, 1, 2, 3, \cdots$

Vector MA(1) Model

Cross Correlation Matrix of $\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$

$$\begin{aligned} \rho(\ell) &= \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \Gamma(\ell) \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{11}(\ell) & \rho_{12}(\ell) \\ \rho_{21}(\ell) & \rho_{22}(\ell) \end{pmatrix} \end{aligned}$$

$$\Gamma(\ell) = \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix}$$

Vector MA(1) Model

Cross Correlation Matrix of $\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$

$$\begin{aligned} \rho(\ell) &= \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \Gamma(\ell) \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{11}(\ell) & \rho_{12}(\ell) \\ \rho_{21}(\ell) & \rho_{22}(\ell) \end{pmatrix} \end{aligned}$$

Def. of Cross correlation matrix with lag ℓ

$$\Gamma(\ell) = \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix}$$

Vector MA(1) Model

Cross Correlation Matrix of $\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$

$$\begin{aligned} \rho(\ell) &= \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \Gamma(\ell) \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{11}(\ell) & \rho_{12}(\ell) \\ \rho_{21}(\ell) & \rho_{22}(\ell) \end{pmatrix} \end{aligned}$$

$$\Gamma(\ell) = \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix}$$

Vector MA(1) Model

Cross Correlation Matrix of $\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$

$$\begin{aligned}
\rho(\ell) &= \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \Gamma(\ell) \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \\
&= \begin{pmatrix} \rho_{11}(\ell) & \rho_{12}(\ell) \\ \rho_{21}(\ell) & \rho_{22}(\ell) \end{pmatrix} \\
\Gamma(\ell) &= \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix} \leftarrow \boxed{\Gamma(\ell) = E(\underset{\sim}{Z_{t-\ell}}\underset{\sim}{Z_t}^\top)}
\end{aligned}$$

Vector MA(1) Model

Cross Correlation Matrix of $\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$

$$\begin{aligned}
\rho(\ell) &= \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \Gamma(\ell) \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \\
&= \begin{pmatrix} \rho_{11}(\ell) & \rho_{12}(\ell) \\ \rho_{21}(\ell) & \rho_{22}(\ell) \end{pmatrix} \\
\Gamma(\ell) &= \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix} \leftarrow \boxed{\Gamma(\ell) = E(\underset{\sim}{Z_{t-\ell}}\underset{\sim}{Z_t}^\top)}
\end{aligned}$$

Vector MA(1) Model

MA(1) :

$$\begin{aligned}
\underset{\sim}{Z_t} &= \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}} \\
\Gamma(0) &= E(\underset{\sim}{Z_t}\underset{\sim}{Z_t}^\top) \\
&= E(\underset{\sim}{a_t} - \theta\underset{\sim}{a_{t-1}})(\underset{\sim}{a_t}^\top - \underset{\sim}{a_{t-1}}^\top\theta^\top) \\
&= \Sigma + \theta\Sigma\theta^\top \\
\Gamma(1) &= E(\underset{\sim}{Z_{t-1}}\underset{\sim}{Z_t}^\top) \\
&= E(\underset{\sim}{a_{t-1}} - \theta\underset{\sim}{a_{t-2}})(\underset{\sim}{a_t}^\top - \underset{\sim}{a_{t-1}}^\top\theta^\top) \\
&= -\Sigma\theta^\top \\
&\neq 0 \\
\Gamma(2) &= \Gamma(3) = \cdots = 0
\end{aligned}$$

Vector MA(1) Model

For MA(1) :

Cross Correlation Matrices :

$$\rho(1) \neq 0$$

$$\rho(2) = \rho(3) = \cdots = 0$$

Vector MA(1) Model

Cross Correlation Matrix of $\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}$

$$\rho(\ell) = \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix} \Gamma(\ell) \begin{pmatrix} (\gamma_{11}(0))^{-1/2} & 0 \\ 0 & (\gamma_{22}(0))^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} \rho_{11}(\ell) & \rho_{12}(\ell) \\ \rho_{21}(\ell) & \rho_{22}(\ell) \end{pmatrix}$$

Def. of Cross correlation matrix with lag ℓ

$$\Gamma(\ell) = \begin{pmatrix} \gamma_{11}(\ell) & \gamma_{12}(\ell) \\ \gamma_{21}(\ell) & \gamma_{22}(\ell) \end{pmatrix}$$

$$\Gamma(\ell) = E(\underset{\sim}{Z_{t-\ell}} \underset{\sim}{Z_t}^\top)$$

Vector MA(1) Model

For MA(1) :

Cross Correlation Matrices :

$$\rho(1) \neq 0$$

$$\rho(2) = \rho(3) = \cdots = 0$$

时间序列分析

第十四讲(下)

Vector MA(1) Model

For MA(1) :

Cross Correlation Matrices :

$$\rho(1) \neq 0$$

$$\rho(2) = \rho(3) = \cdots = 0$$

Vector MA(1) Model

For MA(1) :

Cross Correlation Matrices :

$$\rho(1) \neq 0$$

$$\rho(2) = \rho(3) = \cdots = 0$$

Example: Page 11-12 of Chapter 14 (Pena, Tiao & Tsay)

ACF and Cross Correlation

K=2 3 plots

K=3 6 plots

K=4 10 plots

Vector MA(1) Model

For MA(1) :

Cross Correlation Matrices :

MA(1): $\rho(1) \neq 0$

MA(2): $\rho(2) \neq 0,\ \rho(3) = \rho(4) = \cdots = 0$

Vector MA(1) Model

For MA(1) :

Cross Correlation Matrices :

MA(1): $\rho(1) \neq 0$

MA(2): $\rho(2) \neq 0,\ \rho(3) = \rho(4) = \cdots = 0$

$$\begin{array}{lll} + & \text{if} & \text{coef} \in (\frac{2}{\sqrt{n}}, +\infty) \\ \cdot & \text{if} & \text{coef} \in [-\frac{2}{\sqrt{n}}, \frac{2}{\sqrt{n}}] \\ - & \text{if} & \text{coef} \in (-\infty, -\frac{2}{\sqrt{n}}) \end{array}$$

Vector AR(1)

K=2 : $(I - \phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ Z_1: income Z_2: consumption

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} B\right] \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{aligned} Z_{1t} &= \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ Z_{2t} &= \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{aligned}$$

Vector AR(1)

K=2 : $(I - \phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ Z_1: income Z_2: consumption

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$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{aligned} Z_{1t} &= \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ Z_{2t} &= \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{aligned}$$

Vector AR(1)

K=2 : $(I - \phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$ Z_1: income Z_2: consumption

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} B\right] \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$\begin{aligned} Z_{1t} &= \phi_{11} Z_{1t-1} + \phi_{12} Z_{2t-1} + a_{1t} \\ Z_{2t} &= \phi_{21} Z_{1t-1} + \phi_{22} Z_{2t-1} + a_{2t} \end{aligned}$$

Vector AR(1)

Example :

$$\phi = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

Example: See Page 6 of Chapter 14 (Pena, Tiao & Tsay)

Vector AR(1)

$(I - \phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$\begin{aligned} \underset{\sim}{Z_t} &= (I - \phi B)^{-1}\underset{\sim}{a_t} \\ &= (I + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)\underset{\sim}{a_t} \\ &= \underset{\sim}{a_t} + \phi \underset{\sim}{a_{t-1}} + \phi^2 \underset{\sim}{a_{t-2}} + \phi^3 \underset{\sim}{a_{t-3}} + \cdots \end{aligned}$$

Vector AR(1)

$(I - \phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$\begin{aligned} \underset{\sim}{Z_t} &= (I - \phi B)^{-1}\underset{\sim}{a_t} \\ &= (I + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)\underset{\sim}{a_t} \\ &= \underset{\sim}{a_t} + \phi \underset{\sim}{a_{t-1}} + \phi^2 \underset{\sim}{a_{t-2}} + \phi^3 \underset{\sim}{a_{t-3}} + \cdots \end{aligned}$$

eigen values of ϕ, $|\alpha_i| < 1$, then it will converge

$$(I - \Phi B)^{-1} = \begin{pmatrix} 1 - \phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1 - \phi_{22}B \end{pmatrix}^{-1} = \frac{1}{|I - \Phi B|}\begin{pmatrix} 1 - \phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1 - \phi_{11}B \end{pmatrix}$$

stationarity condition

Vector AR(1)

$(I - \phi B)\underset{\sim}{Z_t} = \underset{\sim}{a_t}$

$$\begin{aligned} \underset{\sim}{Z_t} &= (I - \phi B)^{-1}\underset{\sim}{a_t} \\ &= (I + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots)\underset{\sim}{a_t} \\ &= \underset{\sim}{a_t} + \phi \underset{\sim}{a_{t-1}} + \phi^2 \underset{\sim}{a_{t-2}} + \phi^3 \underset{\sim}{a_{t-3}} + \cdots \end{aligned}$$

eigen values of ϕ, $|\alpha_i| < 1$, then it will converge

$$\begin{aligned} \Gamma(0) &= \mathrm{Cov}(\underset{\sim}{Z_t}\underset{\sim}{Z_t}^{\top}) \\ &= \Sigma + \phi\Sigma\phi^{\top} + \phi^2\Sigma(\phi^{\top})^2 + \phi^3\Sigma(\phi^{\top})^3 + \cdots \\ &= \Sigma + \phi[\Sigma + \phi\Sigma\phi^{\top} + \phi^2\Sigma(\phi^{\top})^2 + \phi^3\Sigma(\phi^{\top})^3 + \cdots]\phi^{\top} \end{aligned}$$

Vector AR(1)

$\Gamma(0) = \Sigma + \phi\Gamma(0)\phi^\top$

Vector AR(1)

$\Gamma(0) = \Sigma + \phi\Gamma(0)\phi^\top$

$$\Gamma(0) = \begin{pmatrix} \Gamma_{11}(0) & \Gamma_{12}(0) \\ \Gamma_{21}(0) & \Gamma_{22}(0) \end{pmatrix}$$

Vector AR(1)

$\Gamma(0) = \Sigma + \phi\Gamma(0)\phi^\top$

$$\Gamma(0) = \begin{pmatrix} \Gamma_{11}(0) & \Gamma_{12}(0) \\ \Gamma_{21}(0) & \Gamma_{22}(0) \end{pmatrix}$$

If Σ, ϕ known, linear in elements of $\Gamma(0)$, we may get $\Gamma(0)$.

Vector AR(1)

$$(I - \phi B)^{-1} = \frac{1}{|I - \phi B|} \begin{pmatrix} 1 - \phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1 - \phi_{11}B \end{pmatrix}$$

Zeros lying outside u.c.

Vector AR(1)

Bivariate AR(1)

$$\begin{pmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{pmatrix} \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

What is the model for $\{Z_{1t}\}$?

Vector AR(1)

Bivariate AR(1)

$$\begin{pmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{pmatrix} \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

What is the model for $\{Z_{1t}\}$?

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{pmatrix}^{-1} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

Vector AR(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \frac{1}{|I-\phi B|} \begin{pmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{pmatrix} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$(1-\phi_{11}B)(1-\phi_{22}B)-\phi_{12}\phi_{21}B^2 = \varphi(B)$$

$$\underbrace{\varphi(B)\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}}_{\text{AR(2)}} = \underbrace{\begin{pmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{pmatrix} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}}_{\text{MA(1)}}$$

AR(2) → ARMA(2,1) ← MA(1)

Vector AR(1)

K=2	ARMA(2,1)
K=3	ARMA(3,2)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \frac{1}{|I-\phi B|} \begin{pmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{pmatrix} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

$$(1-\phi_{11}B)(1-\phi_{22}B)-\phi_{12}\phi_{21}B^2 = \varphi(B)$$

$$\underbrace{\varphi(B)\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix}}_{\text{AR(2)}} = \underbrace{\begin{pmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{pmatrix} \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}}_{\text{MA(1)}}$$

AR(2) → ARMA(2,1) ← MA(1)

Vector AR(1)

In principle, each can be ARMA(2,1)

How to generalize?

Review : SPACF for univariate case ?

Vector AR(1)

$$\left.\begin{array}{lcl} \text{AR}(1) & & \widehat{\phi} \\ \text{AR}(2) & \neq & \widehat{\phi}_2 \\ \text{AR}(3) & \neq & \widehat{\phi}_3 \\ \text{AR}(4) & & \widehat{\phi}_4 \\ & \vdots & \end{array}\right\}$$

此页原为Review of Univariate SPACF, 可划去。

SPACF

Last coefficient estimated!

Vector AR(1)

$$\begin{aligned} y &= \beta x + \epsilon \\ \widehat{\beta} &= (x^\top x)^{-1} x^\top y \end{aligned}$$

simple regression

Vector AR(1)

$$\begin{aligned} y &= \beta x + \epsilon \\ \widehat{\beta} &= (x^\top x)^{-1} x^\top y \end{aligned}$$

simple regression

AR(1) $\quad Z_t = \phi Z_{t-1} + a_t \qquad t = 1, \cdots, T$

$$\widehat{\phi} = \frac{\sum_{t=2}^{T} Z_t Z_{t-1}}{\sum_{t=2}^{T} Z_t^2}$$

Vector AR(1)

$$\text{Vector}: \quad \underset{\sim}{Z}_t = \phi \underset{\sim}{Z}_{t-1} + \underset{\sim}{a}_t$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \qquad t = 1, \cdots, T$$

$$\underset{\sim}{Z}_t^{\top} = \underset{\sim}{Z}_{t-1}^{\top} \phi^{\top} + \underset{\sim}{a}_t^{\top}$$

$$\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} = \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}$$

$$\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array} \begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

Vector AR(1)

$$\text{Vector}: \quad \underset{\sim}{Z}_t = \phi \underset{\sim}{Z}_{t-1} + \underset{\sim}{a}_t$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \qquad t = 1, \cdots, T$$

$$\underset{\sim}{Z}_t^{\top} = \underset{\sim}{Z}_{t-1}^{\top} \phi^{\top} + \underset{\sim}{a}_t^{\top}$$

$$\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} = \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}$$

$$\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array} \underset{\boxed{\underset{\sim}{Z}_1}\ \ \boxed{\underset{\sim}{Z}_2}}{\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix}} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

Vector AR(1)

$$\text{Vector}: \quad \underset{\sim}{Z}_t = \phi \underset{\sim}{Z}_{t-1} + \underset{\sim}{a}_t$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \qquad t = 1, \cdots, T \quad \boxed{t = 2, \cdots, T}$$

$$\underset{\sim}{Z}_t^{\top} = \underset{\sim}{Z}_{t-1}^{\top} \phi^{\top} + \underset{\sim}{a}_t^{\top}$$

$$\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} = \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}$$

$$\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array} \underset{\boxed{\underset{\sim}{y}_1}\ \ \boxed{\underset{\sim}{y}_2}}{\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix}} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

Vector AR(1)

$$\text{Vector}: \quad \underset{\sim}{Z}_t = \phi \underset{\sim}{Z}_{t-1} + \underset{\sim}{a}_t$$

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} \qquad t = 1, \cdots, T \quad \boxed{t = 2, \cdots, T}$$

$$\underset{\sim}{Z}_t^{\top} = \underset{\sim}{Z}_{t-1}^{\top} \phi^{\top} + \underset{\sim}{a}_t^{\top}$$

$$\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} = \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}$$

$$\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array} \underset{\boxed{\underset{\sim}{y}_1}\ \ \boxed{\underset{\sim}{y}_2}}{\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix}} = \underset{\boxed{X_1}\ \ \boxed{X_2}}{\begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix}} \underset{\boxed{\underset{\sim}{\phi}_1}\ \ \boxed{\underset{\sim}{\phi}_2}}{\begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix}} + \underset{\boxed{\underset{\sim}{a}_1}\ \ \boxed{\underset{\sim}{a}_2}}{\begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}}$$

Vector AR(1)

Vector :
$$
\begin{aligned}
\underset{\sim}{Z_t} &= \phi \underset{\sim}{Z_{t-1}} + \underset{\sim}{a_t}\\
\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} &= \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}\\
\underset{\sim}{Z_t}^\top &= \underset{\sim}{Z_{t-1}}^\top \phi^\top + \underset{\sim}{a_t}^\top\\
\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} &= \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}
\end{aligned}
$$

$$
\boxed{\begin{aligned} \underset{\sim}{y_1} &= X_1 \underset{\sim}{\phi_1} + \underset{\sim}{a_1} \\ \underset{\sim}{y_2} &= X_2 \underset{\sim}{\phi_2} + \underset{\sim}{a_2} \end{aligned}}
$$

$t = 1, \cdots, T$

$\boxed{t = 2, \cdots, T}$

$$
\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array}
\underbrace{\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix}}_{\boxed{\underset{\sim}{y_1}}\ \boxed{\underset{\sim}{y_2}}}
= \underbrace{\begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix}}_{\boxed{X_1}\ \boxed{X_2}}
\underbrace{\begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix}}_{\boxed{\underset{\sim}{\phi_1}}\ \boxed{\underset{\sim}{\phi_2}}}
+ \underbrace{\begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}}_{\boxed{\underset{\sim}{a_1}}\ \boxed{\underset{\sim}{a_2}}}
$$

Vector AR(1)

Vector :

$$\boxed{Y = X \underset{\sim}{\phi}^\top + A} \longleftarrow \boxed{\begin{aligned} \underset{\sim}{y_1} &= X_1 \underset{\sim}{\phi_1} + \underset{\sim}{a_1} \\ \underset{\sim}{y_2} &= X_2 \underset{\sim}{\phi_2} + \underset{\sim}{a_2} \end{aligned}}$$

$$
\begin{aligned}
\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} &= \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}\\
\underset{\sim}{Z_t}^\top &= \underset{\sim}{Z_{t-1}}^\top \phi^\top + \underset{\sim}{a_t}^\top\\
\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} &= \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}
\end{aligned}
$$

$t = 1, \cdots, T$

$\boxed{t = 2, \cdots, T}$

$$
\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array}
\underbrace{\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix}}_{\boxed{\underset{\sim}{y_1}}\ \boxed{\underset{\sim}{y_2}}}
= \underbrace{\begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix}}_{\boxed{X_1}\ \boxed{X_2}}
\underbrace{\begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix}}_{\boxed{\underset{\sim}{\phi_1}}\ \boxed{\underset{\sim}{\phi_2}}}
+ \underbrace{\begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}}_{\boxed{\underset{\sim}{a_1}}\ \boxed{\underset{\sim}{a_2}}}
$$

Vector AR(1)

Vector :

$$\boxed{Y = X \underset{\sim}{\phi}^\top + A} \longleftarrow \boxed{\begin{aligned} \underset{\sim}{y_1} &= X_1 \underset{\sim}{\phi_1} + \underset{\sim}{a_1} \\ \underset{\sim}{y_2} &= X_2 \underset{\sim}{\phi_2} + \underset{\sim}{a_2} \end{aligned}}$$

$$
\begin{aligned}
\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} &= \phi \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}\\
\underset{\sim}{Z_t}^\top &= \underset{\sim}{Z_{t-1}}^\top \phi^\top + \underset{\sim}{a_t}^\top\\
\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} &= \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}
\end{aligned}
$$

$t = 1, \cdots, T$

$\boxed{t = 2, \cdots, T}$

横向不独立

纵向独立

$$
\begin{array}{c} t=2 \\ t=3 \\ \vdots \\ t=T \end{array}
\underbrace{\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix}}_{\boxed{\underset{\sim}{y_1}}\ \boxed{\underset{\sim}{y_2}}}
= \underbrace{\begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix}}_{\boxed{X_1}\ \boxed{X_2}}
\underbrace{\begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix}}_{\boxed{\underset{\sim}{\phi_1}}\ \boxed{\underset{\sim}{\phi_2}}}
+ \underbrace{\begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}}_{\boxed{\underset{\sim}{a_1}}\ \boxed{\underset{\sim}{a_2}}}
$$

Vector AR(1)

$$
\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix}
= \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix}
\begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix}
+ \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}
$$

$$
\underbrace{\begin{pmatrix} \underset{\sim}{y_1} & \underset{\sim}{y_2} \end{pmatrix} = X \begin{pmatrix} \underset{\sim}{\phi_1} & \underset{\sim}{\phi_2} \end{pmatrix} + \begin{pmatrix} \underset{\sim}{a_1} & \underset{\sim}{a_2} \end{pmatrix} \Rightarrow \left\{ \begin{aligned} \underset{\sim}{y_1} &= X \underset{\sim}{\phi_1} + \underset{\sim}{a_1} \\ \underset{\sim}{y_2} &= X \underset{\sim}{\phi_2} + \underset{\sim}{a_2} \end{aligned} \right.}
$$

$$
\begin{aligned}
Y &= X\theta + \epsilon \\
\widehat{\theta} &= (X^\top X)^{-1} X^\top Y
\end{aligned}
$$

Vector AR(1)

$$\begin{pmatrix} Z_{12} & Z_{22} \\ Z_{13} & Z_{23} \\ \vdots & \vdots \\ Z_{1T} & Z_{2T} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \\ \vdots & \vdots \\ Z_{1T-1} & Z_{2T-1} \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{21} \\ \phi_{12} & \phi_{22} \end{pmatrix} + \begin{pmatrix} a_{12} & a_{22} \\ a_{13} & a_{23} \\ \vdots & \vdots \\ a_{1T} & a_{2T} \end{pmatrix}$$

$$Y = X\theta + \epsilon$$

$$\widehat{\theta} = (X^\top X)^{-1} X^\top Y$$

fitted $\widehat{Y} = X\widehat{\theta}$

Residual $Y - \widehat{Y}$

Bartlett 1938

LR Test

Test $\underset{\sim}{\theta} = \underset{\sim}{0} \quad v.s. \quad \underset{\sim}{\theta} \neq \underset{\sim}{0}$

Bartlett Test

Regression : $S(0) = Y^\top Y$

residual $(Y - \widehat{Y})^\top (Y - \widehat{Y}) = S(1)$

$$U_1 = \frac{|S(1)|}{|S(0)|}$$

Bartlett Test : $M(1) = -(N - .5 - K)\log(U_1) \overset{\bullet}{\sim} \chi^2(4)$

test: $\theta = 0 \ v.s. \ \theta \neq 0$

Ratio of Determinant : K=2 ⬄ K^2=4

M(1) : corresponds to 1st Partial in Univariate Case

Bartlett Test

AR(2)

$$\underset{\sim}{Z_t} = \phi_1 \underset{\sim}{Z_{t-1}} + \phi_2 \underset{\sim}{Z_{t-2}} + \underset{\sim}{a_t}$$

$$\underset{\sim}{Z_t}^\top = \underset{\sim}{Z_{t-1}}^\top \phi_1^\top + \underset{\sim}{Z_{t-2}}^\top \phi_2^\top + \underset{\sim}{a_t}^\top$$

$$\begin{pmatrix} Z_{1t} & Z_{2t} \end{pmatrix} = \begin{pmatrix} Z_{1t-1} & Z_{2t-1} \end{pmatrix} \phi_1^\top + \begin{pmatrix} Z_{1t-2} & Z_{2t-2} \end{pmatrix} \phi_2^\top + \begin{pmatrix} a_{1t} & a_{2t} \end{pmatrix}$$

$$t = 3, \cdots, T$$

$$Y = X_1\theta_1 + X_2\theta_2 + \underset{\sim}{\epsilon}$$

Bartlett Test

Test $\phi_2^\top = \theta_2 = 0$

$$\widehat{Y_2} = X_1\widehat{\theta_1} + X_2\widehat{\theta_2}$$

$$X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$$

$$S(2) = (Y - \widehat{Y_2})^\top (Y - \widehat{Y_2})$$

$$\frac{|S(2)|}{|S(1)|} = U_2$$

Bartlett Test : $M(2) = -(N - .5 - 2K)\log(U_2) \overset{\bullet}{\sim} \chi^2(4)$

Ratio of Determinant : K=2 ⬄ K^2=4 ***Correspond to 2nd Partial***

Bartlett Test

Test $\phi_2^{\top} = \theta_2 = 0$

$$\widehat{Y_2} = X_1\widehat{\theta}_1 + X_2\widehat{\theta}_2$$

$$X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$$

$$S(2) = (Y - \widehat{Y_2})^{\top}(Y - \widehat{Y_2})$$

$$\frac{|S(2)|}{|S(1)|} = U_2$$

Bartlett Test : $M(2) = -(N - .5 - 2K)\log(U_2) \sim \chi^2(4)$

Ratio of Determinant : K=2 ⇔ K^2=4 ***Correspond to 2nd Partial***

可以类推到 AR(3), AR(4), ……

Example

Quarterly data (SCC)

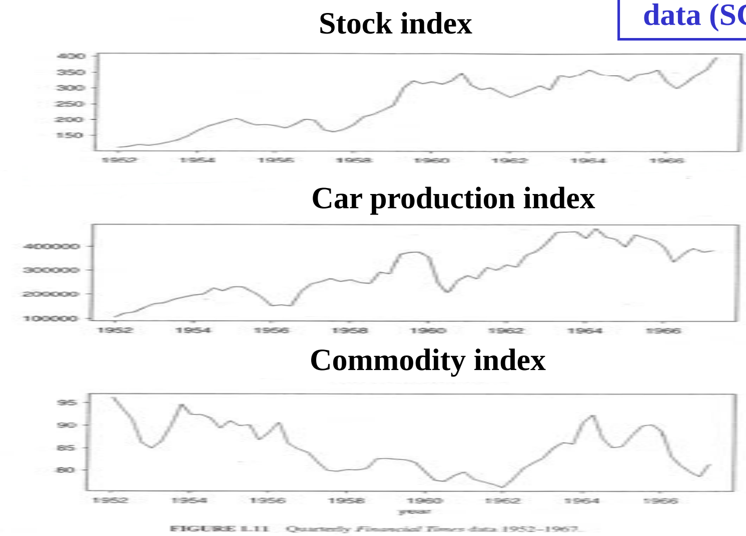

FIGURE 1.11 Quarterly Financial Times data 1952–1967.

Example

$DW \approx 2(1 - \widehat{\rho}_1)$

Quarterly data (SCC)

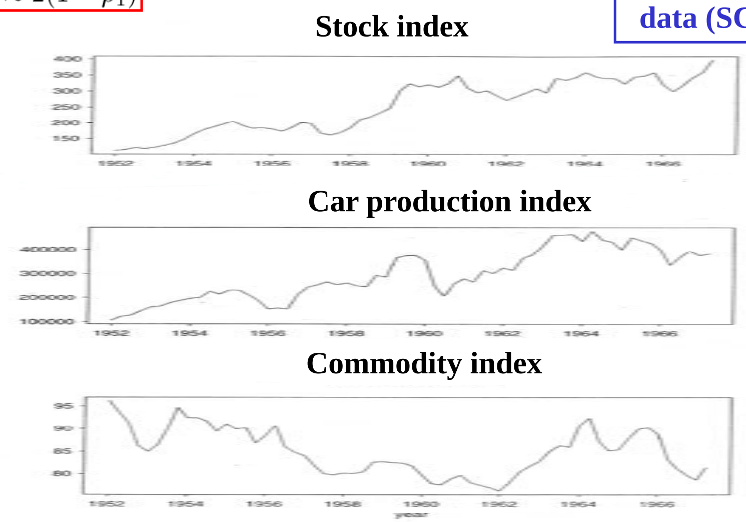

FIGURE 1.11 Quarterly Financial Times data 1952–1967.

Example

$DW \approx 2(1 - \widehat{\rho}_1)$

Quarterly data (SCC)

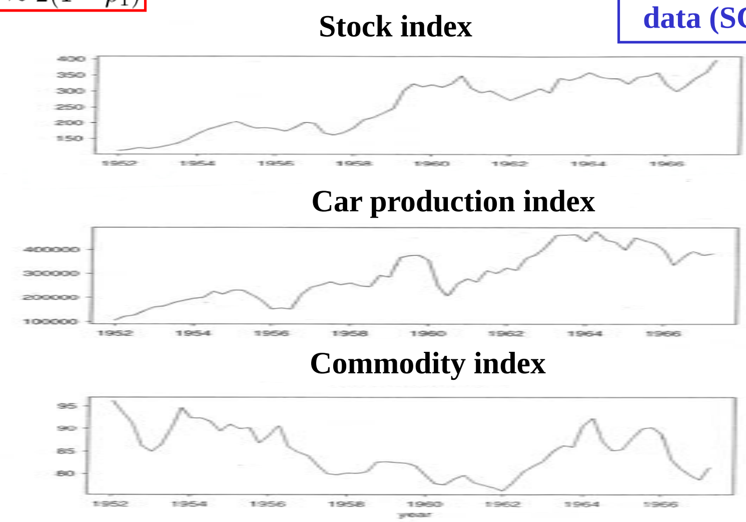

FIGURE 1.11 Quarterly Financial Times data 1952–1967.

$$Stock_t = C + Car_{t-6} + Commodity_{t-7} + (1 - \phi B)a_t$$

时间序列分析

第十五讲(上)

Today's Topic

- Review of Vector MA(1)/AR(1) & their specification
- SCC
- Cross Correlation *v.s.* Dynamic Relationship between 2 Time Series
- Gas Furnace Data
- Structural Analysis – Co-integration
- Variance Change Model
  - ARCH/GARCH Models
  - Connection with ARMA
  - Example

Review

Univariate ARMA:

$$Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

Vector ARMA:

$$\underset{\sim}{Z_t} - \Phi_1 \underset{\sim}{Z_{t-1}} - \cdots - \Phi_p \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \Theta_1 \underset{\sim}{a_{t-1}} - \cdots - \Theta_q \underset{\sim}{a_{t-q}}$$

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ \vdots \\ Z_{kt} \end{pmatrix}, \quad \underset{\sim}{a_t} = \begin{pmatrix} a_{1t} \\ \vdots \\ a_{kt} \end{pmatrix} \overset{i.i.d.}{\sim} N_k(0, \Sigma)$$

Review

Univariate ARMA:

$$Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

Vector ARMA:

$$\underset{\sim}{Z_t} - \underbrace{\Phi_1}_{k \times k} \underset{\sim}{Z_{t-1}} - \cdots - \underbrace{\Phi_p}_{k \times k} \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \underbrace{\Theta_1}_{k \times k} \underset{\sim}{a_{t-1}} - \cdots - \underbrace{\Theta_q}_{k \times k} \underset{\sim}{a_{t-q}}$$

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ \vdots \\ Z_{kt} \end{pmatrix}, \quad \underset{\sim}{a_t} = \begin{pmatrix} a_{1t} \\ \vdots \\ a_{kt} \end{pmatrix} \overset{i.i.d.}{\sim} N_k(0, \Sigma)$$

Review

Univariate ARMA:

$$Z_t - \phi_1 Z_{t-1} - \cdots - \phi_p Z_{t-p} = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

Vector ARMA:

$$\underset{\sim}{Z_t} - \overset{\text{k x k}}{\Phi_1} \underset{\sim}{Z_{t-1}} - \cdots - \overset{\text{k x k}}{\Phi_p} \underset{\sim}{Z_{t-p}} = \underset{\sim}{a_t} - \overset{\text{k x k}}{\Theta_1} \underset{\sim}{a_{t-1}} - \cdots - \overset{\text{k x k}}{\Theta_q} \underset{\sim}{a_{t-q}}$$

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ \vdots \\ Z_{kt} \end{pmatrix}, \quad \underset{\sim}{a_t} = \begin{pmatrix} a_{1t} \\ \vdots \\ a_{kt} \end{pmatrix} \overset{i.i.d.}{\sim} N_k(0, \Sigma)$$

SACF ➔ SCCF

Review

k=2时，考虑MA(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix}$$

Review

k=2时，考虑MA(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix}$$

Review

k=2时，考虑MA(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \Theta \underset{\sim}{a_{t-1}}$$

$$\underset{\sim}{Z_t} = (I - \Theta B) \underset{\sim}{a_t}$$

Review

k=2时，考虑MA(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \Theta \underset{\sim}{a_{t-1}}$$

$$\underset{\sim}{Z_t} = (I - \Theta B)\underset{\sim}{a_t}$$

Invertible form

Review

k=2时，考虑MA(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \Theta \underset{\sim}{a_{t-1}}$$

$$\underset{\sim}{Z_t} = (I - \Theta B)\underset{\sim}{a_t}$$

$\boxed{|I - \Theta B|}$

Invertible form

Review

k=2时，考虑MA(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix}$$

$$\underset{\sim}{Z_t} = \underset{\sim}{a_t} - \Theta \underset{\sim}{a_{t-1}}$$

$$\underset{\sim}{Z_t} = (I - \Theta B)\underset{\sim}{a_t}$$

$\boxed{|I - \Theta B|}$

Invertible form

$$\boxed{\begin{aligned} \rho(\ell) &= coor(Z_t, Z_{t-\ell}) \\ \rho(1) &\neq 0 \\ \rho(2) &= \rho(3) = 0 \end{aligned}}$$

Review

k=2时，考虑AR(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

1) **Individually** **ARMA(2,1)** **most complex case**
2) **Triangular** **uni-directionally**
3) **Stationarity Condition** $|I - \Phi B|$ all zeros outside unit circle

Review

k=2时，考虑AR(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

1) **Individually** **ARMA(2,1)** **most complex case**
2) **Triangular** **uni-directionally**
3) **Stationarity Condition** $|I - \Phi B|$ all zeros outside unit circle

Review

k=2时，考虑AR(1)

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} - \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \end{pmatrix} = \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix}$$

通过**Partial Autoregression**作 **Specification**

1) **Individually** **ARMA(2,1)** **most complex case**
2) **Triangular** **uni-directionally**
3) **Stationarity Condition** $|I - \Phi B|$ all zeros outside unit circle

Review

Example: MA(1) See Page 5 of Chapter 14 (Pena, Tiao & Tsay)

$$\underset{\sim}{Z_t} = \underset{\sim}{\mu} + \underset{\sim}{a_t} - \theta \underset{\sim}{a_{t-1}}$$

$$\Theta = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \quad \underset{\sim}{\mu} = \begin{pmatrix} 17 \\ 25 \end{pmatrix}$$

Example: AR(1) See Page 6 of Chapter 14

$$\Phi = \begin{pmatrix} .2 & .3 \\ -.6 & 1.1 \end{pmatrix} \quad \Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

Example

Example: SSC, From Page 19 of Computer Output

Example

Example: SSC, From Page 19 of Computer Output

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ Z_{3t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \\ a_{3t} \end{pmatrix}$$

If we fit AR(1) to these three series

Example

Example: SSC, From Page 19 of Computer Output

$$\begin{array}{l} \text{stock} \\ \text{car} \\ \text{commodity} \end{array} \begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{pmatrix} = \begin{pmatrix} \boxed{\phi_{11}} & \phi_{12} & \phi_{13} \\ \phi_{21} & \boxed{\phi_{22}} & \phi_{23} \\ \phi_{31} & \phi_{32} & \boxed{\phi_{33}} \end{pmatrix} \begin{pmatrix} Z_{1t-1} \\ Z_{2t-1} \\ Z_{3t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \\ a_{3t} \end{pmatrix}$$

If we fit AR(1) to these three series

$\boxed{}$ 大

小

Cross Correlation

$$y_t = \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\}$$

Cross Correlation

$$y_t = \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\}$$

response

Transfer function model

x_t : continuous RV

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \\
x_t y_t &= (\nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t) x_t \\
&\Downarrow \\
E[x_t y_t] &= E[(\nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t) x_t] \\
&\Downarrow \\
\gamma(x_t y_t) &= \nu_0 E(x_t^2) + \nu_1 E(x_t x_{t-1}) + \nu_2 E(x_t x_{t-2}) + \cdots + E(x_t N_t)
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \\
\gamma(x_t y_t) &= \nu_0 E(x_t^2) + \nu_1 E(x_t x_{t-1}) + \nu_2 E(x_t x_{t-2}) + \cdots + E(x_t N_t)
\end{aligned}
$$

Crucial assumption

$x_t \quad iid \quad N(0, \sigma_a^2)$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \\
\gamma(x_t y_t) &= \nu_0 E(x_t^2) + \nu_1 \cancel{E(x_t x_{t-1})} + \nu_2 \cancel{E(x_t x_{t-2})} + \cdots + \cancel{E(x_t N_t)}
\end{aligned}
$$

Crucial assumption

$x_t \quad iid \quad N(0, \sigma_a^2)$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \quad (x_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)) \\
\gamma(x_t y_t) &= \nu_0 E(x_t^2) + \nu_1 \cancel{E(x_t x_{t-1})} + \nu_2 \cancel{E(x_t x_{t-2})} + \cdots + \cancel{E(x_t N_t)} \\
&\Downarrow \\
\gamma_{xy}(0) &= \nu_0 \sigma_a^2
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \quad (x_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)) \\
\gamma(x_t y_t) &= \nu_0 E(x_t^2) + \cancel{\nu_1 E(x_t x_{t-1})} + \cancel{\nu_2 E(x_t x_{t-2})} + \cdots + \cancel{E(x_t N_t)} \\
&\Downarrow \\
\gamma_{xy}(0) &= \nu_0 \sigma_a^2 \\
\rho_{xy}(0) &= \frac{\nu_0 \sigma_x^2}{\sigma_x \sigma_y} = \nu_0 \frac{\sigma_x}{\sigma_y}
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \quad (x_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)) \\
\gamma(x_{t-1} y_t) &= \nu_0 E(x_{t-1} x_t) + \nu_1 E(x_{t-1}^2) \\
&= 0 + \nu_1 \gamma_{xy}(0) \\
&= 0 + \nu_1 \sigma_x^2
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \quad (x_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)) \\
\gamma(x_{t-1} y_t) &= \nu_0 E(x_{t-1} x_t) + \nu_1 E(x_{t-1}^2) \\
&= 0 + \nu_1 \gamma_{xy}(0) \\
&= 0 + \nu_1 \sigma_x^2 \\
\rho_{xy}(1) &= \nu_1 (\frac{\sigma_x}{\sigma_y})
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \quad (x_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)) \\
\gamma(x_t y_t) &= \nu_0 E(x_t^2) + \cancel{\nu_1 E(x_t x_{t-1})} + \cancel{\nu_2 E(x_t x_{t-2})} + \cdots + \cancel{E(x_t N_t)} \\
&\Downarrow \\
\gamma_{xy}(0) &= \nu_0 \sigma_a^2 \\
\rho_{xy}(0) &= \frac{\nu_0 \sigma_x^2}{\sigma_x \sigma_y} = \nu_0 \frac{\sigma_x}{\sigma_y}
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \qquad \{x_t\} \perp \{N_t\} \\
&\Downarrow \quad (x_t \overset{i.i.d.}{\sim} N(0, \sigma_a^2)) \\
\gamma(x_{t-1} y_t) &= \nu_0 E(x_{t-1} x_t) + \nu_1 E(x_{t-1}^2) \\
&= 0 + \nu_1 \gamma_{xy}(0) \\
&= 0 + \nu_1 \sigma_x^2 \\
\rho_{xy}(1) &= \nu_1 (\frac{\sigma_x}{\sigma_y})
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \\
& \quad (1) \quad \{x_t\} \perp \{N_t\} \\
& \quad (2) \quad \{x_t\} \overset{i.i.d.}{\sim} N(0, \sigma_x^2) \\
&\Downarrow \\
\gamma_{xy}(\ell) &= \nu_\ell \sigma_x^2 \\
\rho_{xy}(\ell) &= \nu_\ell (\frac{\sigma_x}{\sigma_y})
\end{aligned}
$$

Cross Correlation

$$\rho_{xy}(\ell) = \nu_\ell (\frac{\sigma_x}{\sigma_y})$$

cross relationship $\rho_{xy}(\ell)$ are proportional to ν_ℓ

Cross Correlation

$$\rho_{xy}(\ell) = \nu_\ell (\frac{\sigma_x}{\sigma_y})$$

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cross correlation can help specify dynamic relationship

Cross Correlation

$$\rho_{xy}(\ell) \quad = \quad \nu_\ell(\frac{\sigma_x}{\sigma_y})$$

cross relationship $\rho_{xy}(\ell)$ are proportional to ν_ℓ

cross correlation can help specify dynamic relationship

If x_t's are auto correlated: $x_t = \alpha_t - \theta\alpha_{t-1}$ (MA(1))

Cross Correlation

$$\begin{aligned} y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \\ x_t y_t &= \nu_0 x_t^2 + \nu_1 x_{t-1}x_t + \nu_2 x_{t-2}x_t + \cdots + x_t N_t \end{aligned}$$

Cross Correlation

$$\begin{aligned} y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \\ x_t y_t &= \nu_0 x_t^2 + \nu_1 \overset{\neq 0}{x_{t-1}x_t} + \nu_2 \overset{= 0}{x_{t-2}x_t} + \cdots + \overset{= 0}{x_t N_t} \end{aligned}$$

Cross Correlation

$$\begin{aligned} y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \\ x_t y_t &= \nu_0 x_t^2 + \nu_1 \overset{\neq 0}{x_{t-1}x_t} + \nu_2 \overset{= 0}{x_{t-2}x_t} + \cdots + \overset{= 0}{x_t N_t} \\ \gamma_{xy}(0) &= \nu_0 \sigma_x^2 + \nu_1 \gamma_{xx}(1) \\ x_{t-1} y_t &= \nu_0 x_{t-1}x_t + \nu_1 x_{t-1}^2 + \nu_2 x_{t-1}x_{t-2} + \cdots + x_{t-1}N_t \\ & \rho_{xy}(1) \neq \nu_1 \quad \rho_{xy}(0) \neq \nu_0 \end{aligned}$$

If x_t highly auto correlated $\rho_{xy}(\ell)$ last at long lags

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \\
&\quad (1)\ \{x_t\} \perp \{N_t\} \\
&\quad (2)\ x_t = (1-\theta B)\alpha_t \quad \mathrm{MA}(1)
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \\
&\quad (1)\ \{x_t\} \perp \{N_t\} \\
&\quad (2)\ x_t = (1-\theta B)\alpha_t \quad \mathrm{MA}(1)
\end{aligned}
$$

$$
\begin{aligned}
y_t &= \nu_0(1-\theta B)\alpha_t + \nu_1(1-\theta B)\alpha_{t-1} + \nu_2(1-\theta B)\alpha_{t-2} \\
&\quad + \cdots + N_t \\
(1-\theta B)^{-1} y_t &= \nu_0 \alpha_t + \nu_1 \alpha_{t-1} + \nu_2 \alpha_{t-2} + \cdots + (1-\theta B) N_t \\
&\Downarrow \quad (\beta_t = (1-\theta B)^{-1} y_t) \\
\beta_t &= \nu_0 \alpha_t + \nu_1 \alpha_{t-1} + \nu_2 \alpha_{t-2} + \cdots \\
\rho_{xy}(\ell) &= \nu_\ell \left(\frac{\sigma_\alpha}{\sigma_\beta}\right)
\end{aligned}
$$

Cross Correlation

$$
\begin{aligned}
y_t &= \nu_0 x_t + \nu_1 x_{t-1} + \nu_2 x_{t-2} + \cdots + N_t \\
&\quad (1)\ \{x_t\} \perp \{N_t\} \\
&\quad (2)\ x_t = (1-\theta B)\alpha_t \quad \mathrm{MA}(1)
\end{aligned}
$$

$$
\begin{aligned}
y_t &= \nu_0(1-\theta B)\alpha_t + \nu_1(1-\theta B)\alpha_{t-1} + \nu_2(1-\theta B)\alpha_{t-2} \\
&\quad + \cdots + N_t
\end{aligned}
$$

$(1-\theta B)$ ... $-\theta B) N_t$

See Chapter 14 for further reference

$\Downarrow$

$$
\begin{aligned}
\beta_t &= \nu_0 \alpha_t + \nu_1 \alpha_{t-1} + \nu_2 \alpha_{t-2} + \cdots \\
\rho_{xy}(\ell) &= \nu_\ell \left(\frac{\sigma_\alpha}{\sigma_\beta}\right)
\end{aligned}
$$

时间序列分析

第十五讲(下)

Today's Topic

- Review of Vector MA(1)/AR(1) & their specification
- SCC
- Cross Correlation *v.s.* Dynamic Relationship between 2 Time Series

---

- Gas Furnace Data
- Structural Analysis – Co-integration
- Variance Change Model
  - ARCH/GARCH
  - Connection with ARMA
  - Examples

Gas Furnace

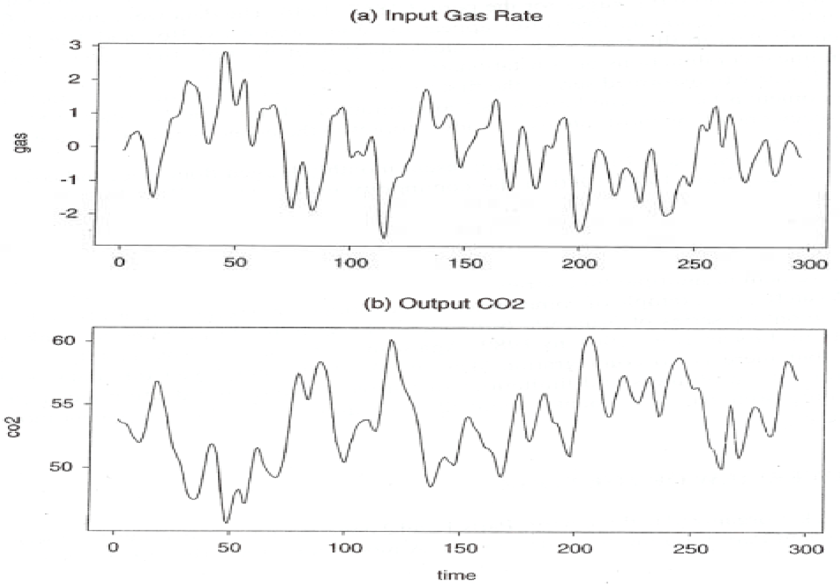

FIGURE 1.10 The gas furnace data: Box–Jenkins series J.

Gas Furnace

See page 7~18 of AO IO handout

See page 24 of Chapter 14

$$\underset{\sim}{Z_t}$$

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \begin{matrix} \leftarrow \text{Input gas rate} \\ \leftarrow \text{Output } CO_2 \end{matrix}$$

Model for Z_{1t}

$$\boxed{(1 - 1.97B - .34B^2)Z_{1t} = a_{1t}} \qquad \hat{\sigma}^2_{a_1} = .0353$$

$$Z_{2t} = \boxed{\frac{\omega(B)}{\delta(B)}} B^3 Z_{1t} + \boxed{\phi^{-1}(B)a_{2t}} \qquad \hat{\sigma}^2_{a_2} = .0561$$

Transfer function

Error term

Gas Furnace

TABLE 14.10. Tentative Identification for the Gas Furnace Data

(a) Pattern of Cross-Correlations of Original Data

$$\begin{matrix} + & - \\ - & + \end{matrix} \Leftarrow$$

	Z_{1t}	Z_{2t}
Z_{1t}	+ + + + + + + + + + +	- - - - - - - - - - -
Z_{2t}	- - - - - - - - - - -	+ + + + + + + + + + +

(b) M Statistic for Partial Autoregression

Lag ℓ	1	2	3	4	5	6	7	8	9	10	11
$M(\ell)$	1650	665	31.7	22.5	5.6	12.9	1.8	8.0	3.5	0	2.0

(c) Pattern of Cross-Correlations of Residuals after AR(6) Fit

	$\hat{a}_{1t}$	$\hat{a}_{2t}$
$\hat{a}_{1t}$	. . . .	. . . .
$\hat{a}_{2t}$	. . . .	. . . .

Gas Furnace

See page 7~18 of AO IO handout

See page 24 of Chapter 14

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \begin{matrix} \leftarrow \text{Input gas rate} \\ \leftarrow \text{Output } CO_2 \end{matrix}$$

Model for Z_{1t}

3 delay periods

$$\boxed{(1 - 1.97B - .34B^2)Z_{1t} = a_{1t}} \qquad \widehat{\sigma}^2_{a_1} = .0353$$

$$Z_{2t} = \boxed{\frac{\omega(B)}{\delta(B)}} B^3 Z_{1t} + \boxed{\phi^{-1}(B)a_{2t}} \qquad \widehat{\sigma}^2_{a_2} = .0561$$

Transfer function

Error term

Gas Furnace

Bivariate AR(6)

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3 - \phi_4 B^4 - \phi_5 B^5 - \phi_6 B^6)\underset{\sim}{Z_t} = c + \underset{\sim}{a_t}$$

$$\phi_1 = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \quad ,\cdots, \quad \phi_6 = \begin{bmatrix} \phi_{11}^{(6)} & \phi_{12}^{(6)} \\ \phi_{21}^{(6)} & \phi_{22}^{(6)} \end{bmatrix}$$

$$\underset{\sim}{a_t} \sim N_2(0, \Sigma) \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

See page 9 of AO IO output

Gas Furnace

AR(1)

$$\begin{matrix} \text{Input gas rate:} & X \\ \text{Output } CO_2: & Y \end{matrix} \begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} - \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} Z_{1t-1} \\ Z_{2t-1} \end{bmatrix} - \cdots$$

Gas Furnace

AR(1)

不显著

$$\begin{array}{ll}\text{Input gas rate:} & X \\ \text{Output } CO_2: & Y\end{array} \begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} - \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} Z_{1t-1} \\ Z_{2t-1} \end{bmatrix} - \cdots$$

Gas Furnace

AR(1)

不显著

$$\begin{array}{ll}\text{Input gas rate:} & X \\ \text{Output } CO_2: & Y\end{array} \begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix} - \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} Z_{1t-1} \\ Z_{2t-1} \end{bmatrix} - \cdots$$

$\phi_{12}^{(l)} \qquad l = 1,2,3,4,5,6$

X不受到过去Y的影响，单向影响

Structural Analysis

Vector model

k=2 $\quad \Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$

k=3

Simple model

No question of p and q

See page 35 of chapter 14

Structural Analysis

Simplest model

k=2 $\quad \Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$

See page 35 of chapter 14

Structural Analysis

Simplest model

k=2 $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$

k=3 $\underset{\sim}{Z_t} = \Phi \underset{\sim}{Z_{t-1}} + \underset{\sim}{\epsilon_t}$

$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{pmatrix}$ $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}$

See page 35 of chapter 14

Structural Analysis

Simplest model

k=2 $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$

k=3 $\underset{\sim}{Z_t} = \Phi \underset{\sim}{Z_{t-1}} + \underset{\sim}{\epsilon_t}$

$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{pmatrix}$ $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}$

See page 35 of chapter 14

Structural Analysis

Simplest model

k=2 $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$

k=3 $\underset{\sim}{Z_t} = \Phi \underset{\sim}{Z_{t-1}} + \underset{\sim}{\epsilon_t}$

$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \\ Z_{3t} \end{pmatrix}$ $\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}$

See page 35 of chapter 14

Structural Analysis

$\underset{\sim}{Z_t} = \begin{bmatrix} Z_{1t} \\ Z_{2t} \end{bmatrix}$ $\underset{\sim}{a_t} \sim N_2(0, \Sigma)$

$(I - \phi B)\underset{\sim}{Z} = (I - \theta B)\underset{\sim}{a_t}$

$\Sigma = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}$ $\phi = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$ $\theta = \begin{bmatrix} -0.5 & 0.5 \\ -1 & 1 \end{bmatrix}$

See page 37 of chapter 14

Structural Analysis

$$(\nu_0^{(1)})^\top = \begin{pmatrix}2 & -1\end{pmatrix} \qquad \begin{pmatrix}2 & -1\end{pmatrix}\underbrace{\begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix}}_{\Phi} = \begin{pmatrix}0 & 0\end{pmatrix}$$

Structural Analysis

$$(\nu_0^{(1)})^\top = \begin{pmatrix}2 & -1\end{pmatrix} \qquad \begin{pmatrix}2 & -1\end{pmatrix}\underbrace{\begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix}}_{\Phi} = \begin{pmatrix}0 & 0\end{pmatrix}$$

$$\begin{pmatrix}2 & -1\end{pmatrix}\underbrace{\begin{pmatrix}-0.5 & 0.5\\ -1 & 1\end{pmatrix}}_{\Theta} = \begin{pmatrix}0 & 0\end{pmatrix}$$

Structural Analysis

$$(\nu_0^{(1)})^\top = \begin{pmatrix}2 & -1\end{pmatrix} \qquad \begin{pmatrix}2 & -1\end{pmatrix}\underbrace{\begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix}}_{\Phi} = \begin{pmatrix}0 & 0\end{pmatrix}$$

$$\begin{pmatrix}2 & -1\end{pmatrix}\underbrace{\begin{pmatrix}-0.5 & 0.5\\ -1 & 1\end{pmatrix}}_{\Theta} = \begin{pmatrix}0 & 0\end{pmatrix}$$

$$(\nu_0^{(2)})^\top = \begin{pmatrix}-1 & 1\end{pmatrix} \qquad \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix} = \begin{pmatrix}-3 & 1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}-0.5 & 0.5\\ -1 & 1\end{pmatrix} = \begin{pmatrix}-0.5 & 0.5\end{pmatrix}$$

Structural Analysis

$$(\nu_0^{(1)})^\top = \begin{pmatrix}2 & -1\end{pmatrix} \qquad \begin{pmatrix}2 & -1\end{pmatrix}\underbrace{\begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix}}_{\Phi} = \begin{pmatrix}0 & 0\end{pmatrix}$$

$$\begin{pmatrix}2 & -1\end{pmatrix}\underbrace{\begin{pmatrix}-0.5 & 0.5\\ -1 & 1\end{pmatrix}}_{\Theta} = \begin{pmatrix}0 & 0\end{pmatrix}$$

$$(\nu_0^{(2)})^\top = \begin{pmatrix}-1 & 1\end{pmatrix} \qquad \begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}3 & -1\\ 6 & -2\end{pmatrix} = \begin{pmatrix}-3 & 1\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\end{pmatrix}\begin{pmatrix}-0.5 & 0.5\\ -1 & 1\end{pmatrix} = \begin{pmatrix}-0.5 & 0.5\end{pmatrix}$$

$$\nu = \begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}$$

Structural Analysis

$$T=\begin{pmatrix} v_0^{(1)'} \\ v_0^{(2)'} \end{pmatrix}=\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \hline 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -0.5 & 0.5 \\ -1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \hline -0.5 & 0.5 \end{pmatrix}$$

$$\begin{aligned} (1-\Phi B)\underset{\sim}{Z_t} &= (I-\Theta B)\underset{\sim}{a_t} \\ &\Downarrow \\ T(1-\Phi B)\underset{\sim}{Z_t} &= T(I-\Theta B)\underset{\sim}{a_t} \end{aligned}$$

Structural Analysis

$$T=\begin{pmatrix} v_0^{(1)'} \\ v_0^{(2)'} \end{pmatrix}=\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \hline 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -0.5 & 0.5 \\ -1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ \hline -0.5 & 0.5 \end{pmatrix}$$

$$\begin{aligned} (1-\Phi B)\underset{\sim}{Z_t} &= (I-\Theta B)\underset{\sim}{a_t} \\ &\Downarrow \\ T(1-\Phi B)\underset{\sim}{Z_t} &= T(I-\Theta B)\underset{\sim}{a_t} \end{aligned}$$

$$\begin{aligned} T(I-\Phi B)\underset{\sim}{Z_t} &= T\underset{\sim}{Z_t} - T\Phi\underset{\sim}{Z_{t-1}} \\ &= T\underset{\sim}{Z_t} - T\Phi T^{-1}T\underset{\sim}{Z_{t-1}} \end{aligned}$$

Structural Analysis

$$\begin{aligned} & \underset{\sim}{Z_t} - \Phi\underset{\sim}{Z_{t-1}} = \underset{\sim}{a_t} - \Theta\underset{\sim}{a_{t-1}} \\ \Rightarrow\ & T(\underset{\sim}{Z_t} - \Phi\underset{\sim}{Z_{t-1}}) = T(\underset{\sim}{a_t} - \Theta\underset{\sim}{a_{t-1}}) \\ \Rightarrow\ & T\underset{\sim}{Z_t} - T\Phi T^{-1}T\underset{\sim}{Z_{t-1}} = T\underset{\sim}{a_t} - T\Theta T^{-1}T\underset{\sim}{a_{t-1}} \end{aligned}$$

Structural Analysis

$$\begin{aligned} & \underset{\sim}{Z_t} - \Phi\underset{\sim}{Z_{t-1}} = \underset{\sim}{a_t} - \Theta\underset{\sim}{a_{t-1}} \\ \Rightarrow\ & T(\underset{\sim}{Z_t} - \Phi\underset{\sim}{Z_{t-1}}) = T(\underset{\sim}{a_t} - \Theta\underset{\sim}{a_{t-1}}) \\ \Rightarrow\ & T\underset{\sim}{Z_t} - T\Phi T^{-1}T\underset{\sim}{Z_{t-1}} = T\underset{\sim}{a_t} - T\Theta T^{-1}T\underset{\sim}{a_{t-1}} \end{aligned}$$

$$\boxed{M = TAT^{-1}\text{: Similar Transformation}}$$

Structural Analysis

$$\underset{\sim}{Z_t} - \Phi \underset{\sim}{Z_{t-1}} = \underset{\sim}{a_t} - \Theta \underset{\sim}{a_{t-1}}$$

$$\Rightarrow \quad T(\underset{\sim}{Z_t} - \Phi \underset{\sim}{Z_{t-1}}) = T(\underset{\sim}{a_t} - \Theta \underset{\sim}{a_{t-1}})$$

$$\Rightarrow \quad T\underset{\sim}{Z_t} - T\Phi T^{-1} T\underset{\sim}{Z_{t-1}} = T\underset{\sim}{a_t} - T\Theta T^{-1} T\underset{\sim}{a_{t-1}}$$

$$\Rightarrow \quad \underset{\sim}{Y_t} - \Phi^* \underset{\sim}{Y_{t-1}} = \underset{\sim}{b_t} - \Theta^* \underset{\sim}{b_{t-1}}$$

$\boxed{M = TAT^{-1}\text{: Similar Transformation}}$

Structural Analysis

$$\Phi^* = T\Phi T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

Structural Analysis

$$\Phi^* = T\Phi T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\Theta^* = T\Theta T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -0.5 & 0.5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix}$$

Structural Analysis

$$\Phi^* = T\Phi T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\Theta^* = T\Theta T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -0.5 & 0.5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} = \begin{pmatrix} b_{1t} \\ b_{2t} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} b_{1t-1} \\ b_{2t-1} \end{pmatrix}$$

Structural Analysis

$$\Phi^* = T\Phi T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\Theta^* = T\Theta T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -0.5 & 0.5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} = \begin{pmatrix} b_{1t} \\ b_{2t} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} b_{1t-1} \\ b_{2t-1} \end{pmatrix}$$

$$\Uparrow$$

$$\utilde{Y}_t - \Phi^* \utilde{Y}_{t-1} = \utilde{b}_t - \Theta^* \utilde{b}_{t-1}$$

$$\Uparrow$$

$$\utilde{Z}_t - \Phi \utilde{Z}_{t-1} = \utilde{a}_t - \Theta \utilde{a}_{t-1}$$

Structural Analysis

$$\Phi^* = T\Phi T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\Theta^* = T\Theta T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -0.5 & 0.5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \end{pmatrix} = \begin{pmatrix} b_{1t} \\ b_{2t} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} b_{1t-1} \\ b_{2t-1} \end{pmatrix}$$

$$\Uparrow$$

$$\utilde{Y}_t - \Phi^* \utilde{Y}_{t-1} = \utilde{b}_t - \Theta^* \utilde{b}_{t-1} \qquad \boxed{Y_{1t} = b_{1t}}$$

$$\Uparrow$$

$$\utilde{Z}_t - \Phi \utilde{Z}_{t-1} = \utilde{a}_t - \Theta \utilde{a}_{t-1}$$

Structural Analysis

$$Y_{1t} = b_{1t}$$

$$Y_{2t} - 2Y_{1t} - Y_{2,t-1} = b_{2t} - 0.5b_{2,t-1}$$

$$\Rightarrow (1-B)Y_{2t} = 2Y_{1t} + (1-0.5B)b_{2t} \qquad (0,1,1)$$

noise

Structural Analysis

$$\begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \rightarrow \begin{pmatrix} Y_{1t} \\ Y_{2t} \end{pmatrix}$$

$$Y_{1t} = b_{1t}$$

$$Y_{2t} - 2Y_{1t} - Y_{2,t-1} = b_{2t} - 0.5b_{2,t-1}$$

$$\Rightarrow (1-B)Y_{2t} = 2Y_{1t} + (1-0.5B)b_{2t} \qquad (0,1,1)$$

noise

See page 37 of chapter 14

Structural Analysis

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{cases} \underset{\sim}{Y_t} = T\underset{\sim}{Z_t} \\ \underset{\sim}{Z_t} = T^{-1}\underset{\sim}{Y_t} \end{cases}$$

Structural Analysis

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{cases} \underset{\sim}{Y_t} = T\underset{\sim}{Z_t} \\ \underset{\sim}{Z_t} = T^{-1}\underset{\sim}{Y_t} \end{cases} \Longrightarrow \begin{cases} \underset{\sim}{Z_t}: & \text{两者都是nonstationary} \\ & \Downarrow \\ \underset{\sim}{Y_t}: & \text{1 nonstationary \& 1 noise} \end{cases}$$

Structural Analysis

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{cases} \underset{\sim}{Y_t} = T\underset{\sim}{Z_t} \\ \underset{\sim}{Z_t} = T^{-1}\underset{\sim}{Y_t} \end{cases} \Longrightarrow \begin{cases} \underset{\sim}{Z_t}: & \text{两者都是nonstationary} \\ & \Downarrow \\ \underset{\sim}{Y_t}: & \text{1 nonstationary \& 1 noise} \end{cases}$$

Engle & Granger (1987), 2 series are co-integrated;

Structural Analysis

$$\underset{\sim}{Z_t} = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{cases} \underset{\sim}{Y_t} = T\underset{\sim}{Z_t} \\ \underset{\sim}{Z_t} = T^{-1}\underset{\sim}{Y_t} \end{cases} \Longrightarrow \begin{cases} \underset{\sim}{Z_t}: & \text{两者都是nonstationary} \\ & \Downarrow \\ \underset{\sim}{Y_t}: & \text{1 nonstationary \& 1 noise} \end{cases}$$

Engle & Granger (1987), 2 series are co-integrated;

Box & Tiao (1977) *Biometrika*, .

Structural Analysis

$$\underset{\sim}{Z}_t = \begin{pmatrix} Z_{1t} \\ Z_{2t} \end{pmatrix} \qquad T = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\left\{ \begin{array}{lcl} \underset{\sim}{Y}_t & = & T \underset{\sim}{Z}_t \\ \underset{\sim}{Z}_t & = & T^{-1} \underset{\sim}{Y}_t \end{array} \right. \Longrightarrow \left\{ \begin{array}{ll} \underset{\sim}{Z}_t : & \text{两者都是nonstationary} \\ & \Downarrow \\ \underset{\sim}{Y}_t : & \text{1 nonstationary \& 1 noise} \end{array} \right.$$

Engle & G... co-integrated;

Box & Tiao (19... .

Basic Idea 完全一样

Structural Analysis

Tiao & Tsay (1989) ***JRSS.B***

Vector ARMA

$\phi_1 \cdots \phi_P$

$\theta_1 \cdots \theta_q$

Simply structure of Φ and Θ

$$\Phi = \begin{pmatrix} O & \cdots & \cdots & O \\ O & \cdots & \cdots & O \\ O & O & O & O \\ \times & \times & \times & \times \end{pmatrix} \qquad \Theta = \begin{pmatrix} O & O & O & O \\ O & O & O & O \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

ARCH &GARCH

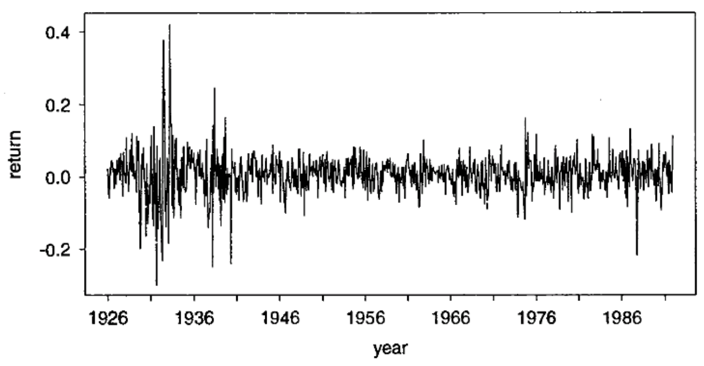

FIGURE 1.8 Value-weighted S&P 500 returns 1926–1991.

see page 8 of chapter 1

ARCH &GARCH

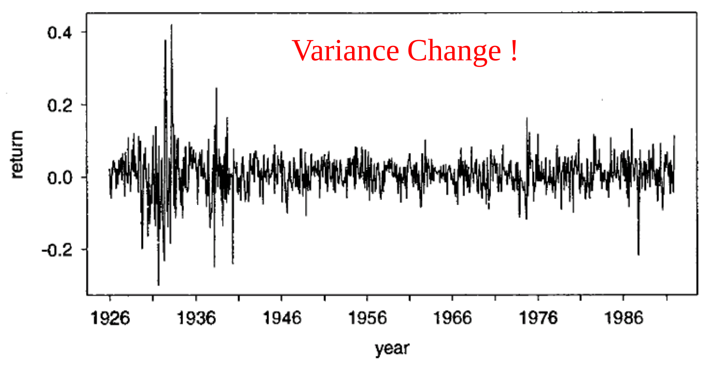

FIGURE 1.8 Value-weighted S&P 500 returns 1926–1991.

see page 8 of chapter 1

时间序列分析

第十六讲(上)

Today's Topic

- Models for Variance
  - Change
  - Dynamics
  - ARCH/GARCH
- Asymptotic Distribution
  - Stationary Case
    - ✓ AR(1)
    - ✓ Box-Pierce-Ljung Q
    - ✓ Auto-correlation for i.i.d. case
  - Nonstationary Case
    - ✓ Unit Root Situation

ARCH &GARCH

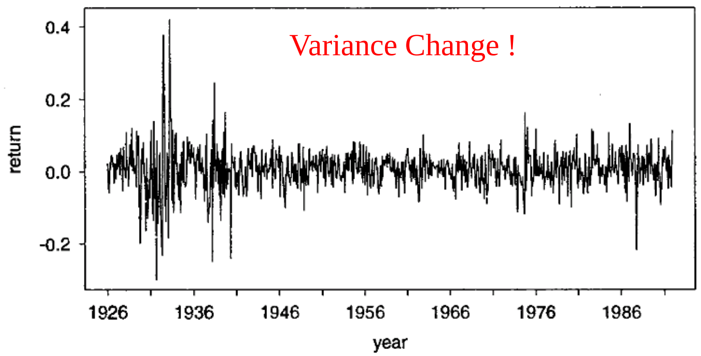

FIGURE 1.8 Value-weighted S&P 500 returns 1926–1991.

see page 8 of chapter 1

ARCH &GARCH

ARMA(p,q) $\phi_q(B)Z_t = \theta_q(B)a_t, \quad \mathrm{Var}(a_t) = \sigma_a^2$

see page 8 of chapter 1

ARCH &GARCH

ARMA(p,q)	$\phi_q(B)Z_t = \theta_q(B)a_t,$	$\mathrm{Var}(a_t) = \sigma_a^2$
Engle (1982)	$\sigma_a^2 \longrightarrow \sigma_t^2$	

see page 8 of chapter 1

ARCH &GARCH

ARMA(p,q)	$\phi_q(B)Z_t = \theta_q(B)a_t,$	$\mathrm{Var}(a_t) = \sigma_a^2$
	$Z_t - \widehat{Z_{t-1}(1)} = a_t$	
	conditional mean:	$\widehat{Z_{t-1}(1)} = \underset{T}{E}(Z_t \vert Z_{t-1}, Z_{t-2}, \cdots)$
Engle (1982)	$\sigma_a^2 \longrightarrow \sigma_t^2$	

see page 8 of chapter 1

ARCH &GARCH

ARMA(p,q)	$\phi_q(B)Z_t = \theta_q(B)a_t,$	$\mathrm{Var}(a_t) = \sigma_a^2$
	$Z_t - \widehat{Z_{t-1}(1)} = a_t$	
	conditional mean:	$\widehat{Z_{t-1}(1)} = \underset{T}{E}(Z_t \vert Z_{t-1}, Z_{t-2}, \cdots)$
Engle (1982)	$\sigma_a^2 \longrightarrow \sigma_t^2$	
	conditional variance:	h

$$\begin{cases} a_t = \sqrt{h}\epsilon_t \\ \mathrm{Var}(\epsilon_t) = 1 \end{cases}$$

see page 8 of chapter 1

ARCH &GARCH

$$\begin{aligned} a_t^2 &= h\epsilon_t^2 \\ E(\epsilon_t^2) &= 1 \\ h_t &= \mathrm{Var}(Z_t|Z_{t-1}, Z_{t-2}, \cdots) \\ &= \omega + \alpha a_{t-1}^2 \end{aligned}$$

see page 8 of chapter 1

ARCH &GARCH

$$\begin{aligned} a_t^2 &= h\epsilon_t^2 \\ E(\epsilon_t^2) &= 1 \\ h_t &= \mathrm{Var}(Z_t|Z_{t-1}, Z_{t-2}, \cdots) \\ &= \omega + \alpha a_{t-1}^2 \end{aligned}$$

$$\begin{aligned} Z_t &= \phi Z_{t-1} + a_t \\ \phi Z_{t-1} &= E(Z_t|Z_{t-1}, \cdots, Z_1) \end{aligned}$$

conditional mean

AR(1)

see page 8 of chapter 1

ARCH &GARCH

$$\begin{aligned} a_t^2 &= h\epsilon_t^2 \\ E(\epsilon_t^2) &= 1 \\ h_t &= \mathrm{Var}(Z_t|Z_{t-1}, Z_{t-2}, \cdots) \\ &= \omega + \alpha a_{t-1}^2 \end{aligned}$$

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conditional mean

AR(1)

see page 8 of chapter 1

ARCH &GARCH

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$$\begin{aligned} Z_t &= \phi Z_{t-1} + a_t \\ \phi Z_{t-1} &= E(Z_t|Z_{t-1}, \cdots, Z_1) \end{aligned}$$

conditional mean

ARCH(1)

AutoRegressive Conditional
Heteroscadastic model

AR(1)

see page 8 of chapter 1

ARCH &GARCH

$$\begin{aligned} a_t^2 &= h\epsilon_t^2 \\ E(\epsilon_t^2) &= 1 \\ h_t &= \mathrm{Var}(Z_t|Z_{t-1}, Z_{t-2}, \cdots) \\ &= \omega + \alpha a_{t-1}^2 \end{aligned}$$

$$\begin{aligned} Z_t &= \phi Z_{t-1} + a_t \\ \phi Z_{t-1} &= E(Z_t|Z_{t-1}, \cdots, Z_1) \end{aligned}$$

conditional mean

ARCH(1)

AutoRegressive Conditional
Heteroscadastic model

ARCH(p)

$$h_t = \omega + \alpha_1 a_{t-1}^2 + \cdots + \alpha_p a_{t-p}^2$$

AR(1)

see page 8 of chapter 1

ARCH &GARCH

1986 Bollerslev *Journal of Econometrics*

Generalized **ARCH**

GARCH(1,1)

$$h_t = \omega + \alpha a_{t-1}^2 + \underline{\beta h_{t-1}}$$

GARCH(p,q)

$$h_t = \omega + \alpha a_{t-1}^2 + \cdots + \alpha_p a_{t-p}^2 + \underline{\beta h_{t-1} + \cdots + \beta_q h_{t-q}}$$

ARCH &GARCH

ARCH(1): $h_t = \omega + \alpha a_{t-1}^2 \qquad h_t = E(a_t^2)$

If we let $a_t^2 = h_t + u_t$

then $h_t = a_t^2 - u_t$

ARCH &GARCH

ARCH(1): $h_t = \omega + \alpha a_{t-1}^2 \qquad h_t = E(a_t^2)$

If we let $a_t^2 = h_t + u_t$

then $h_t = a_t^2 - u_t$

$$a_t^2 - u_t = \omega + \alpha a_{t-1}^2$$

$$a_t^2 = u_t + \omega + \alpha a_{t-1}^2$$

ARCH &GARCH

ARCH(1): $h_t = \omega + \alpha a_{t-1}^2 \qquad h_t = E(a_t^2)$

If we let $a_t^2 = h_t + u_t$

then $h_t = a_t^2 - u_t$

$$a_t^2 - u_t = \omega + \alpha a_{t-1}^2$$

$$a_t^2 = u_t + \omega + \alpha a_{t-1}^2 \Longleftrightarrow \textbf{AR(1)}$$

ARCH &GARCH

ARCH(1): $h_t = \omega + \alpha a_{t-1}^2 \qquad h_t = E(a_t^2)$

If we let $a_t^2 = h_t + u_t$

then $h_t = a_t^2 - u_t$

$$a_t^2 - u_t = \omega + \alpha a_{t-1}^2$$

$$a_t^2 = u_t + \omega + \alpha a_{t-1}^2 \Longleftrightarrow \textbf{AR(1)}$$

can be shown that

u_t uncorrelated with zero mean

& common variance

under certain general condition w.r.t. parameters

ARCH &GARCH

GARCH(1,1): $h_t = \omega + \alpha a_{t-1}^2 + \beta h_{t-1}$

If we let $a_t^2 = h_t + u_t$

then $h_t = a_t^2 - u_t$

$$a_t^2 - u_t = \omega + \alpha a_{t-1}^2 + \beta(a_{t-1}^2 - u_{t-1})$$

$$a_t^2 = u_t + \omega + (\alpha + \beta)a_{t-1}^2 + u_t - \beta u_{t-1} \Longleftrightarrow \textbf{ARMA(1,1)}$$

ARCH &GARCH

Examples: Generated GARCH(1,1)
S&P 500

ARCH &GARCH

A Couple of Further Comments:

(1) outlier checking + variance change model

(2) $\sigma_a^2 \to \sigma_t^2$

- *periodic variance*
- *weather data*
- *水利 data*

(3) level shift in σ_t^2

- *C. Inclan JASA*

Distribution Theory

1. Asymptotic distribution of $\hat{\phi}$ for AR(1) model

2. Box-Pierce Q stat $\chi^2_{m(p+q)}$ special case of AR(1)

Basic result: Asymptotic Distribution of $\dfrac{\sum a_t a_{t-l}}{\sum a_t^2}$

i.i.d. case stationary

3. Unit Root Situation

How stationary result breaks down.

simple AR(1) case with $\phi = 1$

(Notes:p197-p217)

AR(1)

$$Z_t = \phi Z_{t-1} + a_t \;\Rightarrow\; Z_{t-1}Z_t = \phi Z_{t-1}^2 + Z_{t-1}a_t$$

$$\Rightarrow \sum Z_{t-1}Z_t = \sum \phi Z_{t-1}^2 + \sum Z_{t-1}a_t$$

$$\Rightarrow \frac{\sum Z_{t-1}Z_t}{\sum Z_{t-1}^2} = \phi + \frac{\sum Z_{t-1}a_t}{\sum Z_{t-1}^2}$$

$$\Rightarrow \hat{\phi} - \phi = \frac{\sum Z_{t-1}a_t}{\sum Z_{t-1}^2} \qquad \left(\hat{\phi} = \frac{\sum Z_{t-1}Z_t}{\sum Z_{t-1}^2}\right)$$

$$\Rightarrow \sqrt{n}(\hat{\phi} - \phi) = \frac{\frac{1}{n}\sum Z_{t-1}a_t}{\frac{1}{n}\sum Z_{t-1}^2}\sqrt{n}$$

$$\Rightarrow \sqrt{n}(\hat{\phi} - \phi) = \frac{\frac{1}{\sqrt{n}}\sum Z_{t-1}a_t}{\frac{1}{n}\sum Z_{t-1}^2}$$

AR(1)

$$Z_t = \phi Z_{t-1} + a_t \;\Rightarrow\; Z_{t-1}Z_t = \phi Z_{t-1}^2 + Z_{t-1}a_t$$

$$\Rightarrow \sum Z_{t-1}Z_t = \sum \phi Z_{t-1}^2 + \sum Z_{t-1}a_t$$

$$\Rightarrow \frac{\sum Z_{t-1}Z_t}{\sum Z_{t-1}^2} = \phi + \frac{\sum Z_{t-1}a_t}{\sum Z_{t-1}^2}$$

$$\Rightarrow \hat{\phi} - \phi = \frac{\sum Z_{t-1}a_t}{\sum Z_{t-1}^2} \qquad \left(\hat{\phi} = \frac{\sum Z_{t-1}Z_t}{\sum Z_{t-1}^2}\right)$$

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$$\Rightarrow \sqrt{n}(\hat{\phi} - \phi) = \frac{\frac{1}{\sqrt{n}}\sum Z_{t-1}a_t}{\frac{1}{n}\sum Z_{t-1}^2}$$

时间序列分析

第十六讲(下)

Today's Topic

- Models for Variance
  - Change
  - Dynamics
  - ARCH/GARCH
- Asymptotic Distribution
  - Stationary Case
    - AR(1)
    - Box-Pierce-Ljung Q
    - Auto-correlation for i.i.d. case
  - Nonstationary Case
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AR(1)

$$Z_t = \phi Z_{t-1} + a_t \quad \Rightarrow \quad Z_{t-1}Z_t = \phi Z_{t-1}^2 + Z_{t-1}a_t$$

$$\Rightarrow \quad \sum Z_{t-1}Z_t = \sum \phi Z_{t-1}^2 + \sum Z_{t-1}a_t$$

$$\Rightarrow \quad \frac{\sum Z_{t-1}Z_t}{\sum Z_{t-1}^2} = \phi + \frac{\sum Z_{t-1}a_t}{\sum Z_{t-1}^2}$$

$$\Rightarrow \quad \widehat{\phi} - \phi = \frac{\sum Z_{t-1}a_t}{\sum Z_{t-1}^2} \qquad (\widehat{\phi} = \frac{\sum Z_{t-1}Z_t}{\sum Z_{t-1}^2})$$

$$\Rightarrow \quad \sqrt{n}(\widehat{\phi} - \phi) = \frac{\frac{1}{n}\sum Z_{t-1}a_t}{\frac{1}{n}\sum Z_{t-1}^2}\sqrt{n}$$

$$\Rightarrow \quad \sqrt{n}(\widehat{\phi} - \phi) = \frac{\frac{1}{\sqrt{n}}\sum Z_{t-1}a_t}{\frac{1}{n}\sum Z_{t-1}^2}$$

AR(1)

$$Z_t = \phi Z_{t-1} + a_t \quad \Rightarrow \quad Z_{t-1}Z_t = \phi Z_{t-1}^2 + Z_{t-1}a_t$$

$$\Rightarrow \quad \sum Z_{t-1}Z_t = \sum \phi Z_{t-1}^2 + \sum Z_{t-1}a_t$$

$$\Rightarrow \quad \frac{\sum Z_{t-1}Z_t}{\sum Z_{t-1}^2} = \phi + \frac{\sum Z_{t-1}a_t}{\sum Z_{t-1}^2}$$

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$$\Rightarrow \quad \sqrt{n}(\widehat{\phi} - \phi) = \frac{\frac{1}{\sqrt{n}}\sum Z_{t-1}a_t}{\frac{1}{n}\sum Z_{t-1}^2}$$

Assume Stationarity $|\phi| < 1$

$$\frac{1}{n}\sum a_t^2 \xrightarrow{L} \sigma_a^2$$

$$\frac{1}{n}\sum Z_{t-1}^2 \xrightarrow{L} \gamma_0 = \frac{\sigma_a^2}{1-\phi^2}$$

AR(1)

$$
\begin{aligned}
& \sqrt{n}(\widehat{\phi}-\phi) \\
\approx \quad & (1-\phi^2)\sqrt{n}\frac{\sum a_t Z_{t-1}}{\sum a_t^2} \\
= \quad & (1-\phi^2)\sqrt{n}\frac{\sum a_t(a_{t-1}+\phi a_{t-2}+\phi^2 a_{t-3}+\cdots)}{\sum a_t^2} \\
= \quad & (1-\phi^2)\sqrt{n}\left(\frac{\sum a_t a_{t-1}}{\sum a_t^2}+\frac{\sum a_t a_{t-2}}{\sum a_t^2}\phi+\frac{\sum a_t a_{t-3}}{\sum a_t^2}\phi^2+\cdots\right) \\
= \quad & (1-\phi^2)\sqrt{n}(\mathfrak{R}_1^*+\mathfrak{R}_2^*\phi+\mathfrak{R}_3^*\phi^2+\cdots)
\end{aligned}
$$

$$a_t \overset{i.i.d.}{\sim} N(0,\sigma^2)$$

AR(1)

$$
\begin{aligned}
& \sqrt{n}(\widehat{\phi}-\phi) \\
\approx \quad & (1-\phi^2)\sqrt{n}\frac{\sum a_t Z_{t-1}}{\sum a_t^2} \\
= \quad & (1-\phi^2)\sqrt{n}\frac{\sum a_t(a_{t-1}+\phi a_{t-2}+\phi^2 a_{t-3}+\cdots)}{\sum a_t^2} \\
= \quad & (1-\phi^2)\sqrt{n}\left(\frac{\sum a_t a_{t-1}}{\sum a_t^2}+\frac{\sum a_t a_{t-2}}{\sum a_t^2}\phi+\frac{\sum a_t a_{t-3}}{\sum a_t^2}\phi^2+\cdots\right) \\
= \quad & (1-\phi^2)\sqrt{n}(\mathfrak{R}_1^*+\mathfrak{R}_2^*\phi+\mathfrak{R}_3^*\phi^2+\cdots) \\
\text{Let} \quad & X_i=\sqrt{n}\mathfrak{R}_i^* \\
\text{Since} \quad & a_t \overset{i.i.d.}{\sim} N(0,\sigma^2) \\
\text{Then} \quad & \text{Joint asymptotic distribution of } \underset{\sim}{X} \text{ is } N(0,I) \\
& \text{(by R. L. Anderson, 1941)}
\end{aligned}
$$

AR(1)

Assume for large n

$$
\left\{
\begin{aligned}
\begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} &= \sqrt{n}\begin{pmatrix} \mathfrak{R}_1^* \\ \vdots \\ \mathfrak{R}_\ell^* \end{pmatrix} \sim N(0,I) \\
\sqrt{n}(\widehat{\phi}-\phi) &= (1-\phi^2)(x_1+\phi x_2+\phi^2 x_3+\cdots)
\end{aligned}
\right.
$$

$$
\sqrt{n}(\widehat{\phi}-\phi)
\left\{
\begin{array}{l}
\text{Normally Distributed} \\
\mu = 0 \\
\text{Var} = \dfrac{(1-\phi^2)^2}{1-\phi^2} = 1-\phi^2
\end{array}
\right.
$$

AR(1)

Assume for large n

$$
\left\{
\begin{aligned}
\begin{pmatrix} x_1 \\ \vdots \\ x_\ell \end{pmatrix} &= \sqrt{n}\begin{pmatrix} \mathfrak{R}_1^* \\ \vdots \\ \mathfrak{R}_\ell^* \end{pmatrix} \sim N(0,I) \\
\sqrt{n}(\widehat{\phi}-\phi) &= (1-\phi^2)(x_1+\phi x_2+\phi^2 x_3+\cdots)
\end{aligned}
\right.
$$

$$
\sqrt{n}(\widehat{\phi}-\phi)
\left\{
\begin{array}{l}
\text{Normally Distributed} \\
\mu = 0 \\
\text{Var} = \dfrac{(1-\phi^2)^2}{1-\phi^2} = 1-\phi^2
\end{array}
\right.
$$

$$\sqrt{n}(\widehat{\phi}-\phi) \sim N(0, 1-\phi^2)$$

Key simple result !

$$\widehat{\phi}-\phi \sim N\left(0, \frac{1-\phi^2}{n}\right)$$

(Given $|\phi| < 1$)

Box-Pierce Q stat

Box & Pierce (*JASA* 1970)

Residuals from ARMA(p,q) :

$$\begin{aligned}\widehat{a_t} &= Z_t - \widehat{\phi_1} Z_{t-1} - \cdots - \widehat{\phi_p} Z_{t-p} + \widehat{\theta}_1 \widehat{a_{t-1}} + \cdots + \widehat{\theta}_p \widehat{a_{t-p}} \\ Q &= n \sum_{\ell=1}^{m} \Re_\ell^2(\widehat{a_t}) = n \sum_{\ell=1}^{m} x_\ell^2 \\ \Re_\ell(\widehat{a_t}) &= \ell\text{-th sample autocorrelations of residuals } \widehat{a_t}\end{aligned}$$

⟵ Is this a chi-square?

• See Page 205-210 of Notes

Box-Pierce Q stat

Box & Pierce (*JASA* 1970)

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⟵ Is this a chi-square?

$$\begin{aligned}\text{if} \quad & \widehat{a_t} = a_t \\ \text{since} \quad & Z_t = \phi_1 Z_{t-1} + \cdots + \phi_p Z_{t-p} + a_t - \theta_1 a_{t-1} - \cdots \theta_q a_{t-q} \\ \text{we have} \quad & Q \equiv Q^* = \sum_{\ell=1}^{m} n(\Re_\ell^*)^2 \dot\sim \chi_m^2\end{aligned}$$

• See Page 205-210 of Notes

Box-Pierce Q stat

Box & Pierce (*JASA* 1970)

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white noise has lag ℓ autocorrelation with zero mean and asymptotic variance $1/n$

• See Page 205-210 of Notes

Box-Pierce Q stat

Box & Pierce (*JASA* 1970)

Residuals from ARMA(p,q) :

$$\begin{aligned}\widehat{a_t} &= Z_t - \widehat{\phi_1} Z_{t-1} - \cdots - \widehat{\phi_p} Z_{t-p} + \widehat{\theta}_1 \widehat{a_{t-1}} + \cdots + \widehat{\theta}_p \widehat{a_{t-p}} \\ Q &= n \sum_{\ell=1}^{m} \Re_\ell^2(\widehat{a_t}) = n \sum_{\ell=1}^{m} x_\ell^2 \\ \Re_\ell(\widehat{a_t}) &= \ell\text{-th sample autocorrelations of residuals } \widehat{a_t}\end{aligned}$$

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But $\widehat{a_t}$'s are not a_t's but residuals !

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Box-Pierce Q stat

AR(1):

$$\begin{aligned}
\Re_\ell(\widehat{a}_t) &= \frac{\sum \widehat{a}_t \widehat{a_{t-\ell}}}{\sum \widehat{a}_t^{\,2}} \\
Z_t &= \phi Z_{t-1} + a_t \\
\widehat{a}_t &= Z_t - \widehat{\phi} Z_{t-1} \\
&= Z_t - \phi Z_{t-1} - (\widehat{\phi} - \phi) Z_{t-1} \\
&= a_t - \delta Z_{t-1}
\end{aligned}$$

• See Page 205-210 of Notes

Box-Pierce Q stat

AR(1):

$$\Re_\ell(\widehat{a}_t) = \frac{\sum \widehat{a}_t \widehat{a_{t-\ell}}}{\sum \widehat{a}_t^{\,2}} = \frac{\sum (a_t - \widehat{\delta} Z_{t-1})(a_{t-\ell} - \widehat{\delta} Z_{t-\ell-1})}{\sum (a_t - \widehat{\delta} Z_{t-1})^2}$$

Skip the rest, go to page 210

$$\Re_\ell^*(a_t) \implies \sqrt{n}\Re_\ell^*(a_t) = \sqrt{n}\frac{\sum a_t a_{t-\ell}}{\sum a_t^2}$$

$$Q = n \sum_{\ell=1}^{m} \Re_\ell^2(\widehat{a}_t) \sim \chi^2_{m-(p+q)}$$

• See Page 205-210 of Notes

Box-Pierce Q stat

AR(1):

$$\Re_\ell(\widehat{a}_t) = \frac{\sum \widehat{a}_t \widehat{a_{t-\ell}}}{\sum \widehat{a}_t^{\,2}} = \frac{\sum (a_t - \widehat{\delta} Z_{t-1})(a_{t-\ell} - \widehat{\delta} Z_{t-\ell-1})}{\sum (a_t - \widehat{\delta} Z_{t-1})^2}$$

Skip the rest, go to page 210

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$$Q = n \sum_{\ell=1}^{m} \Re_\ell^2(\widehat{a}_t) \sim \chi^2_{m-(p+q)}$$

• See Page 205-210 of Notes

Box-Pierce Q stat

$$\text{if} \quad a_t \overset{i.i.d.}{\sim} N(0,1)$$

$$\text{then} \quad \sqrt{n}\Re_\ell^*(a_t) = \sqrt{n}\frac{\sum a_t a_{t-\ell}}{\sum a_t^2} = \frac{\frac{1}{\sqrt{n}}\sum a_t a_{t-\ell}}{\frac{1}{n}\sum a_t^2}$$

See page 201-204 of Notes

Box-Pierce Q stat

if $\quad a_t \overset{i.i.d.}{\sim} N(0,1)$

then $\quad \sqrt{n}\Re_\ell^*(a_t) = \sqrt{n}\dfrac{\sum a_t a_{t-\ell}}{\sum a_t^2} = \dfrac{\frac{1}{\sqrt{n}}\sum a_t a_{t-\ell}}{\frac{1}{n}\sum a_t^2}$

C.L.T. does NOT seem to work for $a_t a_{t-1}$!

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Box-Pierce Q stat

$t = 1, 2, \cdots, n \qquad \sum a_t a_{t-1}$

$a_1 a_2 \quad a_2 a_3 \quad a_3 a_4 \quad a_4 a_5 \quad a_5 a_6 \quad a_6 a_7 \quad a_7 a_8$

Correlated Correlated Correlated Correlated Correlated Correlated

See page 201-204 of Notes

Box-Pierce Q stat

$t = 1, 2, \cdots, n \qquad \sum a_t a_{t-1}$

$a_1 a_2 \quad a_2 a_3 \quad a_3 a_4 \quad a_4 a_5 \quad a_5 a_6 \quad a_6 a_7 \quad a_7 a_8$

Correlated Correlated Correlated Correlated Correlated Correlated

$a_1 a_2 \quad a_2 a_3 \quad a_3 a_4 \quad a_4 a_5 \quad a_5 a_6 \quad a_6 a_7 \quad a_7 a_8$

Independent!

weak dependence问题

See page 201-204 of Notes

Box-Pierce Q stat

$t = 1, 2, \cdots, n \qquad \sum a_t a_{t-1}$

$a_1 a_2 \quad a_2 a_3 \quad a_3 a_4 \quad a_4 a_5 \quad a_5 a_6 \quad a_6 a_7 \quad a_7 a_8$

Correlated Correlated Correlated Correlated Correlated Correlated

$a_1 a_2 \quad a_2 a_3 \quad a_3 a_4 \quad a_4 a_5 \quad a_5 a_6 \quad a_6 a_7 \quad a_7 a_8$

Independent!

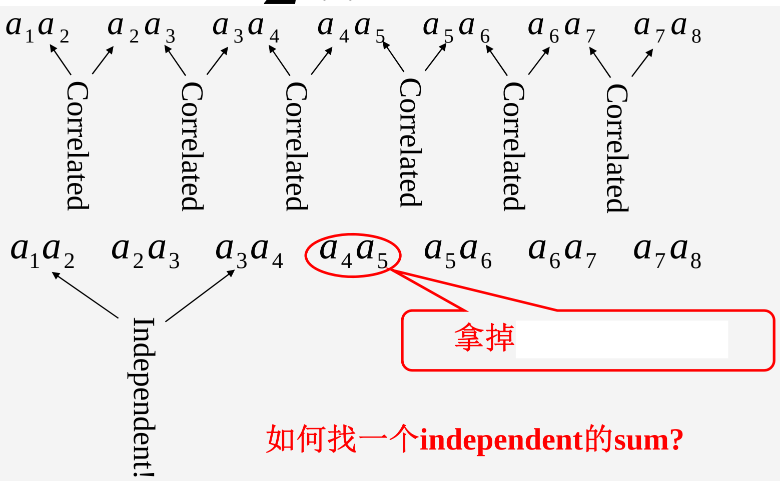
如何找一个independent的sum?

weak dependence问题

See page 201-204 of Notes

Box-Pierce Q stat

$t = 1, 2, \cdots, n \qquad \sum a_t a_{t-1}$

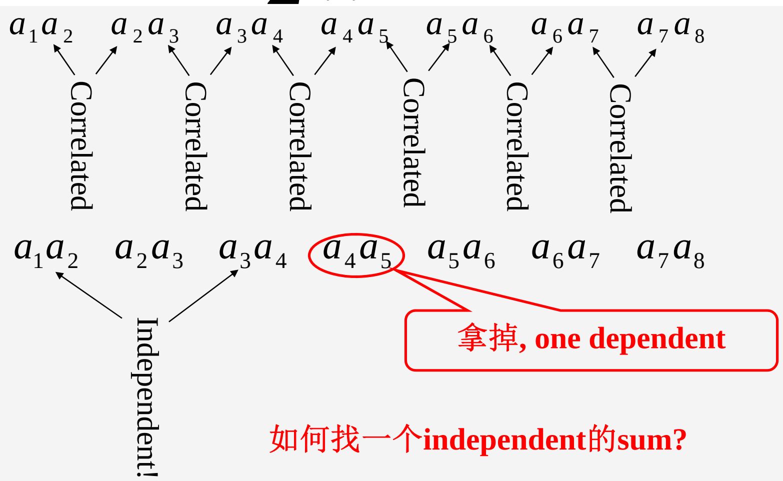


weak dependence问题

See page 201-204 of Notes

Box-Pierce Q stat

$t = 1, 2, \cdots, n \qquad \sum a_t a_{t-1}$



weak dependence问题

See page 201-204 of Notes

Unit Root Problem

Let AR(1) for example

$$Z_t - \phi Z_{t-1} = a_t$$

If $\phi = 1$, then it is a random walk.

Q: What happen to distribution of $\widehat{\phi}$?

A: Take stock for example

Difference is very important in practical sense

Ignore the distribution theory

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Unit Root Problem

- White and M.M. Rao (1950)
- Dickey & Fuller(1979) useful result, very popular in *JASA*
- Chan & Wei(1988) best paper in *Annual of Statistics*

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香港中文大学

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Unit Root Problem

Ahtola

$$\sqrt{n}(\hat{\phi}-\phi)=\frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}}\frac{\sum a_t Z_{t-1}}{\sum Z_{t-1}^2} \longrightarrow ?$$

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时间序列分析

第十七讲(上)

Today's Topic

- Unobserved Component Models
- Seasonal adjustment
  - Filter (Empirical) Approach
  - Model based approach
- X-12-ARIMA
- TRAMO/SEATS
- Kalman Filter
- Review of Course

Some Reflections on Decomposition and Seasonal Adjustment of Economic Time Series

George C. Tiao
The University of Chicago

Useful References

- Bell and Hillmer (*JBES*, 1984)

  Issues Involved with the Seasonal Adjustment of Time Series
- A. Maravall (in *Handbook of Applied Economics*, 1993)

  Unobserved Components in Economic Time Series

  Software: TRAMO/SEATS
- Findley, Monsell, Bell, Otto, Chen (*JBES*, 1998)

  New Capabilities and Methods of the X-12-ARIMA Seasonal Adjustment Program

Outline of Topics

- **Additive Decomposition**
  - **Deterministic components – Regression**
  - **Random components**
  - **Symmetric moving filters**
- **A Model for X-11**
- **Model-based decomposition – pros and cons**
- **Canonical decomposition**
- **X-12-ARIMA *v.s.* TRAMO/SEATS**
- **Forecasting Components**
- **Variances of errors of component forecasts**
- **Unifying X-12-ARIMA and model-based decomposition**

Additive Decomposition

$$Z_t = D_t + R_t$$ (for today, t = month)

D_t : Deterministic components including

Trading day, day of week, holiday, intervention, missing observations, outliers, level shifts and other special effects

R_t : Random components including

trend, seasonality, cycle and irregular

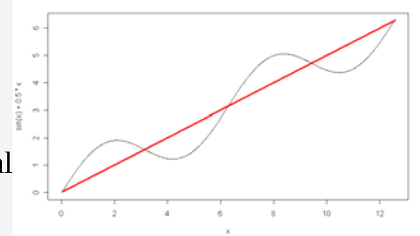
The deterministic components are typically taken care of via linear and nonlinear regression.

The random components are, however, more difficult to handle.

$$Z_t = D_t + R_t$$

$$Z_t = T_t + S_t + N_t$$

T_t : stochastic trend
S_t : stochastic seasonal
R_t : random noise



(unobserved components models, state space models, structural models)

The question is : how to estimate the seasonal component S_t and subtract it from data Z_t to get the 'seasonal adjusted' series

U.S. Bureau of the Census has a computer program, called X-11, to do the seasonal adjustment. (by Julius Shiskin)

Symmetric filter

$$\mathbf{M}: \quad \delta_0 + \delta_1(B+F) + \delta_2(B^2+F^2) + \cdots$$

where B is the back-shift operator and $F=B^{-1}$ the forward-shift operator

Symmetric filter

$$\mathbf{M}: \quad \left\{ \delta_0 + \delta_1(B+F) + \delta_2(B^2+F^2) + \cdots \right\} Z_t$$

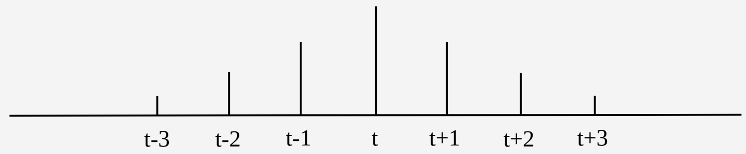
where B is the back-shift operator and $F=B^{-1}$ the forward-shift operator

Symmetric filter

$$\mathbf{M}: \quad \left\{ \delta_0 + \delta_1(B+F) + \delta_2(B^2+F^2) + \cdots \right\} Z_t$$

where B is the back-shift operator and $F=B^{-1}$ the forward-shift operator

$$X_t = MZ_t : \quad \delta_0 Z_t + \delta_1(Z_{t-1}+Z_{t+1}) + \delta_2(Z_{t-2}+Z_{t+2}) + \cdots$$



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Symmetric filter

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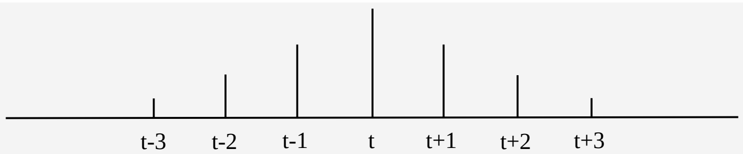

Cleveland & Tiao (1976)

Harvey & Todd (1983)

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$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

See Page 189-196 of Packets

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

1. If we know model for S_t and model for N_t, what is the model for Z_t?

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$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

1. If we know model for S_t and model for N_t, what is the model for Z_t? **E a s y !**
2. If we know model for S_t and model for N_t, from observation Z_t, how to estimate S_t?

See Page 189-196 of Packets

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

1. If we know model for S_t and model for N_t, what is the model for Z_t? **E a s y !**
2. If we know model for S_t and model for N_t, from observation Z_t, how to estimate S_t? **Relatively Easy !**
3. observed $\{Z_t\}$, model $\{Z_t\}$, under what circumstance can we estimate S_t?

See Page 189-196 of Packets

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

1. If we know model for S_t and model for N_t, what is the model for Z_t? **E a s y !**
2. If we know model for S_t and model for N_t, from observation Z_t, how to estimate S_t? **Relatively Easy !**
3. observed $\{Z_t\}$, model $\{Z_t\}$, under what circumstance can we estimate S_t? **Decomposition Problem!!**

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$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$

1. If we know model for S_t and model for N_t, what is the model for Z_t?

Answer: If

$$
\begin{aligned}
(1-\phi B)S_t &= b_t \\
N_t &= e_t \\
\{b_t\} &\perp \{e_t\} \\
\{b_t\},\ \{e_t\} &\text{ are i.i.d. series separately.}
\end{aligned}
$$

Then $\underbrace{(1-\phi B)Z_t}_{\text{AR(1)}} = b_t + \underbrace{(1-\phi B)e_t}_{\text{MA(1)}}$

$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$

1. If we know model for S_t and model for N_t, what is the model for Z_t?

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\end{aligned}
$$

Then $\underbrace{(1-\phi B)Z_t = b_t + (1-\phi B)e_t}_{\{Z_t\}\text{ model: ARMA(1,1)}}$

$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$

2. If we know model for S_t and model for N_t, from observation Z_t, how to estimate S_t?

Answer: If $\{S_t\}$ follows $AR(1)$

$$
\begin{aligned}
N_t &= e_t \quad \text{(white noise)} \\
\text{Var}(e_t) &= \sigma_e^2
\end{aligned}
$$

Then

$$
\begin{aligned}
P(\underset{\sim}{N}) &\propto \sigma_e^{-n}\exp(-\frac{1}{2\sigma_e^2}\underset{\sim}{N}^\top \underset{\sim}{N}) \\
P(\underset{\sim}{S}) &\propto (1-\phi^2)^{1/2}\sigma_b^{-n} \\
&\quad \exp\{-\frac{1}{2\sigma_e^2}[(1-\phi^2)S_1^2 + \sum_{t=2}^{n}(S_t - \phi S_{t-1})^2]\}
\end{aligned}
$$

$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$

2. If we know model for S_t and model for N_t, from observation Z_t, how to estimate S_t?

$$
\begin{aligned}
P(\underset{\sim}{N}) &\propto \sigma_e^{-n}\exp(-\frac{1}{2\sigma_e^2}\underset{\sim}{N}^\top \underset{\sim}{N}) \\
P(\underset{\sim}{S}) &\propto |R^{-1}|^{1/2}\exp(-\frac{1}{2}\underset{\sim}{S}^\top R^{-1}\underset{\sim}{S})
\end{aligned}
$$

$$
\text{where} \quad R = (1-\phi^2)\begin{pmatrix} 1 & \phi & \cdots & \phi^{n-1} \\ \phi & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \phi \\ \phi^{n-1} & \cdots & \phi & 1 \end{pmatrix}
$$

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$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

3. observed $\{Z_t\}$, model $\{Z_t\}$, under what circumstance can we estimate S_t?

We want to know the distributions of $\underset{\sim}{N}$, $\underset{\sim}{S}$, but we only have $\underset{\sim}{Z}$, how to estimate $\underset{\sim}{S}$ from $\underset{\sim}{Z}$, i.e., $P(\underset{\sim}{S}|\underset{\sim}{Z})$?

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

3. observed $\{Z_t\}$, model $\{Z_t\}$, under what circumstance can we estimate S_t?

We want to know the distributions of $\underset{\sim}{N}$, $\underset{\sim}{S}$, but we only have $\underset{\sim}{Z}$, how to estimate $\underset{\sim}{S}$ from $\underset{\sim}{Z}$, i.e., $P(\underset{\sim}{S}|\underset{\sim}{Z})$?

$$P(\underset{\sim}{S}, \underset{\sim}{N}) \quad \rightarrow \quad P(\underset{\sim}{S}, \underset{\sim}{Z})$$

$$\underset{\sim}{Z} = \underset{\sim}{N} + \underset{\sim}{S} \qquad \underset{\sim}{N} = \underset{\sim}{Z} - \underset{\sim}{S}$$

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

3. observed $\{Z_t\}$, model $\{Z_t\}$, under what circumstance can we estimate S_t?

$$\begin{aligned} P(\underset{\sim}{S}|\underset{\sim}{Z}) &= \frac{P(\underset{\sim}{S})P(\underset{\sim}{N})}{P(\underset{\sim}{Z})} \\ &= \frac{P(\underset{\sim}{S})P(\underset{\sim}{Z}-\underset{\sim}{S})}{P(\underset{\sim}{Z})} \qquad (\underset{\sim}{N} = \underset{\sim}{Z} - \underset{\sim}{S}) \end{aligned}$$

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

3. observed $\{Z_t\}$, model $\{Z_t\}$, under what circumstance can we estimate S_t?

$$\begin{aligned} P(\underset{\sim}{S}|\underset{\sim}{Z}) &= \frac{P(\underset{\sim}{S})P(\underset{\sim}{N})}{P(\underset{\sim}{Z})} \\ &= \frac{P(\underset{\sim}{S})P(\underset{\sim}{Z}-\underset{\sim}{S})}{P(\underset{\sim}{Z})} \qquad (\underset{\sim}{N} = \underset{\sim}{Z} - \underset{\sim}{S}) \end{aligned}$$

$$\begin{aligned} P(\underset{\sim}{S}) &\propto |R^{-1}|^{1/2} \exp(-\frac{1}{2}\underset{\sim}{S}^\top R^{-1}\underset{\sim}{S}) \\ P(\underset{\sim}{Z}-\underset{\sim}{S}) &= \sigma_e^{-n} \exp(-\frac{1}{2\sigma_e^2}(\underset{\sim}{Z}-\underset{\sim}{S})^\top(\underset{\sim}{Z}-\underset{\sim}{S})) \end{aligned}$$

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

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$$
\begin{aligned}
P(\underset{\sim}{S}|\underset{\sim}{Z}) &= \frac{P(\underset{\sim}{S})P(\underset{\sim}{N})}{P(\underset{\sim}{Z})} \\
&= \frac{P(\underset{\sim}{S})P(\underset{\sim}{Z}-\underset{\sim}{S})}{P(\underset{\sim}{Z})} \\
P(\underset{\sim}{S}) &\propto |R^{-1}|^{1/2}\exp(-\frac{1}{2}\underset{\sim}{S}^\top R^{-1}\underset{\sim}{S}) \\
P(\underset{\sim}{Z}-\underset{\sim}{S}) &= \sigma_e^{-n}\exp(-\frac{1}{2\sigma_e^2}(\underset{\sim}{Z}-\underset{\sim}{S})^\top(\underset{\sim}{Z}-\underset{\sim}{S}))
\end{aligned}
$$

quadratic forms

A Very Useful Identity in Combing Two Quadratic Forms

Suppose	(1) $\underset{\sim}{x}$, $\underset{\sim}{a}$, $\underset{\sim}{b}$ are $k\times 1$ vectors
	(2) A and B are $k\times k$ matrix
2 quadratic forms	$(\underset{\sim}{x}-\underset{\sim}{a})^\top A(\underset{\sim}{x}-\underset{\sim}{a})$
	$(\underset{\sim}{x}-\underset{\sim}{b})^\top B(\underset{\sim}{x}-\underset{\sim}{b})$

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If $(A+B)^{-1}$ exist, then

$$
\begin{aligned}
&(\underset{\sim}{x}-\underset{\sim}{a})^\top A(\underset{\sim}{x}-\underset{\sim}{a}) + (\underset{\sim}{x}-\underset{\sim}{b})^\top B(\underset{\sim}{x}-\underset{\sim}{b}) \\
= \; &(\underset{\sim}{x}-\underset{\sim}{c})^\top (A+B)(\underset{\sim}{x}-\underset{\sim}{c}) + (\underset{\sim}{a}-\underset{\sim}{b})^\top A(A+B)^{-1}B(\underset{\sim}{a}-\underset{\sim}{b})
\end{aligned}
$$

where $\underset{\sim}{c} = (A+B)^{-1}(A\underset{\sim}{a}+B\underset{\sim}{b})$

$$Z_t = S_t + N_t = \text{Singal} + \text{Noise}$$

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$$
\begin{aligned}
P(\underset{\sim}{S}|\underset{\sim}{Z}) &= \frac{P(\underset{\sim}{S})P(\underset{\sim}{N})}{P(\underset{\sim}{Z})} \\
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P(\underset{\sim}{S}) &\propto |R^{-1}|^{1/2}\exp(-\frac{1}{2}\underset{\sim}{S}^\top R^{-1}\underset{\sim}{S}) \\
P(\underset{\sim}{Z}-\underset{\sim}{S}) &= \sigma_e^{-n}\exp(-\frac{1}{2\sigma_e^2}(\underset{\sim}{Z}-\underset{\sim}{S})^\top(\underset{\sim}{Z}-\underset{\sim}{S}))
\end{aligned}
$$

quadratic forms

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P(\underset{\sim}{Z}-\underset{\sim}{S}) &= \sigma_e^{-n}\exp(-\frac{1}{2\sigma_e^2}(\underset{\sim}{Z}-\underset{\sim}{S})^\top(\underset{\sim}{Z}-\underset{\sim}{S})) \\
\widehat{S}_t &= E(\underset{\sim}{S}|\underset{\sim}{Z}) \\
&= \{\omega_0 + \omega_1(B+F) + \omega_2(B^2+F^2) + \cdots\}Z_t
\end{aligned}
$$

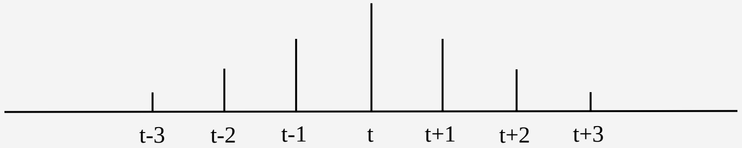
quadratic forms

Symmetric filter

$$\mathbf{M}: \quad \{\delta_0 + \delta_1(B+F) + \delta_2(B^2+F^2) + \cdots\} Z_t$$

where B is the back-shift operator and $F=B^{-1}$ the forward-shift operator

$$X_t = MZ_t: \quad \delta_0 Z_t + \delta_1(Z_{t-1}+Z_{t+1}) + \delta_2(Z_{t-2}+Z_{t+2}) + \cdots$$



Cleveland & Tiao (1976)

Harvey & Todd (1983)

See Page 189-196 of Packets

Symmetric filter

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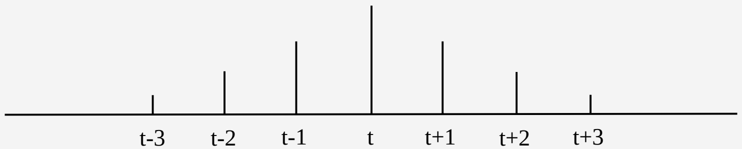

In X-11, specific filters are used (with options) to estimate S_t and T_t :

$$\widehat{S}_t = M^{(S)}Z_t \quad \text{and} \quad \widehat{T}_t = M^{(T)}Z_t$$

with considerable successes.

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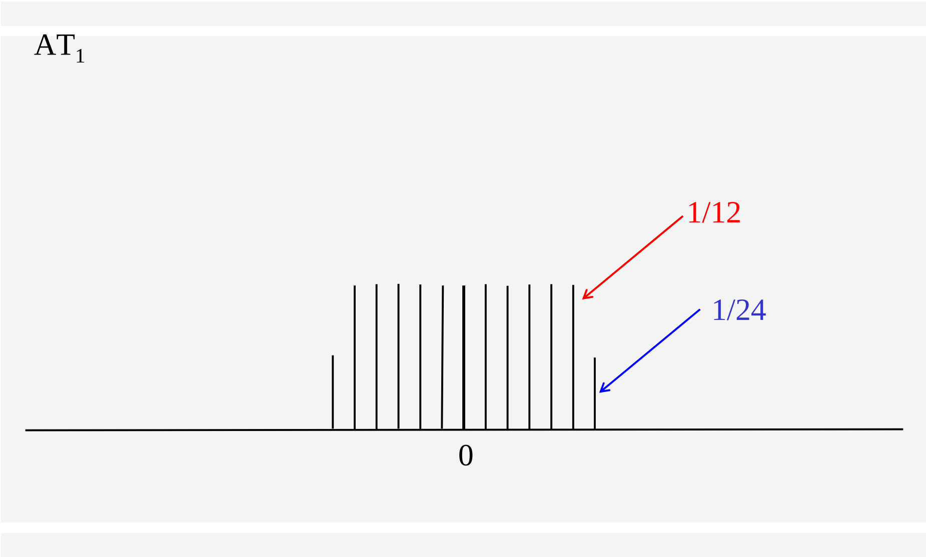
$$
\begin{aligned}
\underset{\sim}{Z_t} &= \underset{\sim}{S_t} + \underset{\sim}{T_t} + \underset{\sim}{N_t} \\
&= \text{Seasonal} + \text{Trend} + \text{Noise}
\end{aligned}
$$

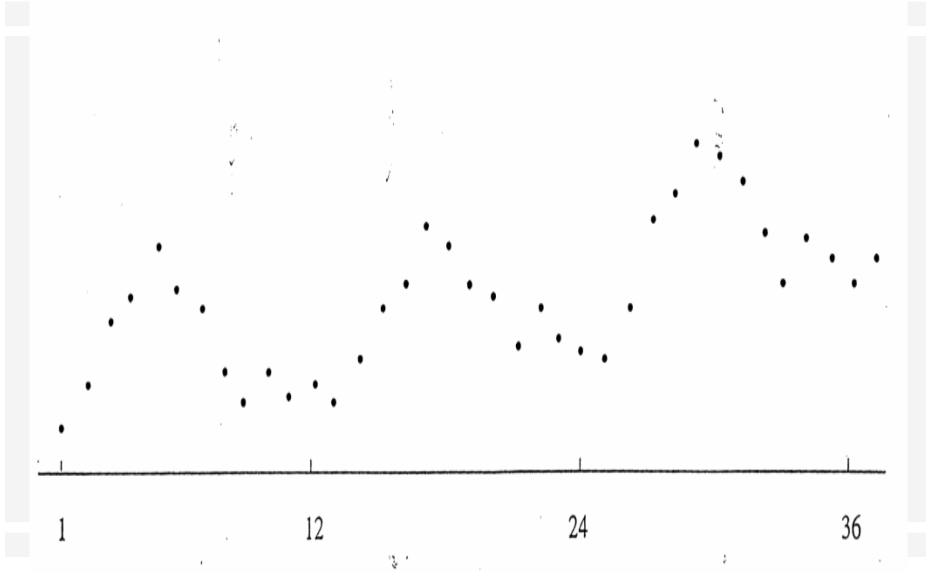
In X-11, specific filters are used (with options) to estimate S_t and T_t :

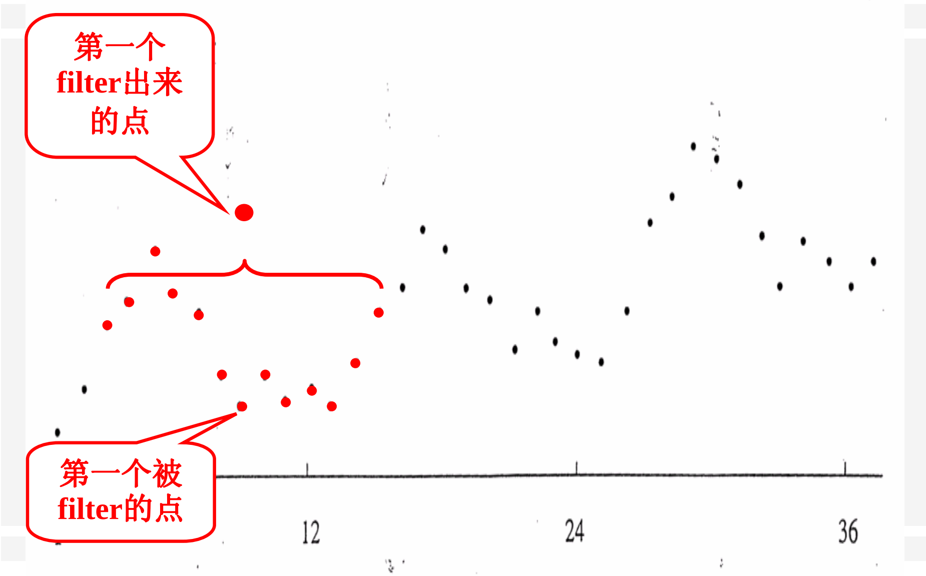
$$\widehat{S}_t = M^{(S)}Z_t \quad \text{and} \quad \widehat{T}_t = M^{(T)}Z_t$$

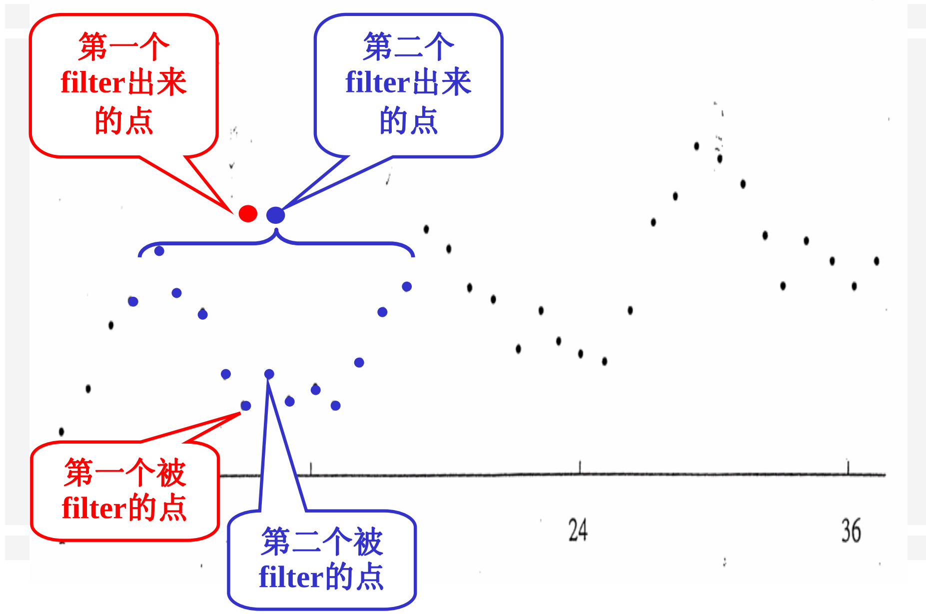
with considerable successes.

Census Filters

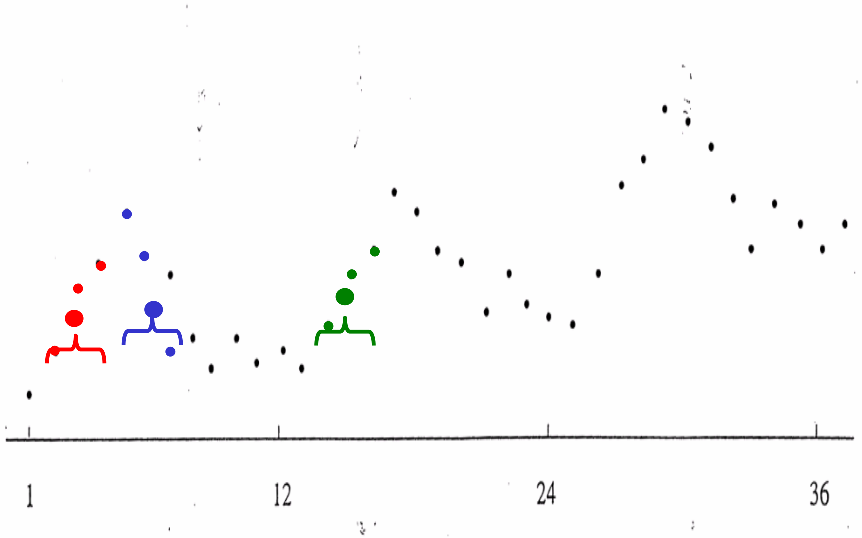


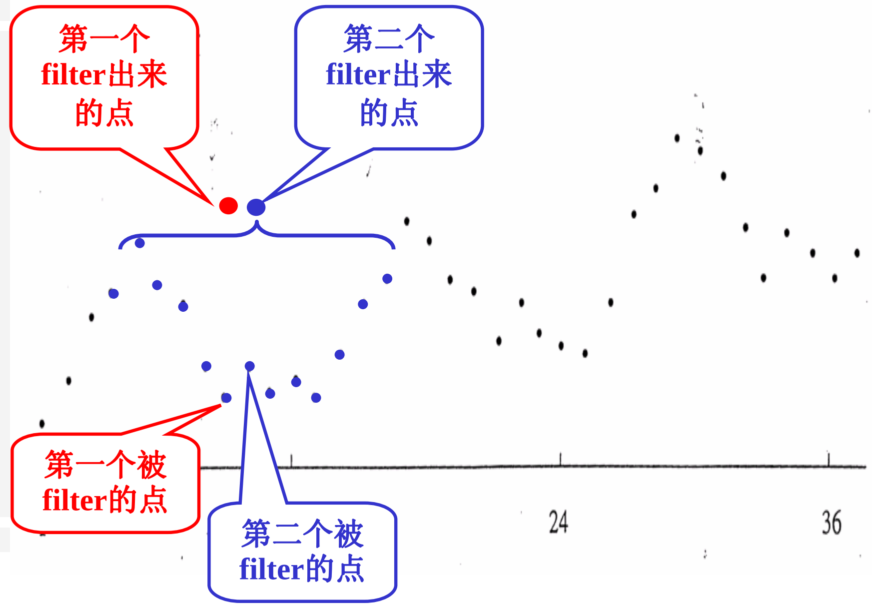




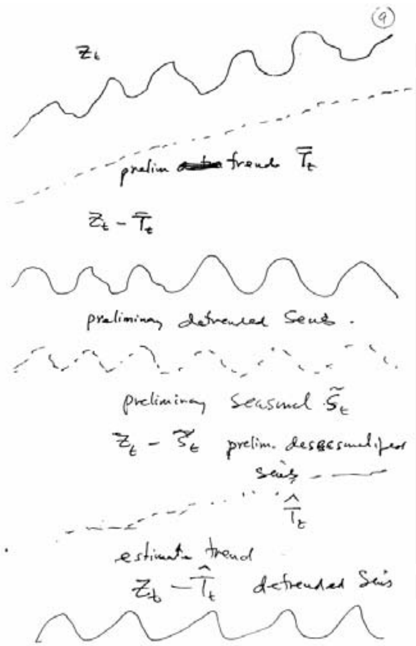


3点平均 (简化版)

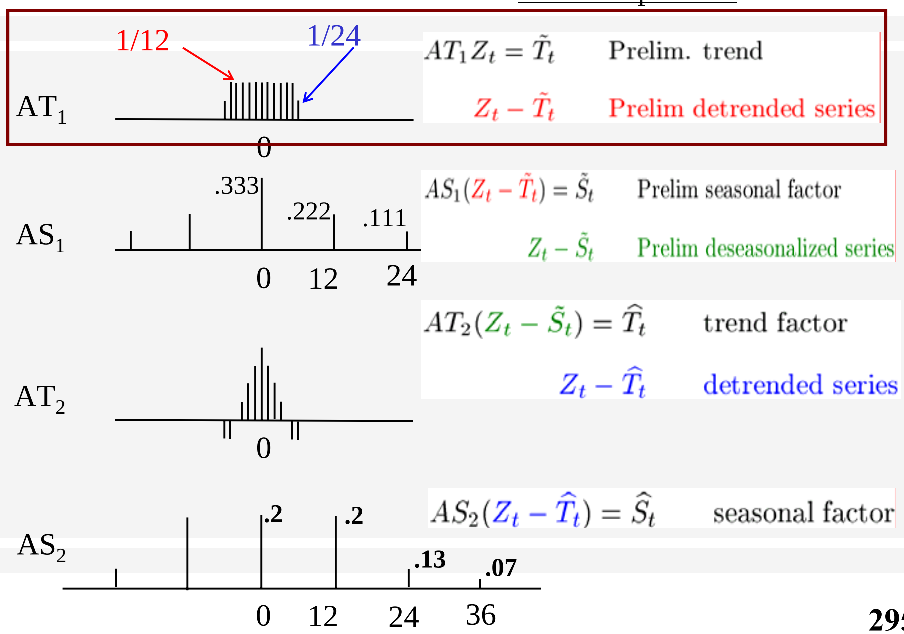


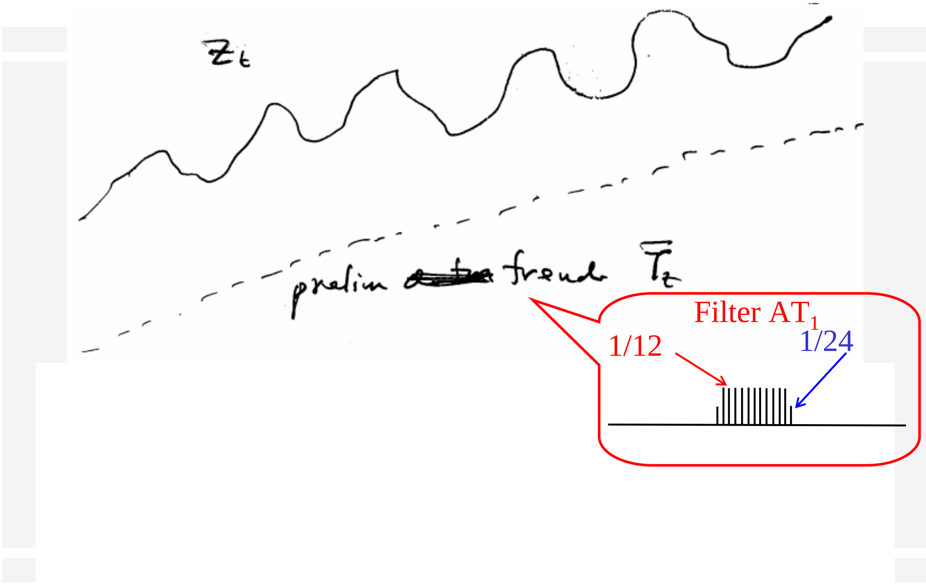


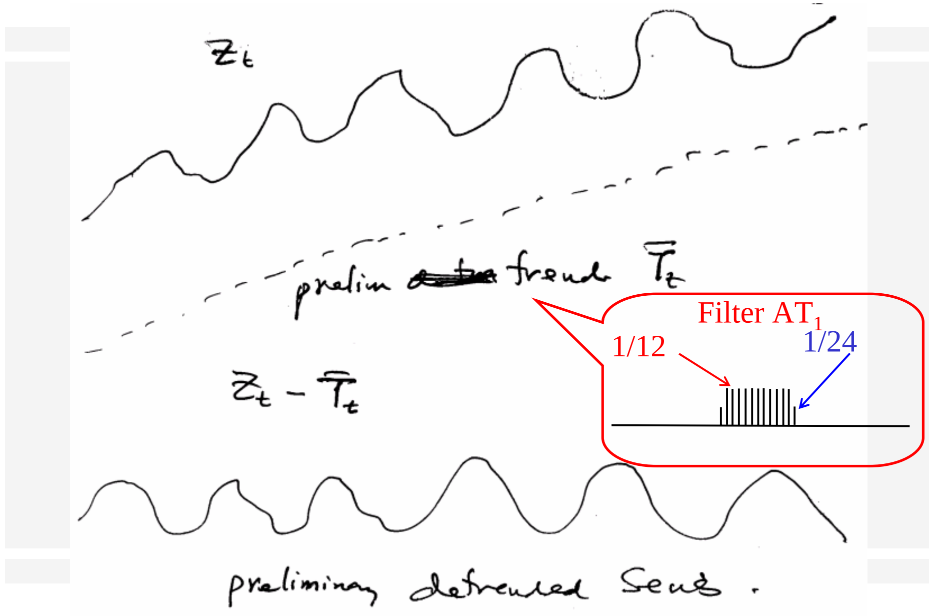
详图请见后面讲解

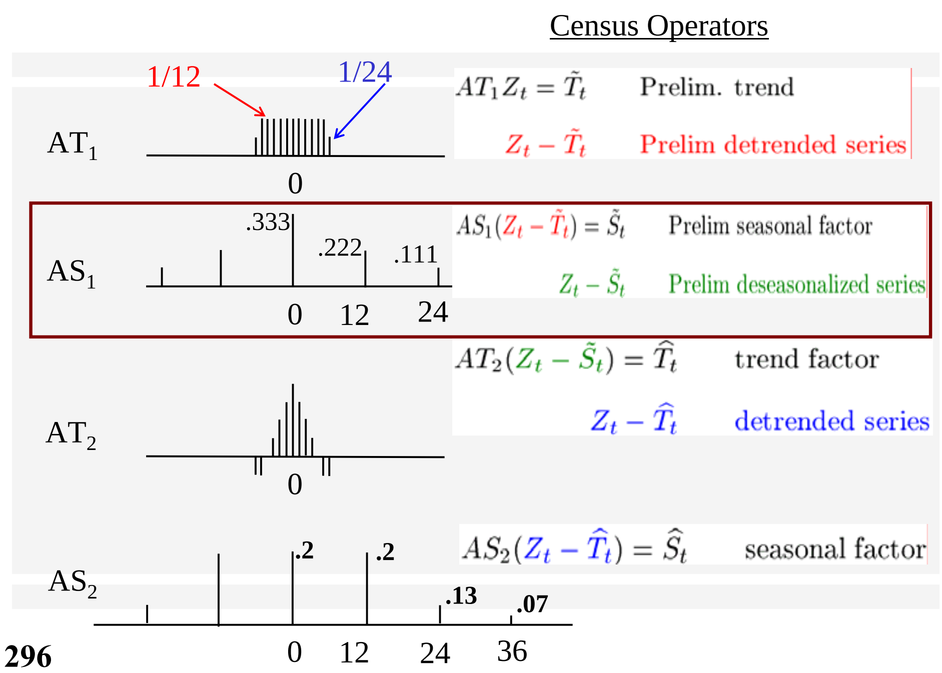


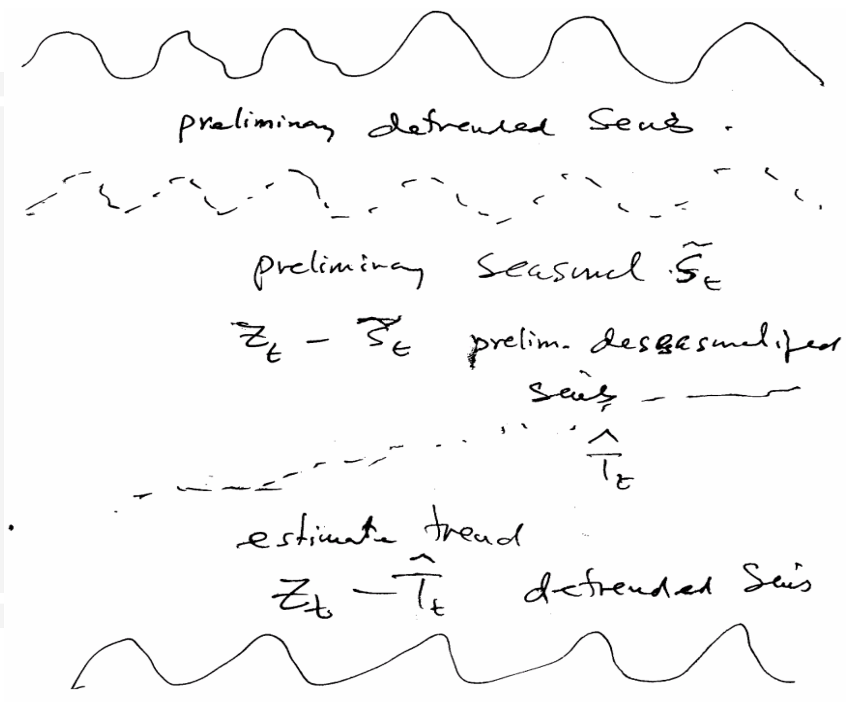
Census Operators



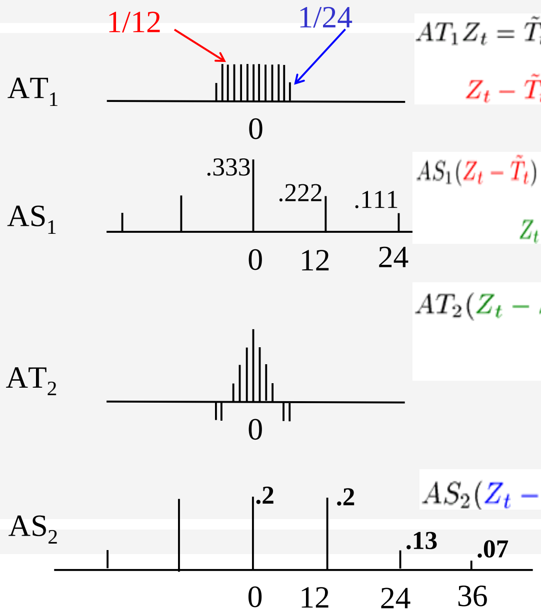

Z_t
prelim ~~data~~ trend $\tilde{T}_t$
Filter AT_1
1/12
1/24


Z_t
prelim ~~data~~ trend $\tilde{T}_t$
Filter AT_1
1/12
1/24
$Z_t - \tilde{T}_t$
preliminary detrended Series.


Census Operators
1/12
1/24
AT_1
0
$AT_1 Z_t = \tilde{T}_t$ Prelim. trend
$Z_t - \tilde{T}_t$ Prelim detrended series
AS_1
.333
.222
.111
0 12 24
$AS_1(Z_t - \tilde{T}_t) = \tilde{S}_t$ Prelim seasonal factor
$Z_t - \tilde{S}_t$ Prelim deseasonalized series
$AT_2(Z_t - \tilde{S}_t) = \widehat{T}_t$ trend factor
$Z_t - \widehat{T}_t$ detrended series
AT_2
0
AS_2
.2
.2
.13
.07
0 12 24 36
$AS_2(Z_t - \widehat{T}_t) = \widehat{S}_t$ seasonal factor


preliminary detrended Series.
preliminary seasonal $\tilde{S}_t$
$Z_t - \tilde{S}_t$ prelim. deseasonalized series
$\widehat{T}_t$
estimate trend
$Z_t - \widehat{T}_t$ detrended Series

Census Operators



$AT_1 Z_t = \tilde{T}_t$ Prelim. trend

$Z_t - \tilde{T}_t$ Prelim detrended series

$AS_1(Z_t - \tilde{T}_t) = \tilde{S}_t$ Prelim seasonal factor

$Z_t - \tilde{S}_t$ Prelim deseasonalized series

$AT_2(Z_t - \tilde{S}_t) = \widehat{T}_t$ trend factor

$Z_t - \widehat{T}_t$ detrended series

$AS_2(Z_t - \widehat{T}_t) = \widehat{S}_t$ seasonal factor

时间序列分析

第十七讲(下)

Symmetric filter

$$\mathbf{M}: \quad \left\{ \delta_0 + \delta_1(B+F) + \delta_2(B^2+F^2) + \cdots \right\} Z_t$$

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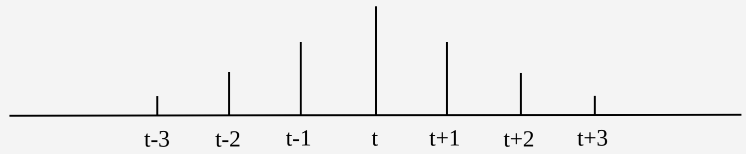
$$X_t = MZ_t: \quad \delta_0 Z_t + \delta_1(Z_{t-1}+Z_{t+1}) + \delta_2(Z_{t-2}+Z_{t+2}) + \cdots$$



In X-11, specific filters are used (with options) to estimate S_t and T_t :

$$\widehat{S}_t = M^{(S)} Z_t \quad \text{and} \quad \widehat{T}_t = M^{(T)} Z_t$$

with considerable successes.

Empirical Method (X-11)

William Cleveland's Work

Theoretical Background

A model for X-11 $(Z_t=T_t+S_t+N_t)$

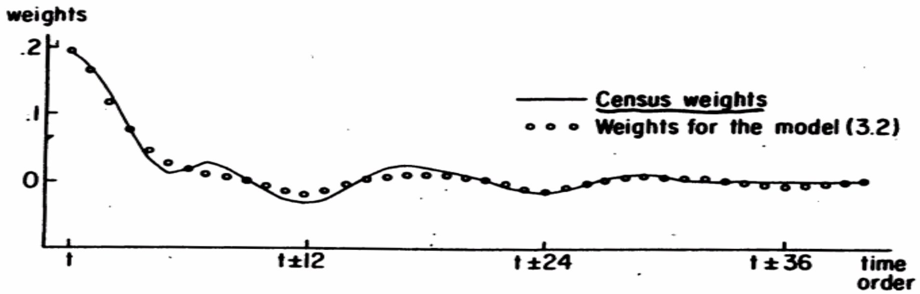
In joint work with Cleveland (1976 *JASA*), we postulated a seasonal ARIMA model for S_t , a regular ARIMA model for T_t and white noise for N_t :

$$\left.\begin{aligned} (1-B)^2 T_t &= (1-\eta_1 B - \eta_2 B^2) b_t \\ (1-B^{12}) S_t &= (1-\delta_1 B^{12} - \delta_2 B^{24}) c_t \\ N_t &= \xi_t \\ (b_t, c_t, \xi_t) &: \text{independent Gaussian noises} \\ & \quad \text{with variance } (\sigma_b^2, \sigma_c^2, \sigma_\xi^2) \end{aligned}\right\} \textbf{model (3.2)}$$

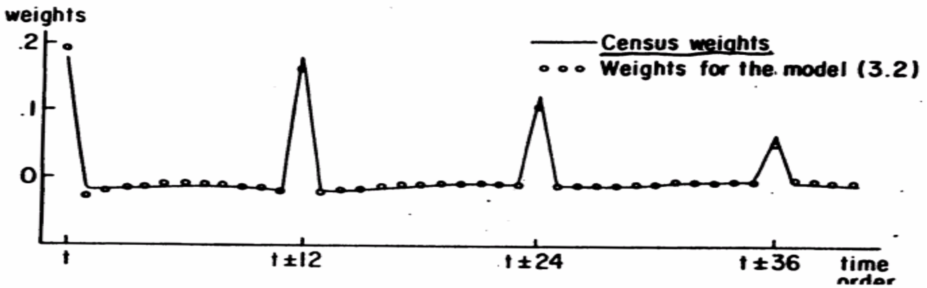
and found that, for specific choice of the parameters,

$$(\eta_1 = -.49, \eta_2 = .49); (\delta_1 = -.64, \delta_2 = -.83)$$
$$\sigma_c^2/\sigma_b^2 = 1.3, \ \sigma_\xi^2/\sigma_b^2 = 14.4$$

A. Weight Functions for the Trend Component



B. Weight Functions for the Seasonal Component



A model for X-11 $(Z_t=T_t+S_t+N_t)$

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Model (3.2)

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(1-B^{12}) S_t &= (1-\delta_1 B^{12}-\delta_2 B^{24}) c_t \\
N_t &= \xi_t \\
(b_t, c_t, \xi_t) &: \text{independent Gaussian noises} \\
& \quad \text{with variance } (\sigma_b^2, \sigma_c^2, \sigma_\xi^2)
\end{aligned}$$

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estimate

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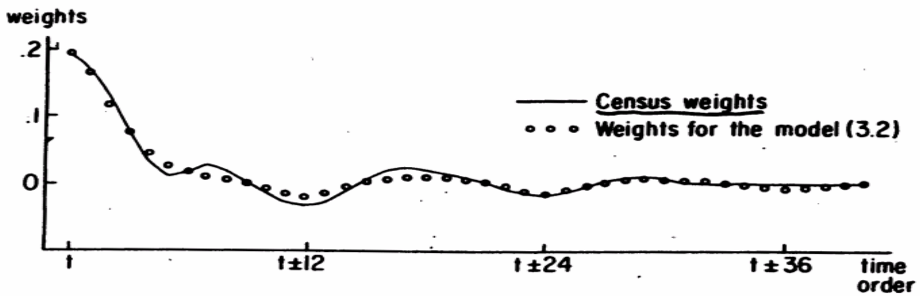
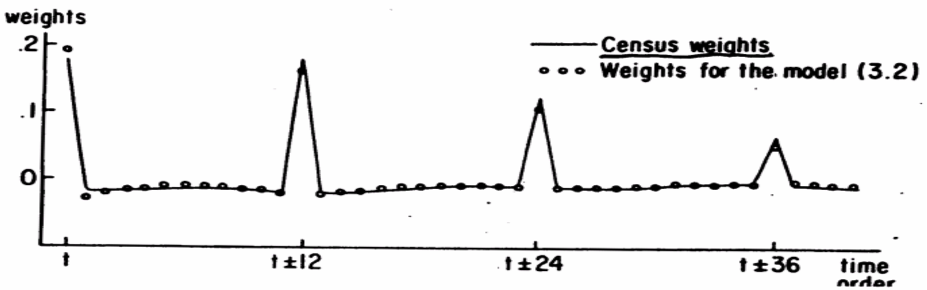


B. Weight Functions for the Seasonal Component



A model for X-11 $(Z_t=T_t+S_t+N_t)$

In joint work with Cleveland (1976 *JASA*), we postulated a seasonal ARIMA model for S_t, a regular ARIMA model for T_t and white noise for N_t:

What is the overall model?

$$\begin{aligned}
(1-B)^2 T_t &= (1-\eta_1 B-\eta_2 B^2) b_t \\
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$$\sigma_c^2/\sigma_b^2=1.3,\ \sigma_\xi^2/\sigma_b^2=14.4$$

Thus, we have an approximate unobserved component model for the X-11 filters

From this unobserved component model, the overall (or marginal) model for Z_t is

$$\begin{aligned}(1-B)(1-B^{12})Z_t &= (1-.34B+.14B^2+.14B^3+.14B^4+.14B^5\\ &\quad +.13B^6+.13B^7+.12B^8+.11B^9+.09B^{10}\\ &\quad +.08B^{11}-.42B^{12}+.23B^{13}+.04B^{24}-.02B^{25})a_t\end{aligned}$$

which is ‘close’ to the model for the famous ‘airline data’ of Box and Jenkins

$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

A model for X-11 ($Z_t=T_t+S_t+N_t$)

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Thus, we have an approximate unobserved component model for the X-11 filters

From this unobserved component model, the overall (or marginal) model for Z_t is

$$
\begin{aligned}
(1-B)(1-B^{12})Z_t &= (1-.34B+\cancel{.14B^2}+\cancel{.14B^3}+\cancel{.14B^4}+\cancel{.14B^5} \\
&\quad +\cancel{.13B^6}+\cancel{.13B^7}+\cancel{.12B^8}+\cancel{.11B^9}+\cancel{.09B^{10}} \\
&\quad +.08B^{11}-.42B^{12}+.23B^{13}+\cancel{.04B^{24}}-\cancel{.02B^{25}})a_t
\end{aligned}
$$

Airline Model?

which is 'close' to the model for the famous 'airline data' of Box and Jenkins

$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

This got us into the model-based approach for decomposition:

Based on the overall model built for the data Zt, back out models for (T_t, S_t, N_t) and estimate the unobserved components accordingly

Some advantages of the model-based approach:

- Make assumptions explicit
- Can make proper inferences about the components
- At least partially verifiable

On the other hand, like many unobserved linear component models in the literature, we run into the problems of

identifiability

Reduced form

i.e. more than one choice of component models leading to the **same overall model**

Simple Example

$$Z_t = T_t + N_t \qquad N_t \overset{i.i.d.}{\sim} N(0, \sigma_N^2)$$

If $\quad (1-B)T_t = b_t \quad$ where $\quad b_t \overset{i.i.d.}{\sim} N(0,\sigma_b^2)$

$$
\begin{aligned}
\text{Then} \quad (1-B)Z_t &= b_t + (1-B)N_t \\
&= a_t - \theta a_t
\end{aligned}
$$

But if $(1-B)T_t = (1-\eta B)b_t$

$$
\begin{aligned}
\text{Then} \quad (1-B)Z_t &= (1-\eta B)b_t + (1-B)N_t \\
&= a_t - \theta a_{t-1}
\end{aligned}
$$

$\therefore$ given (θ, σ_a^2) infinite choices of $(\eta, \sigma_b^2, \sigma_N^2)$

Canonical decomposition (joint work with Hillmer, *Biometrika* 1978)

$$Z_t = T_t + N_t$$

where N_t is gaussian white noise independent of T_t. Consider the spectrum

$$f_z(\omega) = f_T(\omega) + \sigma_N^2$$

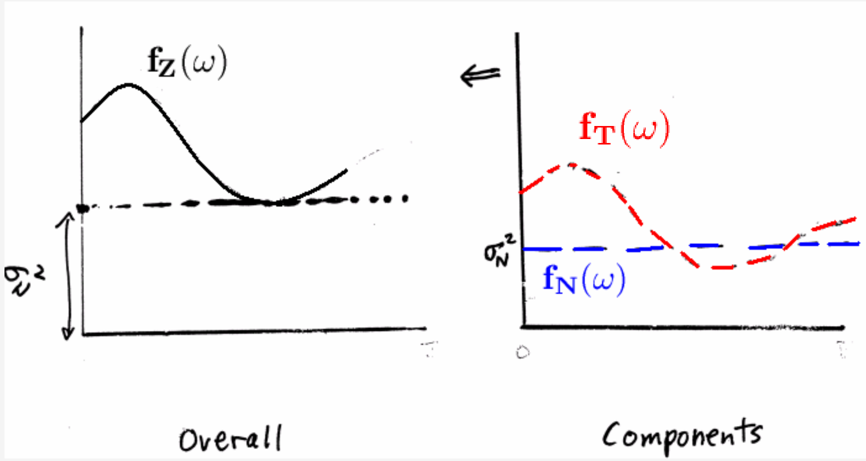
If the models for T_t and N_t are known, we can derive the corresponding overall model for Z_t. On the other hand, if only the overall model for Z_t is known, any choice of σ_N^2 in the range

$$0 \le \sigma_N^2 \le \bar{\sigma}_N^2$$

where $\bar{\sigma}_N^2 = \min_\omega f_z(\omega)$ gives an acceptable decomposition, since $f_T(\omega) = f_z(\omega) - \sigma_N^2 \ge 0$

See Page 193-197 of Notes

$$\mathbf{Z_t = T_t + N_t}$$



The choice

$$\bar{\sigma}_N^2 = \min_\omega f_z(\omega), \quad \bar{f}_T(\omega) = f_z(\omega) - \bar{\sigma}_N^2$$

and this resulting $Z_t = \bar{T}_t + \bar{N}_t$ decomposition is called the canonical decomposition.

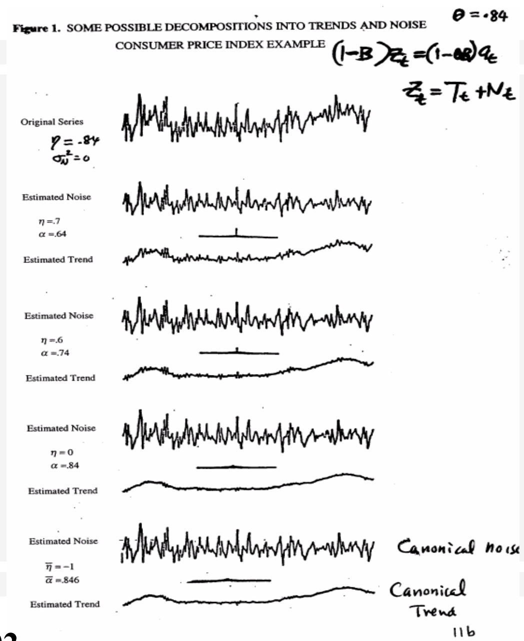
U.S. Inflation Rate of CPI 1/64 to 12/72 (Box, Hillmer & Tiao, 1978):

$$\begin{aligned}(1-B)Z_t &= (1-\theta B)a_t \\ \theta &= .84; \quad \sigma_a^2 = .0019\end{aligned}$$

This implies that the model for T_t must be of the form

$$(1-B)T_t = (1-\eta B)b_t$$

and $$(1-\theta B)(1-\theta F)\sigma_a^2 = (1-\eta B)(1-\eta F)\sigma_b^2 + (1-B)(1-F)\sigma_N^2$$



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The choice

$$\bar{\sigma}_N^2 = \min_\omega f_z(\omega), \quad \bar{f}_T(\omega) = f_z(\omega) - \bar{\sigma}_N^2$$

and this resulting $Z_t = \bar{T}_t + \bar{N}_t$ decomposition is called the canonical decomposition.

U.S. Inflation Rate of CPI 1/64 to 12/72 (Box, Hillmer & Tiao, 1978):

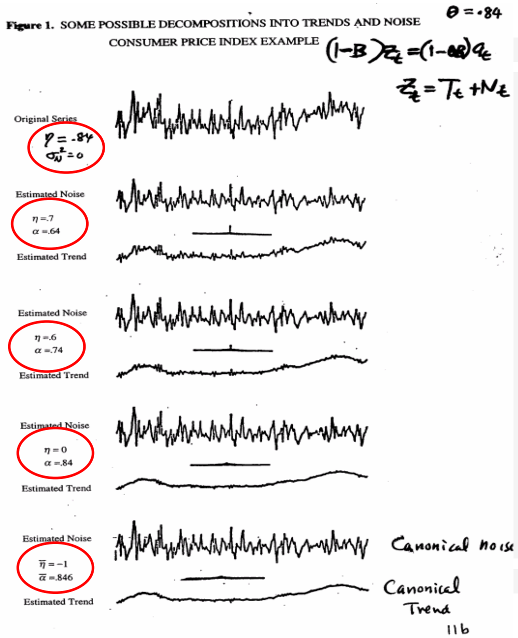
Overall model

$$\boxed{\begin{aligned}(1-B)Z_t &= (1-\theta B)a_t \\ \theta &= .84; \quad \sigma_a^2 = .0019\end{aligned}}$$

This implies that the model for T_t must be of the form

$$(1-B)T_t = (1-\eta B)b_t$$

and $$(1-\theta B)(1-\theta F)\sigma_a^2 = (1-\eta B)(1-\eta F)\sigma_b^2 + (1-B)(1-F)\sigma_N^2$$



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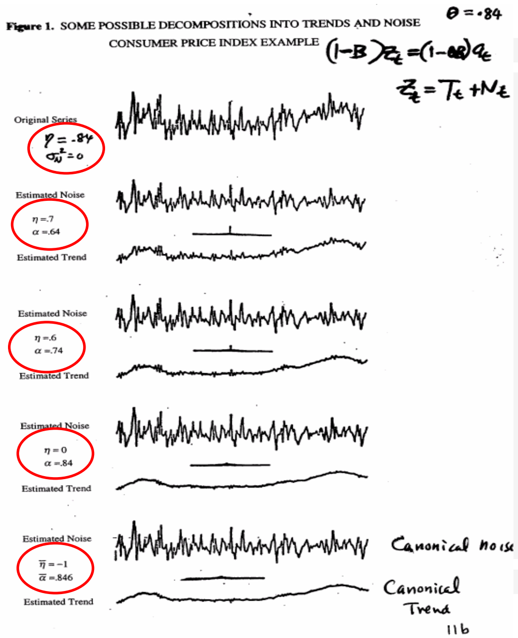
From this example, given the overall model

- Canonical decomposition gives the most deterministic trend component
- It attributes the largest possible variance to the noise component
- The popular choice in econometrics literature, $\eta=0$, gives a estimated trend component very close to that from the canonical decomposition
- But all other acceptable decompositions are of equal claim
- The trend component is clearly not 'robust' over the entire range of acceptable decomposition, but robust over $-1 \leqslant \eta \leqslant 0$



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This formulation was generalized in joint work with Hillmer (1982 *JASA*) to

$$
\begin{aligned}
Z_t &= S_t + T_t + N_t \\
(1-B)^d T_t &= \eta(B) b_t \quad \textbf{(trend model)} \\
U(B) S_t &= \delta(B) c_t \quad \textbf{(seasonal component model)} \\
U(B) &= 1 + B + B^2 + \cdots + B^{11}
\end{aligned}
$$

consistent with an overall model for Z_t of the form

$$(1-B)^d U(B) Z_t = \theta(B) a_t$$

Canonical decomposition gives the most 'deterministic' evolving stochastic seasonal and trend components. See examples in Hillmer, Bell, and Tiao (1983)

X-12-ARIMA and TRAMO/SEATS (A. Maravall)

- Many Similarities
- Each consists of two main parts: a regression-ARIMA modeling program and a decomposition program
- The regression-ARIMA features are very close. Both employ an iterative outlier and level-shift detection procedure initiated in Ih Chang’s thesis (1983)
- The two decomposition programs are built on different foundations; X-12 retains the structures of the X-11-ARIMA (Dagum), an empirically developed procedure; SEATS is a model-based procedure closely related to canonical decomposition.
- For many economic time series, the decomposition results from these two programs are again very close. This is not too surprising since the X-11 and the ‘airline model’ are closely related.

Example: to compare X-12-ARIMA + TRAMO-SEATS

Monthly Logged Airline Passenger Total Data (1949. Jan – 1960. Dec)

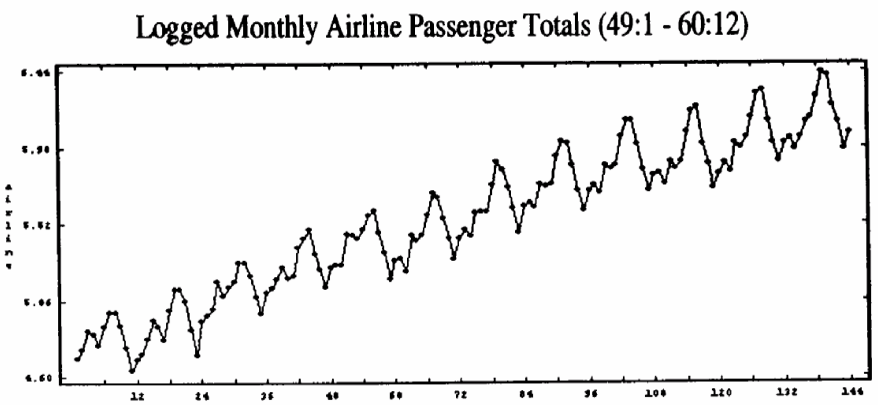
Parameter	SCA	X-12-ARIMA	TRAMO
$\widehat{\theta}$	.402 (.079)*	.401(.079)	.400(.082)
$\widehat{\Theta}$	.560(.071)	.560(.075)	.562(.086)
$\widehat{\sigma}_a$	.0368	.0373	.0375

* standard error of estimates

No outlier is detected for all methods.

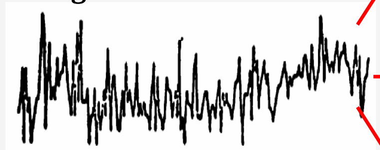
The "Airline Model":

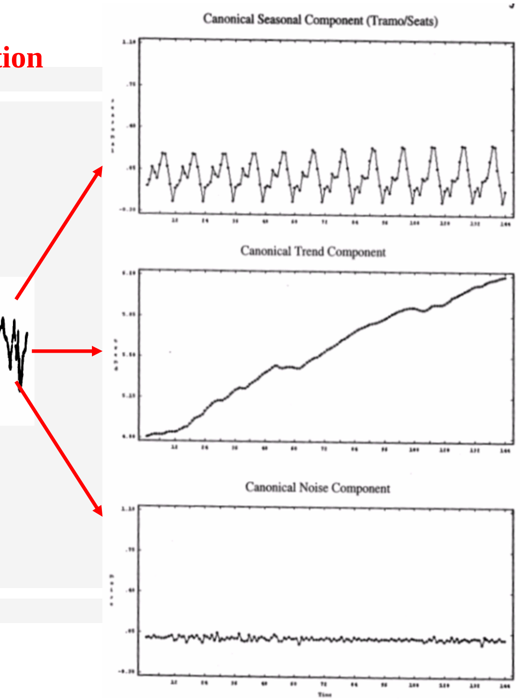
$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$



Tramo/Seats Decomposition

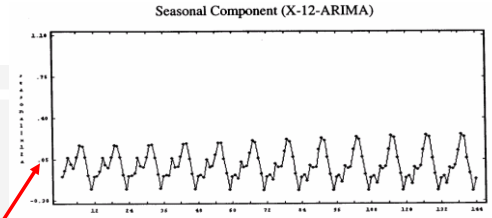
Original Series

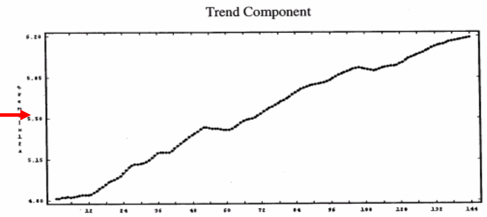


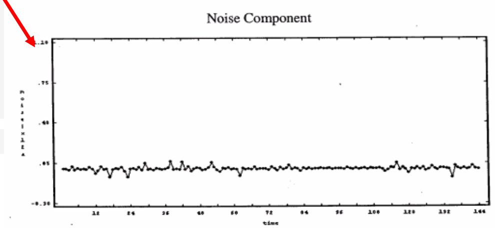


X-12-ARIMA Decomposition

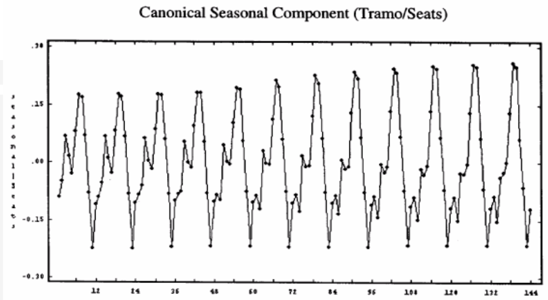
Original Series

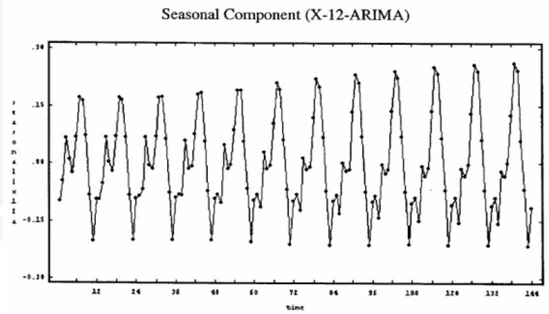




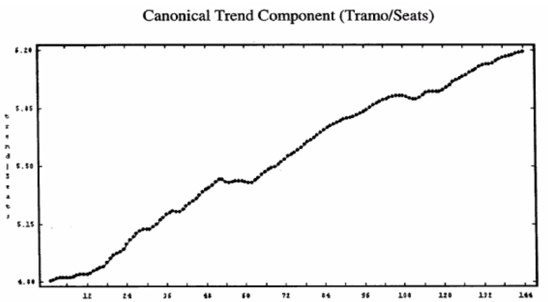


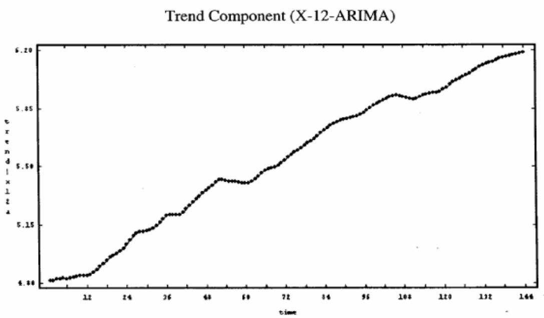
Seasonal component comparison





trend component comparison





Concluding Remarks

- Model based procedures
  - Make assumptions explicit
  - Inferential process straightforward
  - Lend itself to further improvement
- Canonical decomposition exposes explicitly
  - Robustness of seasonal component
  - Robustness of trend component
- Convergence of carious procedures
  - Especially in regression part
- For forecasting future components NO NEED for adjustment
- Measure of strength of seasonal signal and trend
  Signal may be useful

Kalman Filter

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Kalman Filter

observation: $\{y_t\}$

obs. equation: $y_t = f_t\theta_t + \nu_t$ (θ_t is unobservable)

state equation: $\theta_t = g_t\theta_{t-1} + \omega_t$

$\{f_t, g_t\}$ are constants

$\{\nu_t\}, \{\omega_t\}, \theta_0$ are random variable

Kalman Filter

observation: $\{y_t\}$

obs. equation: $y_t = f_t\theta_t + \nu_t$

state equation: $\theta_t = g_t\theta_{t-1} + \omega_t$

Kalman Filter

State Space Model

$\{f_t, g_t\}$ are constants

$\{\nu_t\}, \{\omega_t\}, \theta_0$ are random variable

Kalman Filter

observation: $\{y_t\}$

obs. equation: $y_t = f_t\theta_t + \nu_t$

state equation: $\theta_t = g_t\theta_{t-1} + \omega_t$

Kalman Filter

State Space Model

$\{f_t, g_t\}$ are constants

$\{\nu_t\}, \{\omega_t\}, \theta_0$ are random variable

AIM:

1. distribution of $y_t | y_{t-1}, y_{t-2}, \cdots$
2. distribution of $\theta_t | y_t, y_{t-1}, y_{t-2}, \cdots$

Review of Courses

- ARIMA model
- Model Building
- Seasonal Models
- Intervention and Outlier
- Multivariate Time Series
- ARCH/GARCH
- Seasonal Adjustment