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Paper List

1. Distribution of the Estimators for Autoregressive Time Series with a Unit Root
David A. Dickey; Wayne A. Fuller
Journal of the American Statistical Association, Vol. 74, No. 366. (Jun., 1979), pp. 427-431.
2. Intervention Analysis with Applications to Economic and Environmental Problems
G. E. P. Box; G. C. Tiao
Journal of the American Statistical Association, Vol. 70, No. 349. (Mar., 1975), pp. 70-79.
3. Outliers, Level Shifts, and Variance Changes in Time Series
Tsay, Ruey S.
Journal of Forecasting; Jan-Mar 1988; 7, 1
4. Consistent Estimates of Autoregressive Parameters and Extended Sample Autocorrelation Function for Stationary and Nonstationary ARMA Models
Ruey S. Tsay; George C. Tiao
Journal of the American Statistical Association, Vol. 79, No. 385. (Mar., 1984), pp. 84-96.
5. Parameter Inference for a Nearly Nonstationary First-Order Autoregressive Model
J. Ahtola; G. C. Tiao
Biometrika, Vol. 71, No. 2. (Aug., 1984), pp. 263-272.
6. Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation
Robert F. Engle
Econometrica, Vol. 50, No. 4. (Jul., 1982), pp. 987-1007.
7. Autoregressive Moving Average Models, Intervention Problems and Outlier Detection in Time Series
G. C. Tiao
Handbook of Statistics, Vol 5, No.3, Elsevier, B.V.(1985)85-118
8. Consistency Properties of Least Squares Estimates of Autoregressive Parameters in ARMA Models
George C. Tiao; Ruey S. Tsay
The Annals of Statistics, Vol. 11, No. 3. (Sep., 1983), pp. 856-871.

A Brief Introduction to

Nonlinear Least Square

13.1 Review of nonlinear least square

We begin with a brief review of nonlinear least squares. Let y_t be the response and $x_{1t}, \dots, x_{kt}$ be k functions of r inputs $\xi_{1t}, \dots, \xi_{rt}$, $t = 1, \dots, n$. A model of the form

$$y_t = \beta_1 x_{1t} + \dots + \beta_k x_{kt} + a_t \quad (13.1)$$

where $\beta_1, \dots, \beta_k$ are k constants and a_t is the error term with zero mean and independent of the X_{jt} 's, is called a linear model. That is, the expected value of y_t is linear in the parameters $\beta = (\beta_1, \dots, \beta_k)^\top$. Suppose that the a_t 's are identically and independently distributed as $N(0, \sigma_a^2)$. Then the likelihood function is

$$\ell(\beta, \sigma_a^2 | \underline{y}) \propto \sigma_a^{-n} e^{-\frac{1}{2\sigma_a^2} S(\beta)} \quad (13.2)$$

where the sum of square function in the exponent $S(\beta) = \underline{a}^\top \underline{a} = \sum_{t=1}^n a_t^2$, $\underline{a}^\top = (a_1, \dots, a_n)$ and $\underline{y}^\top = (y_1, \dots, y_n)$. It is clear that the maximum likelihood estimate (MLE) $\hat{\beta}$ of β is such that

$$S(\hat{\beta}) = \min_{\beta} S(\beta) \quad (13.3)$$

and that the MLE of σ_a^2 is

$$\widehat{\sigma_a^2} = \frac{1}{n} S(\beta) \quad (13.4)$$

We can write

$$\underline{a} = \underline{y} - \underline{X}\underline{\beta} \quad (13.5)$$

where

$$\underline{X} = \begin{pmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & \ddots & \vdots \\ x_{k1} & \cdots & x_{nk} \end{pmatrix}$$

Assuming that $\underline{X}$ is of rank k , the MLE $\widehat{\underline{\beta}}$ is also the least squares estimate

$$\widehat{\underline{\beta}} = (\underline{X}^\top \underline{X})^{-1} \underline{X}^\top \underline{y} \quad (13.6)$$

and the estimated covariance matrix of $\widehat{\underline{\beta}}$ is

$$\widehat{V}(\widehat{\underline{\beta}}) = \widehat{\sigma_a^2} (\underline{X}^\top \underline{X})^{-1} \quad (13.7)$$

Notice that from (13.1) we have that

$$\frac{\partial a_t}{\partial \beta_j} = -x_{jt}, \quad j = 1, \dots, k$$

independent of $\underline{\beta}$. Any model which does not have this property is called a nonlinear model, namely, nonlinear in the parameters. As an example, the model

$$a_t = y_t - \beta_1 \xi_{1t} - \beta_1^2 \xi_{2t} - \beta_2^2 \xi_{3t} \quad (13.8)$$

is nonlinear in β_1 and β_2 . In general, a nonlinear model can be written as

$$a_t = y_t - f(\underline{\beta}, \underline{\xi}_t) \quad (13.9)$$

where $\underline{\xi}_t^\top = (\xi_{1t}, \dots, \xi_{rt})$ and the functional form f is known. Assuming again that the a_t 's are i.i.d. $N(0, \sigma_a^2)$, the likelihood function of $\underline{\beta}$ and σ_a^2 is

$$\ell(\underline{\beta}, \sigma_a^2 | \underline{y}) \propto \sigma_a^{-n} e^{-\frac{1}{2\sigma_a^2} S(\underline{\beta})} \quad (13.10)$$

which is of exactly the same form as that in (13.2) except that the sum of squares function $S(\underline{\beta})$ in the exponent now is

$$S(\underline{\beta}) = \sum_{t=1}^n a_t^2 = \sum_{t=1}^n [y_t - f(\underline{\beta}, \underline{\xi}_t)]^2 \quad (13.11)$$

so that (13.5)-(13.7) no longer apply.

For low dimensional vector $\underline{\beta}$, it is best to study the sum of squares function $S(\underline{\beta})$ by plotting contours in one, two or three dimensions. For higher dimensional situations, the MLE of $\underline{\beta}$ can be obtained via nonlinear least squares. For a given guess value of $\underline{\beta}$, $\underline{\beta} = \underline{\beta}_{(0)}$, the model (13.9) can be locally linearized yielding

$$a_t \doteq a_{t,(0)} + \sum_{j=1}^k \left(\frac{\partial a_t}{\partial \beta_j} \right)_{\beta_{(0)}} (\beta_j - \beta_{j(0)}) \quad (13.12)$$

where $a_{t,(0)} = y_t - f(\underline{\beta}_{(0)}, \underline{\xi}_t)$, which can be written as

$$a_t \doteq a_{t,(0)} + \sum_{j=1}^k x_{jt}^{(0)} \beta_j \quad (13.13)$$

with $x_{jt}^{(0)} = -(\frac{\partial a_t}{\partial \beta_j})_{\beta_{(0)}}$. Expression (13.13) is now in the same form as (13.1). Thus, the least squares estimates of $\underline{\delta}^\top = (\delta_1, \dots, \delta_k)$ is

$$\widehat{\underline{\delta}} = (\underline{X}_{\underline{\beta}_{(0)}}^\top \underline{X}_{\underline{\beta}_{(0)}})^{-1} \underline{X}_{\underline{\beta}_{(0)}}^\top \underline{a}_{(0)} \quad (13.14)$$

where $\underline{a}_{(0)}^\top = (a_{1,(0)}, \dots, a_{n,(0)})$ and

$$\underline{X}_{\beta_{(0)}} = \begin{pmatrix} x_{11}^{(0)} & \cdots & x_{1k}^{(0)} \\ \vdots & \ddots & \vdots \\ x_{n1}^{(0)} & \cdots & x_{nk}^{(0)} \end{pmatrix}$$

We can now take

$$\underline{\beta}_{(1)} = \underline{\beta}_{(0)} + \widehat{\underline{\delta}} \quad (13.15)$$

as the next guess value of $\underline{\beta}$ and repeat the process until convergence occurs. Finally, when $\widehat{\underline{\beta}}$ is obtained, i.e.

$$\min_{\underline{\beta}} S(\underline{\beta}) = S(\widehat{\underline{\beta}})$$

We have that

$$\sigma_a^2 = \frac{1}{n} S(\widehat{\underline{\beta}}) \quad (13.16)$$

and the approximate estimated covariance matrix of $\widehat{\underline{\beta}}$ is

$$\widehat{V}(\widehat{\underline{\beta}}) \doteq \widehat{\sigma_a^2} (\widehat{X}_{\widehat{\underline{\beta}}}^\top \widehat{X}_{\widehat{\underline{\beta}}})^{-1} \quad (13.17)$$

For a more detailed discussion of the theory and methods of nonlinear least squares, see e.g. Draper and Smith (1981).

Linear Difference Equation

Consider the relationship

$$y_t - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p} = 0$$

which can be written in backshift operator notation as

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p) y_t = 0$$

In many time series situations we will be interested in the behavior of data which follow difference equations. Let's consider some specific examples:

1. $y_t - \phi_1 y_{t-1} = 0$

A. $y_t - .5y_{t-1} = 0$

$$\begin{aligned} y_t &= .5y_{t-1} \\ &= (.5)^2 y_{t-2} \\ &\vdots \\ &= (.5)^t y_{t-t} \\ &= (.5)^t y_0 \quad (\text{see Figure 1}) \end{aligned}$$

B. $y_t + .5y_{t-1} = 0$

$$\begin{aligned} y_t &= -.5y_{t-1} \\ &= (-.5)^t y_0 \quad (\text{see Figure 2}) \end{aligned}$$

C. $y_t - 2y_{t-1} = 0$

$$\begin{aligned} y_t &= 2y_{t-1} \\ &= (2)^t y_0 \quad (\text{see Figure 3}) \end{aligned}$$

Question: In general, how can we describe the behavior of y_t ?

Answer: By using 'characteristic' equations:

Example: For the 1st degree homogeneous equation:

$$y_t - \phi_1 y_{t-1} = 0 \quad (3)$$

It's characteristic equation is defined as

$$m - \phi_1 = 0 \Rightarrow m = \phi_1$$

Claim: The general solution to (3) is then

$$y_t = h(t) = y_0 m^t = y_0 \phi_1^t$$

Note that the behavior of y_t depends on ϕ_1 . If

$$\begin{aligned} \phi_1 > 1 & \quad \text{then } y_t \quad \text{explodes} \\ 0 < \phi_1 < 1 & \quad \text{then } y_t \quad \text{dies out} \\ -1 < \phi_1 < 0 & \quad \text{then } y_t \quad \text{damped oscillation} \\ \phi_1 < -1 & \quad \text{then } y_t \quad \text{explosive oscillation} \end{aligned}$$

Therefore for y_t to exhibit stationary behavior, the roots of the characteristic equation must be less than one in absolute value.

Consider a 2nd degree equation

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} = 0$$

Its characteristic equation is defined as

$$m^2 - \phi_1 m - \phi_2 = 0$$

Solving for m:

$$m = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2} \quad (4)$$

or

$$(m - k_1)(m - k_2) = 0 \quad \text{where} \quad k_1 + k_2 = \phi_1$$
$$k_1 * k_2 = -\phi_2$$

- In general, three things can happen:
- (1) Distinct Roots
 - (2) Equal Roots
 - (3) Complex Roots ($i = \sqrt{-1}$)

1. Distinct Roots

Example: (See Figure 4)

Homogeneous Equation: $y_t - .3y_{t-1} - .4y_{t-2} = 0$

Characteristic Equation:

$$\begin{aligned}m^2 - .3m - .4 &= 0 \\(m - .8)(m + .5) &= 0 \\m &= .8, \quad -.5\end{aligned}$$

Solution:

$$\begin{aligned}y_t = h(t) &= c_1(m_1)^t + c_2(m_2)^t \\&= c_1(.8)^t + c_2(-.5)^t\end{aligned}$$

where c_1 and c_2 can be determined by the initial conditions:

$$\text{at } t=0 \quad y_0 = c_1 + c_2 \quad 2 \text{ eqn's, 2 unknowns}$$

$$\text{at } t=1 \quad y_1 = .8c_1 - .5c_2 \quad \text{solve for } c_1 \text{ and } c_2$$

Proof of solution: $y_t = c_1(.8)^t + c_2(-.5)^t$

$$\begin{aligned}y_t - .3y_{t-1} - .4y_{t-2} &= [c_1(.8)^t + c_2(-.5)^t] - .3[c_1(.8)^{t-1} + c_2(-.5)^{t-1}] - .4[c_1(.8)^{t-2} + c_2(-.5)^{t-2}] \\&= c_1(.8)^{t-2} [.8^2 - .3(.8) - .4] + c_2(-.5)^{t-2} [(-.5)^2 - .3(-.5) - .4] \\&= c_1(.8)^{t-2} [.64 - .24 - .4] + c_2(-.5)^{t-2} [.25 + .15 - .4] \\&= 0\end{aligned}$$

2. Equal Roots

Example: (See Figure 5)

Homogeneous equation: $y_t - y_{t-1} + .25y_{t-2} = 0$

Characteristic equation:

$$m^2 - m + .25 = 0$$

$$m = m_1 = m_2 = .5$$

Solution:

$$\begin{aligned} y_t = h(t) &= (c_1 + c_2 t)(m)^t \\ &= (c_1 + c_2 t)(.5)^t \end{aligned}$$

again, c_1 and c_2 can be determined by the initial conditions:

$$y_0 = c_1 \quad 2 \text{ eqn's, 2 unknowns}$$

$$y_1 = (c_1 + c_2) \cdot .5 \quad \text{solve for } c_1 \text{ and } c_2$$

Proof of solution: $y_t = (c_1 + c_2 t)(.5)^t$

$$\begin{aligned} y_t - y_{t-1} + .25y_{t-2} &= (c_1 + c_2 t)(.5)^t - (c_1 + c_2(t-1))(.5)^{t-1} + .25(c_1 + c_2(t-2))(.5)^{t-2} \\ &= (.25)^{t-2} [.25(c_1 + c_2 t) - .5(c_1 + c_2(t-1)) + .25(c_1 + c_2(t-2))] \\ &= (.25)^{t-2} [c_1(.25 - .5 + .25) + c_2(t-2)(.25 - .5 + .25) + c_2(.25(2) - .5)] \\ &= 0 \end{aligned}$$

3. Complex Roots

Example: (See Figure 6)

Homogeneous equation: $y_t + .4y_{t-1} + .13y_{t-2} = 0$

Characteristic equation:

$$\begin{aligned}m^2 + .4m + .13 &= 0 \\m &= \frac{-.4 \pm \sqrt{.16 - 4(.13)}}{2} \\&= \frac{-.4 \pm .6i}{2} \\m_1 &= -.2 + .3i \quad \text{complex} \\m_2 &= -.2 - .3i \quad \text{conjugates}\end{aligned}$$

Lets plot these solutions on the complex plane:

I. In general $e^{i\theta} = \cos \theta + i \sin \theta$

Proof: Take Taylor Series expansion of e^x about zero:

$$\begin{aligned}e^x &= 1 + x + x^2/2! + x^3/3! + \cdots + x^n/n! + \cdots \\e^{ix} &= 1 + (ix) + (ix)^2/2! + (ix)^3/3! + \cdots \\&= 1 + ix - x^2/2! - ix^3/3! + x^4/4! + \cdots \\&= (1 - x^2/2! + x^4/4! - \cdots) + i(x - x^3/3! + x^5/5! - \cdots) \\&= \cos x + i \sin x\end{aligned}$$

II. For

$$\begin{aligned}m_1 &= -.2 + .3i = a + bi \\m_2 &= -.2 - .3i = a - bi \\a + bi &= d(\cos \theta + i \sin \theta) = de^{i\theta} = m_1 \\a - bi &= d(\cos \theta - i \sin \theta) = de^{-i\theta} = m_2\end{aligned}$$

where $d = \sqrt{a^2 + b^2} = \sqrt{.2^2 + .3^2} = \sqrt{.13}$

Now, using the solution for distinct roots:

$$\begin{aligned}y_t = h(t) &= c_1(de^{i\theta})^t + c_2(de^{-i\theta})^t \\&= d^t(c_1e^{i\theta t} + c_2e^{-i\theta t}) \\&= d^tc_1(\cos \theta t + i \sin \theta t) + d^tc_2(\cos \theta t - i \sin \theta t) \\&= d^t(c_1 + c_2) \cos \theta t + d^t(c_1 - c_2)i \sin \theta t \\&= d^t(h_1 \cos \theta t + h_2i \sin \theta t)\end{aligned}$$

In summary, for a second order homogeneous equation, three possibilities arise when solving the characteristic equation:

1. Distinct Roots
2. Equal Roots
3. Complex Roots

<u>Type of Roots</u>	<u>Solution</u>	<u>Comments</u>
Distinct	$c_1(m_1)^t + c_2(m_2)^t$	
Equal	$(c_1 + c_2 t)(m)^t$	
Imaginary	$d^t(h_1 \cos \theta t + h_2 i \sin \theta t)$	$ d = m_1 = m_2 $

Note the for stationarity, the roots of the characteristic equation must be "inside the unit circle" (e.g. $|m_1| < 1$)

Sometimes the homogeneous equation is expressed in terms of the backshift operator where

$$By_t = y_{t-1}$$

Then $y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} = 0$

can be expressed as

$$(1 - \phi_1 B - \phi_2 B^2)y_t = 0$$

Treating B as an unknown variable and solving:

$$y_t - .3y_{t-1} - .4y_{t-2} = 0$$

$$(1 - .3B - .4B^2)y_t = 0$$

$$(1 - .8B)(1 + .5B)y_t = 0$$

$$B = 1/.8, \quad 1/-.5$$

Note that these roots are the inverse of on page 4. Therefore, an equivalent condition for stationarity is that the roots of the AR polynomial lie outside the unit circle.

In general, the p-th order homogeneous equation

$$y_t - \phi_1 y_{t-1} - \cdots - \phi_p y_{t-p} = 0$$

has a characteristic equation of the form

$$m^p - \phi_1 m^{p-1} - \cdots - \phi_p = 0$$

solving for m we have

$$(m_1 - k_1)(m_2 - k_2) \cdots (m_p - k_p) = 0$$

where

- the first i are distinct root
- the next j are equal
- the last $k/2$ are complex pairs

(Note that $i + j + k = p$)

The general solution is then

$$\begin{aligned}
 y_t = h(t) &= c_1 m_1^t + c_2 m_2^t + \cdots + c_i m_i^t \quad (\text{distinct}) \\
 &+ (c_{i+1} + c_{i+2}t + c_{i+3}t^2 + \cdots + c_{i+j}t^{j-1})m^t \quad (\text{equal}) \\
 &+ d_1^t [h_1 \cos \theta_1 t + h'_1 i \sin \theta_1 t] \\
 &+ d_2^t [h_2 \cos \theta_2 t + h'_2 i \sin \theta_2 t] \quad (k/2 \text{ complex pairs}) \\
 &+ \cdots \\
 &+ d_{k/2}^t [h_{k/2} \cos \theta_{k/2} t + h'_{k/2} i \sin \theta_{k/2} t]
 \end{aligned}$$

FIGURE 1

 $Y(t) - .5Y(t-1) = 0$

$Y(0) = 5$

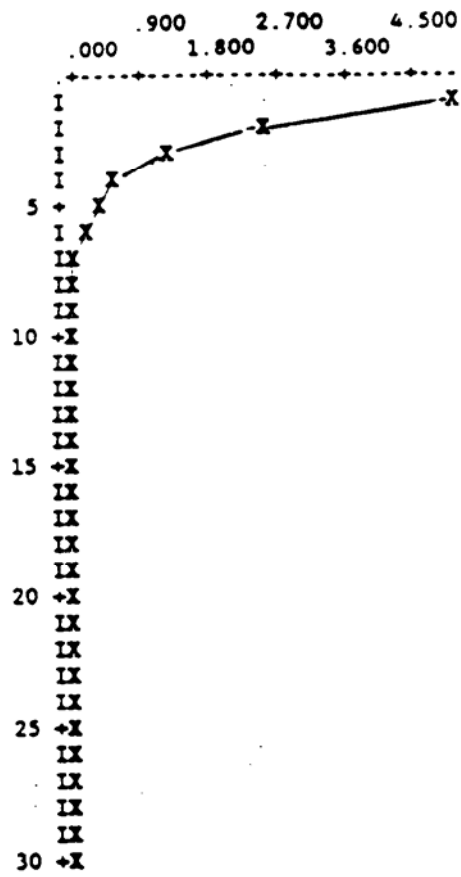


FIGURE 2

 $Y(t) + .5Y(t-1) = 0$

$Y(0) = 5$

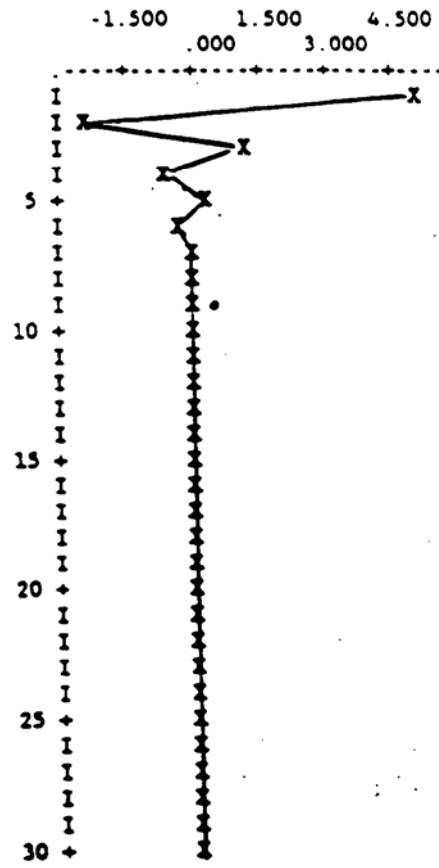


FIGURE 3

$$Y(\tau) - 2Y(\tau-1) = 0$$

$$Y(0) = .01$$

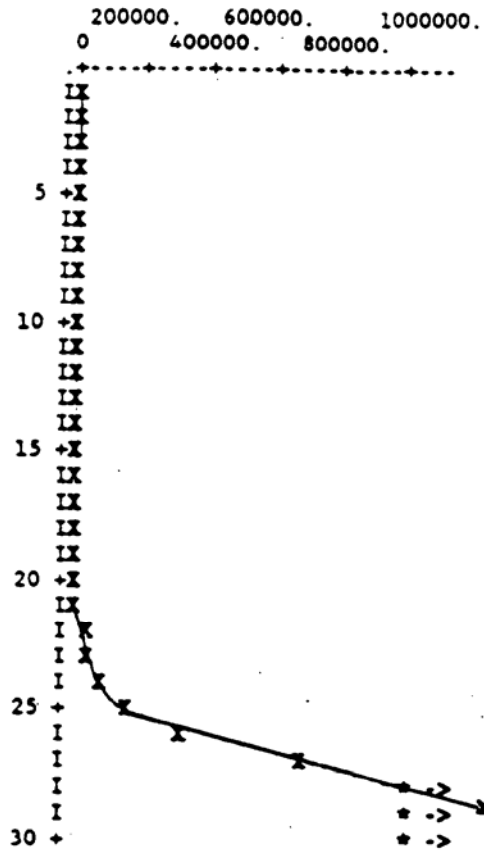


FIGURE 4

$$Y(\tau) - .3Y(\tau-1) - .4Y(\tau-2) = 0$$

$$Y(0) = 5 ; Y(1) = 7$$

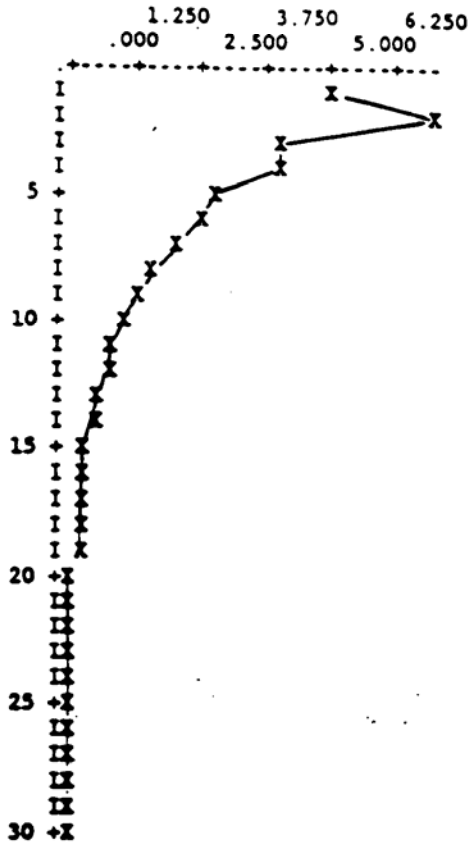


FIGURE 5

$$Y(\tau) - Y(\tau-1) + .25Y(\tau-2) = 0$$

$$Y(0) = 5 ; Y(1) = 7$$

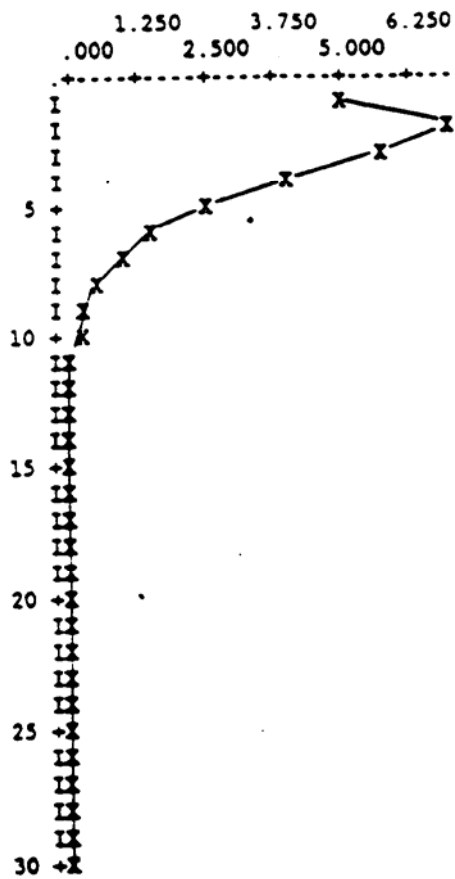
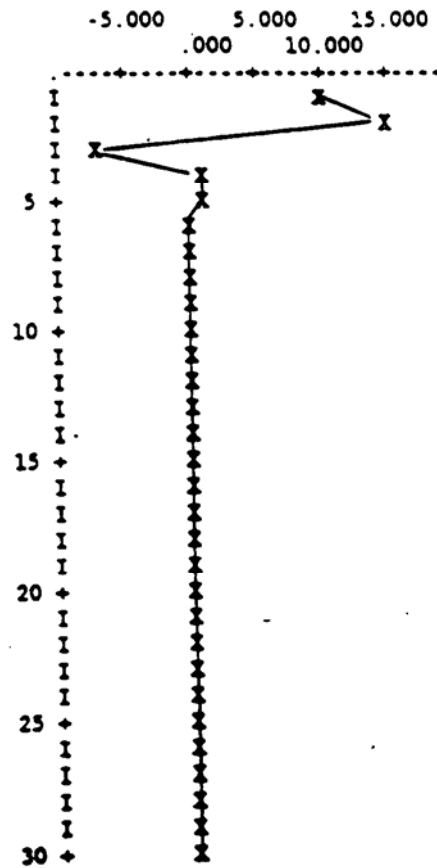


FIGURE 6

$$Y(\tau) + .4Y(\tau-1) + .13Y(\tau-2) = 0$$

$$Y(0) = -10 ; Y(1) = 15$$



Time Series Analysis for Forecasting and Model Building

Reading Wei(1989) Ch 1-2, 3.1, 3.2

An example to introduce the course

Com. Edison – Forecasting future electricity demand

short term prediction

long term prediction

- Simplest – first comes to mind

Historical data

Patterns: stationary, nonstationary, trend, changing trend, seasonal, changing seasonal.

Show WEPCO Data – Intervention

- Exogenous variables

Weather, economic indicators, population growth, leading and lagging, feedback.

Examples of times series in practice: variables patterns (see handout)

How to model such data for forecasting?

Outline of course objectives

- Linear dynamic models

ARIMA models

Forecasting

Model Building

Dynamic transfer function model

Intervention and outliers

- Unit root test

- Signal extraction, unobserved components, seasonal adjustment

- Models with nonhomogeneous variable: ARCH, GARCH, and others.

- Adaptive modeling

Nonlinear models (introduction)

Dynamics, inertia and memory

Most systems (physical, biological, social) have inertia.

ie. The impact of stimuli will take time to materialize.

Chemical and environmental

Social change induced by, say, Federal policy.

e.g. education, welfare

Economic: effect of advertising promotion, government fiscal policy.

- Simple Aspirin example – pain relief

- $Y_t = v_0 X_t + v_1 X_{t-1} + \dots$

X_{t-j} : impulse, stimulus at $t-j$

Y_t : impact at t

v_j : effect of X_{t-j} on Y_t

Impulse responses (v_0, v_1, v_2)

- Backshift or lag operator $B \equiv L$

$$BX_t = X_{t-1} \quad B^2 X_t = X_{t-2}$$

$$Y_t = (v_0 + v_1 B + \dots) X_t = V(B) X_t$$

- $V(B)$: transfer function

distributed lag

Clearly, pattern in Y_t depends on

◆ nature of $v(B)$

◆ nature of $\{X_t\}$

Linear dynamic system

How does pattern in time series arise?

One conceptual formulation:

z : observable variables has some long run equilibrium level η

z_t : observation of Z at time t ; $z_t - \eta$ deviation from η

$$\dot{z}_t = \dot{z}_t - \eta$$

$$\dot{z}_t = a_t + \psi_1 + \psi_2 a_{t-2} + \dots$$

a_t : white noise hitting the system at time t . a_t, a_{t-j} independent, $j \neq 0$

with $E(a_t) = 0$ $\text{var}(a_t) = \sigma_a^2$ (homogeneous)

$$a_t \sim N(0, \sigma_a^2)$$

Pattern arises due to $(1, \Psi_1, \Psi_2, \Psi_3, \dots)$

In practice, Ψ_j unknown but $j=1, \dots$ can be a large number. On the other hand, data(observations) are limited.

➔ Model the pattern of time series → model the Ψ 's, i.e. make them functional related.

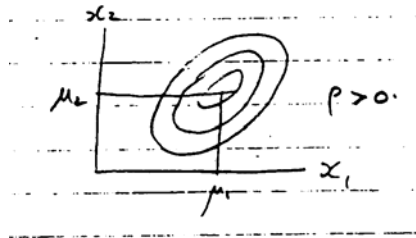
e.g. $\Psi_j = 0 \quad j=2, 3,$
 $\Psi_2 = \Psi_1^2, \Psi_3 = \Psi_1^3, \dots$
 $\Psi_1 = \Psi_2 = \Psi_3$

ARIMA models: a class of finite-parameter representations of the Ψ 's

Review of normal distribution: $P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

(μ, σ^2) completely specify the distribution.

Bivariate:



$$P(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho)^{1/2}} e^{\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]}$$

$$E(x_i) = \mu_i$$

$$\text{var}(x_i) = \sigma_i^2, i = 1, 2$$

$$\text{cov}(x_1, x_2) = \sigma_1\sigma_2\rho = \sigma_{12} \quad |\rho| < 1$$

write:

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \quad (\text{symmetric})$$

$$P(\underset{\sim}{x}) = \frac{1}{2\pi} |\Sigma| e^{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu})}$$

Multivariate:

$$P(\underset{\sim}{x}) = \frac{1}{2\pi^{2/k}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu})} \quad N_k(\underset{\sim}{\mu}, \Sigma)$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} \quad \underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1k} \\ & \ddots & & \\ & & \ddots & \\ & & & \sigma_{kk} \end{bmatrix}$$

Σ is p.d. i.e. $\underset{\sim}{c}' \Sigma \underset{\sim}{c} > 0$, $\underset{\sim}{c}' \Sigma \underset{\sim}{c} = 0$ iff $\underset{\sim}{c} = 0$

$$E(\underset{\sim}{x}) = \underset{\sim}{\mu}, \quad E(\underset{\sim}{x}-\underset{\sim}{\mu})(\underset{\sim}{x}-\underset{\sim}{\mu})' = \Sigma = \text{cov}(\underset{\sim}{x}), \quad (\underset{\sim}{x}-\underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu}) \sim \chi_k^2$$

Decomposition

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ \sim \\ x_2 \\ \sim \end{bmatrix} \quad \underset{\sim}{\mu} = \begin{bmatrix} \mu_1 \\ \sim \\ \mu_2 \\ \sim \end{bmatrix} \quad \underset{\sim}{\Sigma} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

Then $\underset{\sim}{x}_j \sim N(\underset{\sim}{\mu}_j, \Sigma_{jj}) \quad j = 1, 2$

$$\underset{\sim}{x}_2 \mid \underset{\sim}{x}_1 \sim N(\underset{\sim}{\mu}_{2 \cdot 1}, \Sigma_{22 \cdot 1})$$

$$\Sigma_{22 \cdot 1} = \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}$$

$$\underset{\sim}{\mu}_{2 \cdot 1} = \underset{\sim}{\mu}_2 - \Sigma_{21} \Sigma_{11}^{-1} (\underset{\sim}{x}_1 - \underset{\sim}{\mu}_1)$$

Similarly for $\underset{\sim}{x}_1 \mid \underset{\sim}{x}_2$; of course, $P(\underset{\sim}{x}) = P(\underset{\sim}{x}_1)P(\underset{\sim}{x}_2 \mid \underset{\sim}{x}_1) = P(\underset{\sim}{x}_2)P(\underset{\sim}{x}_1 \mid \underset{\sim}{x}_2)$

For k dimension normal, $(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ completely specify it.

Stochastic process

$$\{z_t\} \quad t = 0, \pm 1, \pm 2, \dots$$

sequence of random variable with specified probability structure.

Stationary process

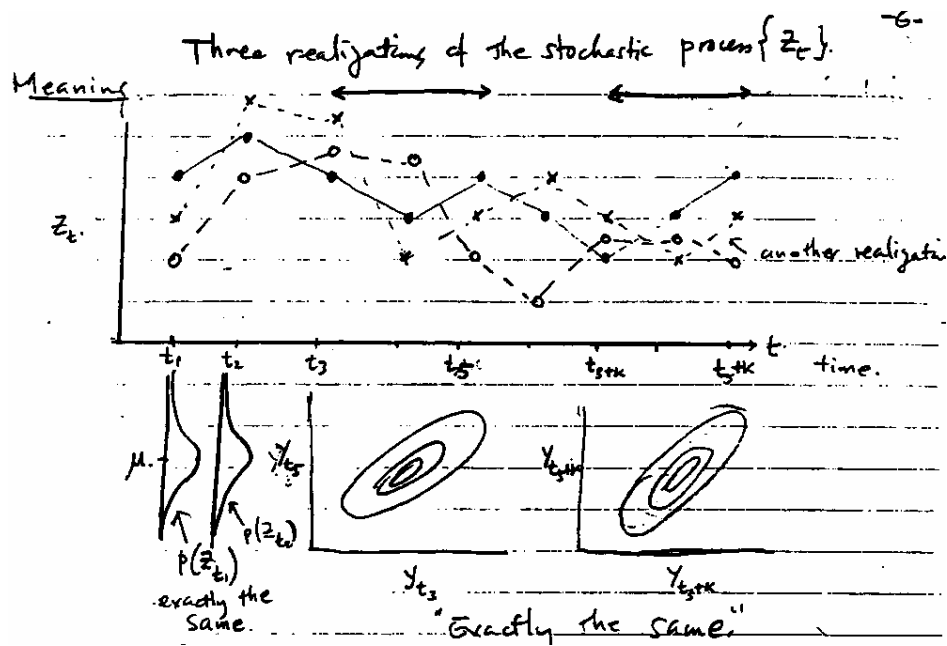
$$P(z_{t_1}, z_{t_2}, \dots, z_{t_n}) \sim P(z_{t_1+k}, z_{t_2+k}, \dots, z_{t_n+k})$$

For any choice of $(t_1, \dots, t_n; k)$. In words, probability structure invariant to shift in time origin (see BJ-approach). This means that,

$$E(z_t) = \mu \quad \forall t$$

$$\text{cov}(z_t, z_{t+k}) = \gamma_k = \gamma_{-k} \quad \text{function of } |k| \text{ only}$$

We also assume Normality.



Stationary Gaussian

We only need to discuss the process in terms of μ and $\gamma_k, k=0,1,2,\dots$

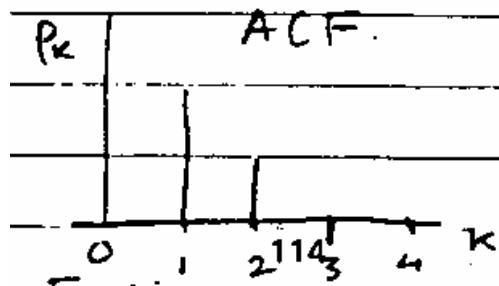
If we have n observations $z_t, t=1,\dots,n$

$$\text{cov}_{\tilde{z}} = \text{cov} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k & \cdots & \gamma_{n-1} \\ \gamma_1 & \ddots & & & & \gamma_k \\ & \ddots & \ddots & & & \\ \gamma_k & & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \gamma_1 \\ \gamma_{n-1} & & & \gamma_k & \gamma_1 & \gamma_0 \end{bmatrix} \quad \sigma^2 = \gamma_0 \quad \rho_k = \frac{\gamma_k}{\gamma_0} \quad (\text{ACF})$$

$$\text{corr}(\tilde{z}) = \rho(\tilde{z}) = \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{n-1} \\ \rho_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \rho_1 \\ \rho_{n-1} & \cdots & \rho_1 & 1 \end{bmatrix} \quad \sigma^2 \text{ and } \rho_k, k=0,\pm 1,\dots$$

$$\rho_1 = \text{corr}(z_t, z_{t+1}) = \text{corr}(z_t, z_{t-1})$$

$$\rho_2 = \text{corr}(z_t, z_{t+2}) = \text{corr}(z_t, z_{t-2}) \quad \text{Autocorrelation Function}$$



Wold Decomposition

The representation

$$\dot{z}_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots \quad (\text{A})$$

is in fact valid under very general conditions. Wold showed (193) that for any stationary process with finite variance, (A) holds with

$$(i) \quad \psi_1^2 + \psi_2^2 + \dots < \infty$$

$$(ii) \quad E(a_t) = 0, \quad \text{var}(a_t) = \sigma_a^2, \quad E(a_t a_{t-j}) = 0 \quad j \neq 0$$

uncorrelated but not necessarily normal.

(This means that even nonlinear stationary process can have this representation)

This is known as Wold Decomposition Theorem (a simplified form here)

In what follows, we assume normality for the a_t 's and regard (A) as linear model.

Note that for the white noise process $\{a_t\}$ process ,

$$\gamma_0 = \sigma_a^2, \quad \gamma_j = 0 \quad j \neq 0 \quad \rho_j = 0$$

Now we are ready to introduce stationary ARMA models

Yule (1927), Slutsky (1927)

AR MA

Sunspot (see Fig. chapter 1, p.12) 100 obs. 1770-1869

MA(1) $\dot{z}_t = a_t - \theta a_{t-1}$

Here, $\psi_1 = -\theta$, $\psi_2 = \dots = 0$ Effect of a_t short-lived, lasting only one lag period.

See Fig. p.236 for generated MA(1) $\theta = .7$

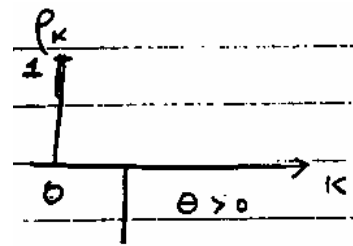
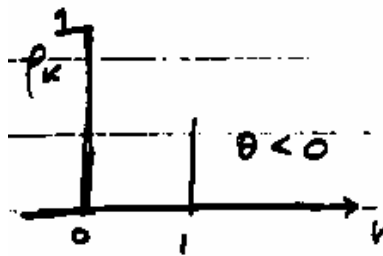
Basic properties:

$$\dot{z}_t = a_t - \theta a_{t-1}$$

$$\dot{z}_{t-1} = a_{t-1} - \theta a_{t-2}$$

$$E(z_t) = \mu$$

$$\text{cov} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \sigma_a^2 \begin{bmatrix} 1+\theta^2 & -\theta & & \\ -\theta & \ddots & \ddots & \\ & \ddots & \ddots & -\theta \\ & & -\theta & 1+\theta^2 \end{bmatrix} \Rightarrow \begin{cases} \gamma_0 = \sigma_a^2(1+\theta^2) \\ \rho_1 = \frac{-\theta}{1+\theta^2} \end{cases} \quad \begin{matrix} \gamma_1 = -\theta\sigma_a^2 \\ \rho_2 = \dots = 0 \end{matrix} \quad \begin{matrix} \gamma_2 = \dots = 0 \\ \text{ACF} \end{matrix}$$



(i) Stationary

(ii) z_t, z_{t-j} independent for $|j| > 1$

(iii) a_t independent of $z_{t-j}, j > 0$

$$(iv) \text{cov}(z) = \sigma_a^2 \theta^2 \begin{bmatrix} 1+\frac{1}{\theta^2} & -\frac{1}{\theta} & & \\ -\frac{1}{\theta} & \ddots & \ddots & \\ & \ddots & \ddots & -\frac{1}{\theta} \\ & & -\frac{1}{\theta} & 1+\frac{1}{\theta^2} \end{bmatrix} = \sigma_b^2 \begin{bmatrix} 1+\eta^2 & -\eta & & \\ -\eta & \ddots & \ddots & \\ & \ddots & \ddots & -\eta \\ & & -\eta & 1+\eta^2 \end{bmatrix}$$

equivalent representation:

$$\dot{z}_t = b_t - \eta b_{t-1} \quad \sigma_b^2 = \sigma_a^2 \theta^2 \quad \eta = \frac{1}{\theta}$$

$$|\theta| \leq 1 \quad \Leftrightarrow \quad |\eta| \geq 1$$

$\therefore$ we have 2 equivalent representations:

$$\dot{z}_t = a_t - \theta a_{t-1} \quad \text{or} \quad \dot{z}_t = b_t - \eta b_{t-1}$$

(v) When $|\theta| \leq 1$ process called “invertible”

$$\dot{z}_t = (1 - \theta B)a_t \quad a_t = (1 - \theta B)^{-1} \dot{z}_t$$

$$(1 - \theta B)^{-1} = (1 + \theta B + \theta^2 B^2 + \dots)$$

$$\dot{z}_t = -\theta \dot{z}_{t-1} - \theta^2 \dot{z}_{t-2} - \dots + a_t$$

The meaning and usefulness of this representation will be discussed later. For $|\theta| = 1$, not invertible, i.e. this expression is meaningless.

(vi) Relationship between (ρ_1, γ_0) and (σ_a^2, θ)

(a) $|\rho_1| \leq 0.5$ (exercise)

(b) From $\rho_1 = \frac{-\theta}{1+\theta^2} \Rightarrow (1+\theta^2)\rho + \theta = 0 \Rightarrow \theta = \frac{-1 \pm \sqrt{1-4\rho_1^2}}{2\rho_1}$ for $\rho \neq 0$

invertible solution is $\theta = \frac{1}{2\rho_1}(-1 + \sqrt{1-4\rho_1^2})$ (verify)

and $\sigma_a^2 = \frac{\gamma_0}{1+\theta^2}$

Here we express (θ, σ_a^2) in terms of (γ_0, ρ_1)

Practical usefulness: later we will discuss estimating ρ_k by Ω_k --sample autocorrelation function. The shape of ρ_k will help specify an MA(1) model.

(vii) Auto-covariance and Auto-correlation generating function

Define $\gamma(B) = \sigma_a^2(1 - \theta B)(1 - \theta F)$

$$= \sigma_a^2 [1 - \theta(B + F) + \theta^2 BF]$$

$$= \sigma_a^2 [1 + \theta^2 - \theta(B + F)]$$

$$= \gamma_0 + \gamma_1(B + F)$$

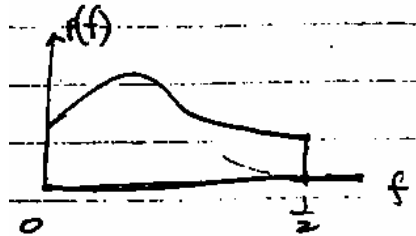
$$\rho(B) = \frac{1}{\gamma_0} \gamma(B) = 1 + \rho_1(B + F)$$

(viii) Put $B = e^{i2\pi f}$ $0 \leq f \leq \frac{1}{2}$

Wei(1989), chapter 3, 4

$$\text{Let } P(f) = 2\gamma(e^{i2\pi f}) = 2[\gamma_0 + \gamma_1 2 \cos 2\pi f] \quad 0 \leq f \leq \frac{1}{2}$$

Called spectrum density function



$$P(f) \geq 0 \quad \int_0^{\frac{1}{2}} P(f) = \gamma_0$$

$P(f)$: allocation of total variance to different frequency.

$$\text{MA}(2) \quad \dot{z}_t = (1 - \theta_1 B - \theta_2 B^2) a_t \quad \psi_1 = -\theta_1 \quad \psi_2 = -\theta_2$$

1) clearly stationary $\gamma_j = 0, j > 2$, also $a_t \perp Z_{t-j}$

$$2) \quad \gamma(B) = \sigma_a^2 (1 - \theta_1 B - \theta_2 B^2)(1 - \theta_1 F - \theta_2 F^2)$$

$$= \gamma_0 + \gamma_1 (B + F) + \gamma_2 (B^2 + F^2)$$

verify indeed that $\gamma_0 = \text{var}(Z_t) = (1 + \theta_1^2 + \theta_2^2) \sigma_a^2$

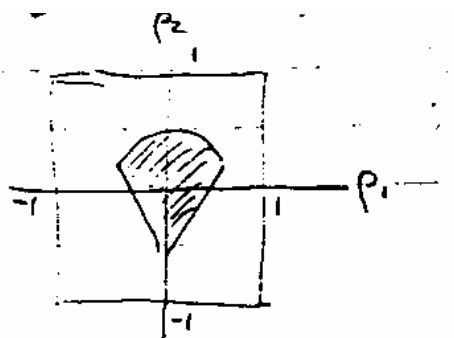
$$\gamma_1 = \text{cov}(z_t, z_{t-1}) = -\theta_1(1 - \theta_2) \sigma_a^2 \quad \text{and} \quad \gamma_2 = \text{cov}(z_t, z_{t-2}) = -\sigma_a^2$$

$$\Rightarrow \quad \rho_1 = \frac{-\theta_1(1 - \theta_2)}{1 + \theta_1^2 + \theta_2^2} \quad \rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

Key: $\rho_2 \neq 0$ and $\rho_j = 0 \quad j > 2$

Admissible region for (ρ_1, ρ_2)

$$\rho_1 + \rho_2 \geq 0.5, \quad \rho_2 - \rho_1 \geq -0.5, \quad \rho_1^2 \leq 4\rho_2(1 - 2\rho_2) \quad \text{see BJR p.85}$$



3) invertible representation (polynomial of degree 2 in B)

$$(1 - \theta_1 B - \theta_2 B^2) = (1 - \alpha_1 B)(1 - \alpha_2 B), \quad \alpha_1 + \alpha_2 = \theta_1, \quad \alpha_1 \alpha_2 = -\theta_2$$

If $|\alpha_i| < 1, \quad i = 1, 2$ model is invertible, i.e. $(1 - \theta_1 B - \theta_2 B^2)^{-1} z_t = a_t$

$$\text{write } (1 - \theta_1 B - \theta_2 B^2)^{-1} = (1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \dots) = \pi(B) \quad (*)$$

$\pi(B)$ will converge for $|B| \leq 0$

$$\text{Then } z_t = \pi_1 z_{t-1} + \pi_2 z_{t-2} + \pi_3 z_{t-3} + \dots + a_t$$

expressing z_t as a linear combination of past values of z_t plus current shock .

$$\text{From } (*) \quad (1 - \theta_1 B - \theta_2 B^2)(1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \dots) = 1$$

Equating coefficient of powers of B

$$B : \quad -\pi_1 - \theta_1 = 0 \Rightarrow \pi_1 = -\theta_1$$

$$B^2 : \quad -\pi_2 + \theta_1 \pi_1 - \theta_2 = 0 \Rightarrow \pi_2 = \theta_1 \pi_1 - \theta_2 = -\theta_1^2 - \theta_2$$

$$B^3 : \quad -\pi_3 + \theta_1 \pi_2 + \theta_2 \pi_1 = 0 \Rightarrow \pi_3 = \theta_1 \pi_2 + \theta_2 \pi_1$$

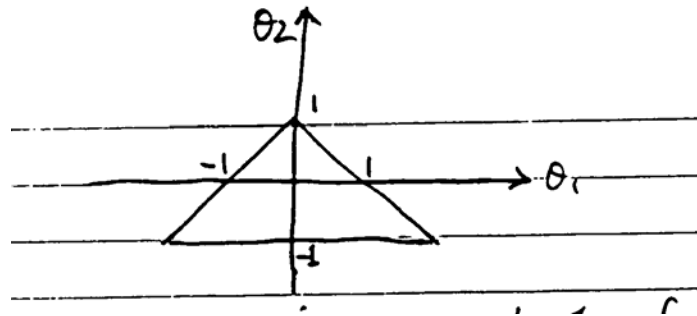
$$B^j : \quad j \geq 3 \Rightarrow \pi_j = \theta_1 \pi_{j-1} + \theta_2 \pi_{j-2} = 0 \quad \text{Difference equation}$$

To study nature of π 's , solve difference equation.

For actual computation—recursive using,

$$\pi_j = \theta_1 \pi_{j-1} + \theta_2 \pi_{j-2} \quad \text{with } \pi_l = 0, \quad l < 0, \quad \pi_0 = -1$$

Invertible region for (θ_1, θ_2) , $\theta_1 + \theta_2 \leq 1$, $\theta_2 - \theta_1 \leq 1$, $|\theta_2| \leq 1$



- 4) For any MA(2) model of the form

$$\dot{z}_t = (1 - \eta_1 B - \eta_2 B^2) b_t, \quad \sigma_b^2 = \text{var}(b_t)$$

Then exists an equivalent representation

$$\dot{z}_t = (1 - \theta_1 B - \theta_2 B^2) a_t$$

such that the zeros of $(1 - \theta_1 B - \theta_2 B^2)$ are on or outside the unit circle.

(exercise)

- 5) put $b = e^{i2\pi f}$ $p(f) = 2\gamma(e^{i\pi f}) = 2[\gamma_0 + 2\gamma_1 \cos 2\pi f + 2\gamma_2 \cos 4\pi f]$

MA(q) $\dot{z}_t = (1 - \theta_1 B - \dots - \theta_q B^q) a_t$ or $\dot{z}_t = \theta(B) a_t$

- 1) stationary
- 2) $\left. \begin{array}{l} \rho_l \neq 0, l = q \\ \rho_l = 0, l > q \end{array} \right\}$ key feature. "Cutting off" property.
- 3) If zeros of $\theta(B)$ all outside unit circle, invertible.

$$\theta(B)\pi(B) = 1, \quad \pi(B) = 1 - \pi_1 B - \pi_2 B^2 \dots \text{converge}$$

$$\pi_j = \theta_1 \pi_{j-1} + \dots + \theta_q \pi_{j-q}, \quad \pi_0 = -1, \quad \pi_l = 0, \quad l < 0, j = 1, \dots$$

- 4) Note the 1-1 correspondence between length of memory and length of autocorrelation.

Stationary Gaussian linear process

$$z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots \quad a_t \sim N(0, \sigma_a^2)$$

$$\text{stationary if } \sum_{j=0}^{\infty} \psi_j^2 < \infty \quad \psi_0 = 1$$

ACF

$$E(z_t) = \mu$$

$$\text{var}(z_t) = \gamma_0 = \sigma_a^2(1 + \psi_1^2 + \psi_2^2 + \dots)$$

$$\text{cov}(z_t, z_{t-1}) = (\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 + \dots)\sigma_a^2 = \gamma_1$$

$$\text{cov}(z_t, z_{t-k}) = (\psi_k + \psi_1\psi_{k+1} + \psi_2\psi_{k+2} + \dots)\sigma_a^2 = \gamma_k$$

$$\rho_k = \frac{\gamma_k}{\gamma_0} \quad \text{clearly, } \rho_k \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

SACF Sample ACF $z_t, \quad t = 1, \dots, n$

$$r_k = \frac{\frac{1}{n} \sum_{t=1}^{n-k} (z_t - \bar{z})(z_{t+k} - \bar{z})}{\frac{1}{n} \sum_{t=1}^n (z_t - \bar{z})^2} = \frac{c_k}{c_0} \quad k = 1, 2, \dots \quad \bar{z} = \frac{1}{n} \sum_{t=1}^n z_t$$

Various versions slightly different from this, but this one is the most commonly used.

$$c_k \xrightarrow{P} \gamma_k$$

$$r_k \xrightarrow{P} \rho_k \quad \text{for stationary Gaussian process.}$$

(validity more general)

(exercise: do one for MA(1) model) use Tchebycheff inequality.

Show generated MA(1) example – SACF

$\text{var}(r_k) = ?$ Bartlett (1946) (see 1955, An Introduction to Stochastic. P256, p289)

For stationary process

$$\text{cov}(r_k, r_{k+l}) \doteq \frac{1}{n} \sum_{v=-\infty}^{\infty} (\rho_v \rho_{v+l} + \rho_v \rho_{v+2k+l} + 2\rho_k \rho_{k+l} \rho_v^2 - 2\rho_k \rho_v \rho_{v+k+l} - 2\rho_{k+l} \rho_v \rho_{v+k})$$

Put $l = 0$, we get $\text{var}(r_k)$

Complicated in general. Special case: suppose $\rho_l = 0$, $l > q$

i.e. MA(q) situation then

$$\text{var}(r_k) \doteq \frac{1}{n} [1 + 2(\rho_1^2 + \dots + \rho_q^2)] \text{ for } k > q$$

Practical use:

1st assume model is MA(0), i.e. white noise.

$$\text{var}(r_k) \doteq \frac{1}{n} \quad k = 1, 2, \dots$$

If r_1 is large, $(|r_1| > \frac{2}{\sqrt{n}})$ then assume model is MA(1), Then

$$\text{var}(R_k) \doteq \frac{1}{n} [1 + 2\rho_1^2] \doteq \frac{1}{n} [1 + 2R_1^2] \quad k = 2, 3, \dots$$

etc. Show generated MA(1) example's SACF

$$\text{AR}(1) \quad \dot{z}_t = a_t + \psi_1 a_{t-1} + \dots$$

$$= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} + \dots \quad (**)$$

stationary for $|\phi| < 1$. Alternately, we can write as

$$\dot{z}_t - \phi \dot{z}_{t-1} = a_t \text{ or } \dot{z}_t = \phi \dot{z}_{t-1} + a_t$$

Show generated AR(1) example; more momentum.

Note that AR(1) is a “high” order MA model or model with a long but exponentially decaying memory.

$$(1 - \phi B) \dot{z}_t = a_t$$

$$\dot{z}_t = (1 - \phi B)^{-1} a_t = (1 + \phi B + \phi^2 B^2 + \dots) a_t$$

Covariance structure use difference equation approach.

$$E \dot{z}_{t-l} (\dot{z}_t - \phi \dot{z}_{t-1}) = \dot{z}_{t-l} a_t$$

$$l = 0 \quad \gamma_0 - \phi \gamma_1 = \sigma_a^2$$

$$l = 1 \quad \gamma_1 - \phi \gamma_0 = 0$$

$$l=2 \quad \gamma_2 - \phi\gamma_1 = 0$$

$$\gamma_k - \phi\gamma_{k-1} = 0 \quad k = 1, 2, \dots$$

Also $\gamma_0 = \frac{\sigma^2}{1 - \phi^2}$ (Do this directly from $(**)$)

Note again the correspondence between ρ_k and ψ 's.

Here $\rho_k \rightarrow$ exponential decay between ψ 's $\rightarrow$ exponential decay.

Unlike MA model for which ρ_k “cuts off”

SACF: Show generated AR(1) example.

Covariance generating function:

$$\gamma(B) = \sigma_a^2 \frac{1}{(1 - \phi B)(1 - \phi F)} = \sigma_a^2 (1 + \phi B + \phi^2 B^2 + \dots)(1 + \phi F + \phi^2 F^2 + \dots)$$

Spectrum:

$$p(f) = \frac{2\sigma_a^2}{(1 - \phi e^{i2\pi f})(1 - \phi e^{-i2\pi f})} = \frac{2\sigma_a^2}{1 + \phi^2} \left[1 - \frac{2\phi}{1 + \phi^2} \cos 2\pi f \right] \quad 0 \leq f \leq \pi$$

Exercise: (plot this for $\phi = -0.5, 0.5, 0.9$)

Joint distribution if $z_1, \dots, z_n$, since $a_t \perp z_{t-j}, j > 0$,

$$\begin{bmatrix} 1 & & & & \\ -\phi & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\phi & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ \vdots \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$z_1 = a_1 + \phi z_0$$

$$z_0 = a_0 + \phi a_{-1} + \phi^2 a_{-2} + \dots$$

implies that $z_t | z_1, \dots, z_{t-1} \rightarrow z_t | z_{t-1}$ Markovian property.

And $P(z_t | z_{t-1}) \sim N(\phi z_{t-1}, \sigma_a^2)$

AR(2) $(1 - \phi_1 B - \phi_2 B^2) \dot{z}_t = a_t$ Yule's model

Memory $\psi's$

$$\dot{z}_t = (1 - \phi_1 B - \phi_2 B^2)^{-1} a_t = (1 + \psi_1 B + \dots) a_t$$

$$\therefore (1 - \phi_1 B - \phi_2 B^2)(1 + \psi_1 B + \psi_2 B^2 \dots) = 1$$

$$\Rightarrow \psi_l = \phi_1 \psi_{l-1} + \phi_2 \psi_{l-2} \quad l = 1, 2, \dots$$

$$\text{with } \psi_0 = 1, \quad \psi_l = 0 \quad l < 0$$

Thus $\psi_1 = \phi_1$, $\psi_2 = \phi_1^2 + \phi_2$ etc.

Difference equation solution:

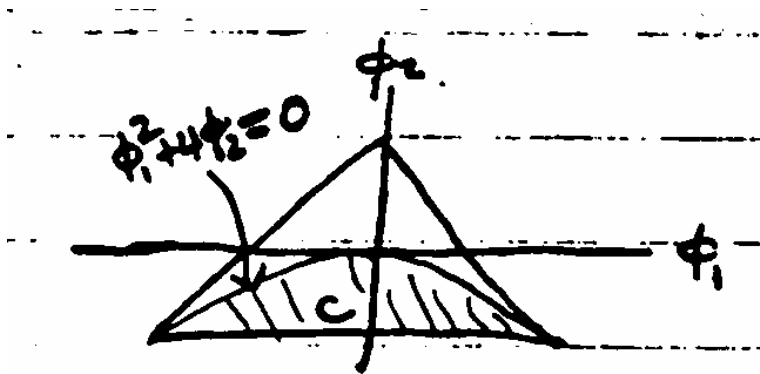
$$\psi_l = A_1 \alpha_1^l + A_2 \alpha_2^l$$

For recursive calculation, don't use the solution. This is only for understanding the nature of $\psi's$.

$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \alpha_1 B)(1 - \alpha_2 B) \quad \alpha_1 + \alpha_2 = \phi_1$$

stationary condition $|\alpha_i| < 1 \Rightarrow (\phi_1, \phi_2)$ in the triangle.

$$\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1, \quad |\phi_2| < 1$$



Note the mirror image between AR(2) stationary and MA(2) invertibility.

Covariance structure: $z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots$

$$z_{t-l}(z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2}) = z_{t-l} a_t$$

Taking expectation:

$$l = 0 \quad \gamma_0 - \phi_1 \gamma_1 - \phi_2 \gamma_2 = \sigma_a^2$$

$$l = 1 \quad \gamma_1 - \phi_1 \gamma_0 - \phi_2 \gamma_1 = 0$$

$$l = 2 \quad \gamma_2 - \phi_1 \gamma_1 - \phi_2 \gamma_0 = 0$$

$$l \geq 2 \quad \gamma_l - \phi_1 \gamma_{l-1} - \phi_2 \gamma_{l-2} = 0$$

From $l=0$ $\gamma_0 = \sigma_a^2(1 - \phi_1\rho_1 - \phi_2\rho_2)^{-1}$

$$\begin{cases} l=1 \\ l=2 \end{cases} \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \quad \text{Yule-Walker equation}$$

$$\rho_l - \phi_1\rho_{l-1} - \phi_2\rho_{l-2} = 0$$

$$\rho_l = c_1\alpha_1^l + c_2\alpha_2^l$$

same nature as the ψ' 's .

Mixture of 2 exponentials or damped sine wave.

$$\text{Also, } \rho_1(1 - \rho_2) = \phi_1 \quad \rho_1 = \phi_1/(1 - \phi_2)$$

$$\rho_2 - \phi_1\rho_1 - \phi_2 = 0 \quad \rho_2 = \phi_2 + \phi_1^2/(1 - \phi_2)$$

Thus, we can express $\gamma_0, \rho_1, \rho_2, \dots$ in terms of $\sigma_a^2, \phi_1, \phi_2, \dots$ and vice versa.

Covariance generating function and spectrum. (exercise)

Joint distribution of $z_1, \dots, z_n$. Again note $a_t \perp z_{t-j}, j > 0$,

$$\begin{bmatrix} 1 & & & & & \\ -\phi_1 & \ddots & & & & \\ -\phi_2 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\phi_2 & -\phi_1 & 1 & \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$$

$$\omega_1 = a_1 + \phi_1 z_0 + \phi_2 z_{-1}$$

$$\omega_2 = a_2 + \phi_2 z_0$$

$$(\omega_1, \omega_2) \perp (a_3, \dots, a_n)$$

$$\Rightarrow z_t | z_1, \dots, z_n \text{ depend only on } z_{t-1}, z_{t-2}$$

$$z_t | z_{t-1}, z_{t-2} \sim N(\phi_1 z_{t-1} + \phi_2 z_{t-2}, \sigma_a^2)$$

$$\text{AR(P)} \quad (1 - \phi_1 B - \dots - \phi_p B^p) \dot{z}_t = a_t$$

$$\text{Memory} \quad \psi'(s) = \phi_p(B) = (1 - \alpha_1 B) \dots (1 - \alpha_p B)$$

$$\text{And stationary} \quad |\alpha_j| < 1 \quad \text{stationarity condition} \quad j = 1, \dots, p$$

$$(1 - \phi_1 B - \dots - \phi_p B^p) \psi(B) = 1$$

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \dots$$

$$\Rightarrow \psi_l = \phi_1 \psi_{l-1} + \dots + \phi_p \psi_{l-p} \quad \text{Difference equation.}$$

$$l = 1, 2, \dots$$

$$\text{with } \psi_0 = 1, \quad \psi_l = 0, \quad l < 0$$

$$\gamma_l \text{ and } \rho_l \text{ structure: } \dot{z}_t = a_t + \psi_1 a_{t-1} + \dots$$

$$\dot{z}_{t-l} (\dot{z}_t - \phi_1 \dot{z}_{t-1} - \dots - \phi_p \dot{z}_{t-p}) = \dot{z}_{t-l} a_t$$

Taking expectation:

$$l=1 \quad \gamma_0 - \phi_1 \gamma_1 - \dots - \phi_p \gamma_p = \sigma_a^2$$

$$l=1 \quad \gamma_1 - \phi_1 \gamma_0 - \dots - \phi_p \gamma_{p-1} = 0$$

$$\vdots$$

$$l=p \quad \gamma_p - \phi_1 \gamma_{p-1} - \dots - \phi_p \gamma_0 = 0$$

$$l > p \quad \gamma_l - \phi_1 \gamma_{l-1} - \dots - \phi_p \gamma_{l-p} = 0$$

$$\text{From } \gamma_0 = \sigma_a^2 (1 - \phi_1 \rho_1 - \dots - \phi_p \rho_{l-p})^{-1}$$

$$\rho_l = \phi_1 \rho_{l-1} + \dots + \phi_p \rho_{l-p} \quad l > p \quad \text{Difference equation.}$$

For $l = 1, \dots, p$

$$\begin{bmatrix} 1 & \rho_1 & \dots & \rho_{p-1} \\ \rho_1 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \rho_1 \\ \rho_{p-1} & \rho_1 & 1 & \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \vdots \\ \phi_p \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \vdots \\ \vdots \\ \rho_p \end{bmatrix} \quad \text{Yule-Walker equation.}$$

PACF motivation: ACF complex for AR(P) model but simple for MA(q) model. Can we use something of a similar nature for AR?

From Yule-Walker equation, we see that

$$P=1 \quad \phi_1 = \rho_1$$

$$P=2 \quad \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \quad \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{1-\rho_1^2} \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & 1 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

$$\phi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}}$$

$$P=3 \quad \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}$$

$$\phi_3 = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} \quad \text{Cramer's rule}$$

Partial ACF P_l , Define

$$\phi_l = \frac{\begin{vmatrix} 1 & \cdots & \rho_{l-2} & \rho_1 \\ \rho_1 & \ddots & \rho_{l-3} & \rho_2 \\ \vdots & & \vdots & \vdots \\ \rho_{l-1} & & \rho_1 & \rho_l \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \cdots & \rho_l \\ \rho_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ \rho_{l-1} & & \rho_1 & 1 \end{vmatrix}}$$

Then for AR(p)

$$\phi_l = \begin{cases} \phi_p & l = p \\ 0 & l > p \end{cases} \quad \text{In Wei's book as in BJ } P_l = \phi_{ll}$$

Thus $\phi_l = f(\rho_1, \dots, \rho_l)$ has the nice property for AR as the ACF for MA's.

Sample PACF: Do L.S. regression successively write AR(1), AR(2), ... and for each fit, retain the estimate of the coefficient of the highest order. (This is better than using R_k , to be explained later.)

$$\hat{\phi}_l \quad l = 1, \dots$$

$$\text{var}(\hat{\phi}_l) \doteq \frac{1}{n}, \quad l > p \quad \text{if the model is AR}(p)$$

show output of AR(1), MA(1), BJ-F,

Practice ϕ_l for MA(1) $\rho_1 \neq 0 \quad \rho_2 = \dots = 0$

$$\phi_1 = \rho_1 \neq 0$$

$$\phi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{-\rho_1^2}{1 - \rho_1^2} \neq 0$$

$$\phi_3 = \frac{-\rho_1^3}{1 - 2\rho_1^2} \neq 0 \quad \text{but getting smaller (tailing off)}$$

Reason $z_t = (1 - \theta B)a_t \Rightarrow (1 - \theta B)^{-1}z_t = a_t$ high order AR.

ARMA(p,q) show BJ-A SACF and SPACF

$$(1 - \phi_1 B - \dots - \phi_p B^p) \dot{z}_t = (1 - \theta_1 B - \dots - \theta_q B^q) a_t$$

$$\phi_p(B) = (1 - \phi_1 B - \dots - \phi_p B^p) \quad \theta_q(B) = (1 - \theta_1 B - \dots - \theta_q B^q)$$

no cancellation $\Leftrightarrow$ no common factors for these 2 polynomials.

ARMA(1,1)

$$(1 - \phi B) z_t = (1 - \theta B) a_t \quad \dot{z}_t = (1 - \phi B)^{-1} (1 - \theta B) a_t$$

$$\psi's \quad (1 - \phi B)^{-1} (1 - \theta B) = \psi(B) \quad \text{or} \quad (1 - \phi B)(1 + \psi_1 B + \psi_2 B^2 + \dots) = (1 - \theta B)$$

$$\psi_l - \phi \psi_{l-1} = -\theta_l \quad \theta_l = \begin{cases} \theta & l=1 \\ 0 & l>1 \end{cases}$$

$$l=1,2,\dots; \quad \psi_0 = 1$$

clearly stationary requires $|\phi| < 1$

$$\pi's \quad a_t = (1 - \theta B)^{-1} (1 - \phi B) z_t = \pi(B) z_t$$

$$(1 - \theta B)(1 - \pi_1 B - \pi_2 B^2 - \dots) = (1 - \phi B)$$

$$\pi_l - \theta \pi_{l-1} = \phi_l \quad \phi_l = \begin{cases} \phi & l=1 \\ 0 & l>1 \end{cases}$$

$$l=1,2,\dots, \quad \pi_0 = -1$$

$\therefore$ invertibility requires $|\theta| < 1$ Note: both $\psi's$ and $\pi's$ tail off.

Covariance structure: $z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots$

$$z_{t-l}(z_t - \phi z_{t-1}) = z_{t-l}(a_t - \theta a_{t-1})$$

$$l=0 \quad \gamma_0 - \phi \gamma_1 = \sigma_a^2 - \theta \psi_1 \sigma_a^2$$

$$l=1 \quad \gamma_1 - \phi \gamma_0 = -\theta \sigma_a^2$$

$$l \geq 2 \quad \gamma_l - \phi \gamma_{l-1} = 0$$

thus, $\rho_l - \phi\rho_{l-1} \quad l = 2, 3, \dots$

or $\rho_l = \phi^{l-1}\rho_1 \quad l = 2, 3, \dots$

we also have $\gamma_0(1 - \phi\rho_1) = \sigma_a^2(1 - \theta\phi_1) \quad \phi_1 = \phi - \theta$

$$\gamma_0(\rho_1 - \phi) = -\sigma_a^2$$

$\therefore$ We can express (γ_0, ρ_1) in terms of $(\sigma_a^2, \phi, \theta)$

and also solve $(\sigma_a^2, \phi, \theta)$ in terms of $(\gamma_0, \rho_1, \rho_2)$ --exercise.

As another exercise, work out P_l for ARMA(1,1), $l = 1, 2, 3, \dots$

Thus both ρ_l and P_l tail-off hard to specify via sample ACF and PACF.

Example see BJ-A SACF and SPACF.

Covariance generating:

$$\gamma(B) = \sigma_a^2 \left(\frac{1 - \theta B}{1 - \phi B} \right) \left(\frac{1 - \theta F}{1 - \phi F} \right)$$

(p,q) in general.

$$\psi's \quad (1 - \phi_1 B - \dots - \phi_p B_{l-p})(1 + \theta_1 B + \theta_2 B^2 + \dots) = (1 - \theta_1 B - \dots - \theta_q B_q) \quad \text{For}$$

$$l = 2, \dots$$

$$\psi_l - \phi_1 \psi_{l-1} - \dots - \phi_p \psi_{l-p} = -\theta_l \quad \theta_l = 0 \quad l > q$$

$$\psi_0 = 1 \quad \psi_j = 0 \quad j < 0$$

Thus $\psi's$ converge if $\phi_p(B)$ has also zeros outside unit circle $\leftrightarrow$ stationary.

Similarly,

$$\pi's \quad (1 - \theta_1 B - \dots - \theta_q B^q)(1 - \pi_1 B - \pi_2 B^2 - \dots) = (1 - \phi_1 B - \dots - \phi_p B_{l-p}) \quad \text{For}$$

$$l = 2, \dots$$

$$\pi_l - \phi_1 \pi_{l-1} - \dots - \phi_p \pi_{l-p} = -\phi_l \quad \phi_l = 0 \quad l > q$$

$$\pi_0 = -1 \quad \pi_j = 0 \quad j < 0 \quad \text{etc.}$$

Covariance Structure: $z_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots$

$$z_{t-l}(z_t - \phi_1 z_{t-1} - \dots - \phi_p z_{t-p}) = z_{t-l}(a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q})$$

For $l = 0, \dots, q$

$$\gamma_l - \phi_1 \gamma_{l-1} - \dots - \phi_p \gamma_{l-p} = \sigma_a^2 [\phi_{-l} - \theta_1 \psi_{1-l} - \dots - \theta_q \psi_{q-l}]$$

$$\text{where } \psi_j = 0 \quad j < 0 \quad \psi_0 = 1$$

$$l > q \quad \gamma_l - \phi_1 \gamma_{l-1} - \dots - \phi_p \gamma_{l-p} = 0$$

$$\text{or } \rho_l - \phi_1 \rho_{l-1} - \dots - \phi_p \rho_{l-p} = 0 \quad \text{where } \rho_{-j} = \rho_j$$

Thus for $l > \max(p, q) = R$

$$(1 - \phi_1 B - \dots - \phi_p B^p) \rho_l = 0 \quad \text{Difference equation to solve for}$$

Also solve $\phi_1, \dots, \phi_p$, using the p equation $l = q+1, \dots, q+p$

$$\rho_{q+1} = \phi_1 \rho_q + \dots + \phi_p \rho_{q+1-p}$$

$$\rho_{q+2} = \phi_1 \rho_{q+1} + \dots + \phi_p \rho_{q+2-p}$$

$\vdots$

$$\rho_{q+p} = \phi_1 \rho_{q+p-1} + \dots + \phi_p \rho_q$$

$$\begin{bmatrix} \rho_q & \rho_{q-1} & \dots & \rho_{q+1-p} \\ \rho_{q+1} & \rho_q & & \rho_{q+2-p} \\ \vdots & & \ddots & \\ \rho_{q+p-1} & & & \rho_q \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \vdots \\ \phi_p \end{bmatrix} = \begin{bmatrix} \rho_{q+1} \\ \vdots \\ \vdots \\ \rho_{q+p} \end{bmatrix}$$

Generalized Yule-Walker equation

$q = 0$ reduced to Y-W. This will be used later to develop EACF for specifying ARMA model.

Nonstationary Models

Random Walk

$$z_t = a_t + a_{t-1} + \cdots + a_1$$

$$z_t = z_{t-1} + a_t \quad \text{or} \quad z_t - z_{t-1} = a_t$$

compared with AR(1)

$$z_t = a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} \cdots$$

as if $\phi = 1$

$$\text{Now for } |\phi| < 1 \quad \psi_j = \phi^j \rightarrow 0 \quad \text{as } j \rightarrow \infty \quad \text{var}(z_t) = \frac{\sigma_a^2}{1 - \phi^2}$$

For $\phi = 1$, this does not converge. (i.e. $\psi_j = 1, j = 1, 2, \dots$)

Now Assume series starts at some fixed time point $t=m$.

$$\begin{cases} z_m = w_m + a_m & t = m \\ z_t = \phi z_{t-1} + a_t & t > m \end{cases}$$

Then $Z_m = w_m + a_m$

$$Z_{m+1} = a_{m+1} + \phi a_m + \phi w_m$$

$$Z_{m+2} = a_{m+2} + \phi a_{m+1} + \phi^2 a_{m+2} + \phi^2 w_m$$

$\vdots$

$$z_t = a_t + \phi a_{t-1} + \cdots + \phi^{t-m} a_m + \phi^{t-m} w_m$$

Assume $w_m \perp a's$

$$\gamma_0(t) = \text{var}(z_t) = \frac{\sigma_a^2}{1 - \phi^2} (1 - \phi^{2(t-m+1)}) + \phi^{2(t-m)} \text{var}(w_m)$$

$$|\phi| < 1 \quad \gamma_0(t) \rightarrow \frac{\sigma_a^2}{1 - \phi^2} \quad \text{as } t-m \rightarrow \infty \quad \text{stationary}$$

$$\phi = 1 \quad \gamma_0(t) = (t - m + 1)\sigma_a^2 + \text{Var}(w_m) \quad O(t)$$

$\therefore$ Variance increase with t ; series exhibits drifting and “mean” or “equilibrium level” become useless concepts.

However, if it is still true that

$$1) \quad a_t \quad z_{t-j} \text{ independent for } j > 0$$

$$2) \quad z_t | z_{t-1}, \dots, z_m \sim N(\phi z_{t-1}, \sigma_a^2)$$

$$\sim N(z_{t-1}, \sigma_a^2)$$

$$1^{\text{st}} \text{ Diff} \quad W_t = (1-B)z_t \text{ then } W_t = a_t$$

$$\text{The model can be written as } (1-B)z_t = a_t$$

i.e. z_t nonstationary, but $1^{\text{st}} \text{ Diff } (1-B)z_t$ is stationary and white noise.

Transforming (Differencing) to produce stationary.

$$\text{Generalization } (1-B)z_t = (1-\theta B)a_t \text{ starts at } t=m$$

$$z_t - z_{t-1} = a_t - \theta a_{t-1}$$

$$z_{t-1} - z_{t-2} = a_{t-1} - \theta a_{t-2}$$

$\vdots$

$$z_{m+1} - z_m = a_{m+1} - \theta a_m$$

$$z_m = w_m + a_m$$

$$z_t = a_t + (1-\theta)(a_{t-1} + \dots + a_m) + w_m$$

$$\psi_j = (1-\theta) \quad j=1,2,\dots \quad \gamma_0(t) = \sigma_a^2 [1 + (t-m)(1-\theta)^2] + \text{var}(w_m)$$

$$\text{or } (1-\theta B)(1 + \psi_1 B + \psi_2 B^2 + \dots) = (1-\theta B)$$

$$\psi_l - \psi_{l-1} = -\theta_l \quad \theta_l = \begin{cases} \theta & l=1 \\ 0 & l>1 \end{cases}$$

$$\Rightarrow \psi_l = 1 - \theta_l$$

Interpretation: Suppose we regard $(1-\theta)(a_{t-1} + \dots + a_m) + W_m$ as the level of z_t at time t . Then effects of past shocks are permanent; $(1-\theta)$ portion of the shocks get absorbed into the level and carry into the future. (Muth, 1960, JASA)

Consider now the π 's

$$z_t - z_{t-1} = a_t - \theta a_{t-1}$$

$$\theta(z_{t-1} - z_{t-2}) = \theta(a_{t-1} - \theta a_{t-2})$$

$$\theta^2(z_{t-1} - z_{t-2}) = \theta^2(a_{t-1} - \theta a_{t-2})$$

$\vdots$

$$\theta^{t-m-1}(z_{m+1} - z_m) = \theta^{t-m-1}(a_{m+1} - \theta a_m)$$

$$\theta^{t-m} z_m = \theta^{t-m} (w_m + a_m)$$

$$z_t = (1-\theta)[z_{t-1} + \theta z_{t-2} + \dots + \theta^{t-m-1} z_m] + \theta^{t-m} w_m + a_t$$

Alternatively, $(1-B)z_t = (1-\theta B)a_t$

$$(1-\theta B)(1-\pi_1 B - \pi_2 B^2 - \dots) = (1-B)$$

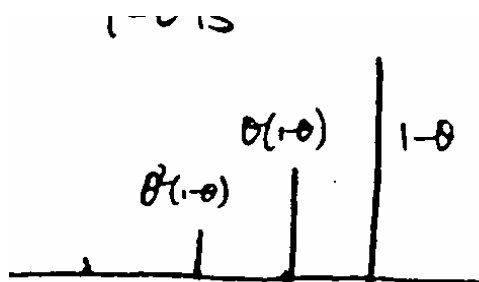
$$\pi_l - \theta \pi_{l-1} = \phi_l \quad \begin{cases} \phi_l = 1 & l=1 \\ \phi_l = 0 & l>1 \end{cases}$$

$$\Rightarrow \pi_l = (1-\theta)\theta^{l-1} \quad l=1,2,\dots \text{ same answer}$$

Exponential Smoothing – as $t-m \rightarrow \infty$ (assume $|\theta| < 1$)

$$a_t = \frac{1-B}{1-\theta B} z_t = (1 - \frac{(1-\theta)B}{1-\theta B}) z_t$$

$$\text{or } Z_t = \frac{1-\theta}{1-\theta B} z_{t-1} + a_t \quad \text{suppose } \theta = 0.4 \quad (1-\theta) = 0.6$$



BJ called this model integrated MA or IMA. Show generated (0,1,1)

Discuss SACF and SPACF of generated (0,1,1)

Rationale for differencing.

More generally we can have $(1-B)z_t$ follows an ARMA(p,q)

$$(1-\phi_1 B - \dots - \phi_p B^p)(1-B)z_t = (1-\theta_1 B - \dots - \theta_q B^q)a_t$$

(p,1,q) model $\phi_p(B)$ and $\theta_q(B)$ have all zeros outside the unit circle.

e.g. (1,1,1) model

$$(1-\phi B)(1-B)z_t = (1-\theta B)a_t$$

series starts at t=m

$$z_t - (1+\phi)z_{t-1} + \phi z_{t-2} = a_t - \theta a_{t-1}$$

$$z_m = w_m + a_m$$

$$w_m = (1+\phi)z_{m-1} - \phi z_{m-2} - \theta a_{m-1}$$

$$z_{m+1} - (1+\phi)z_m = w_{m+1} + a_{m+1} - \theta a_m$$

$$w_{m+1} = -\phi z_{m-1}$$

$$z_{m+2} - (1+\phi)z_{m+1} + \phi z_m = a_{m+2} - \theta a_{m+1}$$

$$\text{etc.} \quad w_m, w_{m+1} \perp a's$$

Then for $t > m+2$

$$Z_t = a_t + \psi_1 a_{t-1} + \dots + \psi_{t-m} a_m + (\delta_{1t} w_{m+1} + \delta_{2t} w_m)$$

where $(1-\phi B)(1-B)(1+\psi_1 B + \psi_2 B^2 + \dots) = (1-\theta B)$

$$\psi_l = \frac{1-\theta}{1-\phi} + \frac{\theta-\phi}{1-\phi} \phi^l \quad \delta_{1t} \rightarrow c \quad \delta_{2t} \rightarrow c \quad \therefore \text{var}(Z_t) = \gamma_0(t) = o(t)$$

Also, we can express z_t in the form

$$z_t = \pi_1 z_{t-1} + \dots + \pi_{t-m} z_{t-m} + (\delta_{1t} w_{m+1} + \delta_{2t} w_m)$$

$$(1-\theta B)(1-\pi_1 B - \pi_2 B^2 - \dots) = (1-\phi B)(1-B)$$

In general for the (p,1,q) model

$$z_t = \pi_1 z_{t-1} + \dots + \pi_{t-m} z_m + a_t + (\text{linear combination of } p+1 \text{ } w's)$$

invertible $\Rightarrow$ the last term of the 2nd representation goes to 0 as $t - m \rightarrow \infty$

$$\phi_p(B)(1-B)\psi(B) = \theta_q(B)$$

$$\theta_q(B)\pi(B) = \phi_p(B)(1-B) \quad \text{same way as in the stationary case.}$$

Key thing is that $\sum \psi_j^2 \rightarrow \infty$ ($\sum \pi_j = 1$) because of the (1-B)

We can also write $\Phi(B) = \phi_p(B)(1-B)$ as long as we allow $\Phi(B)$ to have roots on

or outside unit circle. The above class is called ARIMA(p,1,q), integrated ARMA.

Now consider the generated (0,2,0) series.

$$a_1, \dots, a_{250}$$

$$u_t = a_t + \dots + a_1 \quad t = 1, \dots, 250$$

$$z_t = u_1 + \dots + u_t \quad t = 1, \dots, 250$$

$$(1-B)z_t = u_t \quad (1-B)^2 z_t = (1-B)u_t = a_t$$

2nd difference is white noise(stationary)

Also $u_1 = a_1$

$$u_2 = a_1 + a_2$$

$\vdots$

$$u_t = a_1 + a_2 + \dots + a_t$$

$$\therefore z_t = a_t + 2a_{t-1} + \dots + ta_{t-1} \quad \psi_1 = 2, \quad \psi_2 = 3, \text{ etc}$$

$$\gamma_0(t) = \text{var}(Z_t) = \sigma_a^2 [1 + 2^2 + \dots + t^2] = \sigma_a^2 \frac{1}{6} t(t+1)(2t+1) = O(t^3)$$

$\therefore$ variance of z_t increase very fast!

Alternatively, we can write

$$z_t = a_t + \psi_1 a_{t-1} + \dots + \psi_{t-1} a_1$$

$$\text{where } (1 - 2B + B^2)(1 + \psi_1 B + \dots) = 1$$

$$\psi_l - 2\psi_{l-1} + \psi_{l-2} = 0 \quad \psi_l = 2\psi_{l-1} + \psi_{l-2}$$

$$\psi_0 = 1, \quad \psi_1 = 2, \quad \psi_l = A_1 + A_2 l$$

$$l = 0 \quad A_1 = \psi_0 = 1$$

$$l = 1 \quad A_1 + A_2 = 2 \quad A_2 = 1 \quad A_1 = A_2 = 1 \quad \psi_l = l + 1$$

Next consider

$$(1 - B)^2 z_t = (1 - \theta_1 B - \theta_2 B^2) a_t$$

assume series starts at $m=1$ (for convenience) $w_1 = w_2 = 0$

$$\psi_0 = 1, \quad \psi_1 = 2 - \theta_1, \quad \psi_l = 2(2 - \theta_1) - (1 + \theta_2)$$

$$\psi_l - 2\psi_{l-1} + \psi_{l-2} = 0 \quad \psi_l = 2\psi_{l-1} + \psi_{l-2}$$

$$l = 1 \quad A_1 + A_2 = 2 - \theta_1$$

$$l = 2 \quad A_1 + 2A_2 = 2(2 - \theta_1) - (1 + \theta_2)$$

$$\Rightarrow \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 + \theta_2 \\ 1 - \theta_1 - \theta_2 \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \end{bmatrix}$$

Then $z_t = a_t + (\lambda_0 \lambda_1) a_{t-1} + (\lambda_0 \lambda_1^2) a_{t-2} + \dots + (\lambda_0 \lambda_1^{t-1}) a_1$

$$\begin{aligned} \frac{\gamma_0(t)}{\sigma_a^2} &= 1 + \sum_{j=1}^{t-1} (\lambda_0 + j \lambda_1)^2 = 1 + \sum_{j=1}^{t-1} \lambda_0^2 + 2 \lambda_0 \lambda_1 \sum_{j=1}^{t-1} j + \lambda_1^2 \sum_{j=1}^{t-1} j^2 \\ &= 1 + (t-1) \lambda_0^2 + 2 \lambda_0 \lambda_1 t(t-1) + \frac{\lambda_1^2}{6} (t-1)t(2t-1) = O(t^3) \end{aligned}$$

$\therefore$ order of $\gamma_0(t)$ not affected by the MA part.

Also, we can express $z_t = \pi_1 z_{t-1} + \pi_2 z_{t-2} + \dots + \pi_{t-1} z_1 + a_t$

$$(1 - \theta_1 B - \theta_2 B^2)(1 - \pi_1 B - \pi_2 B^2 - \dots) = (1 - 2B + B^2)$$

$$\pi_0 = -1, \quad \pi_1 = 2 - \theta_1, \quad \pi_2 = (2 - \theta_1)\theta_1 - (1 + \theta_2)$$

$$\pi_l - \theta_1 \pi_{l-1} - \theta_2 \pi_{l-2} = 0 \quad l > 2$$

$$\therefore \pi_l = c_1 \alpha_1^l + c_2 \alpha_2^l$$

$$\text{where } (1 - \theta_1 B - \theta_2 B^2) = (1 - \alpha_1 B)(1 - \alpha_2 B)$$

$$\begin{aligned} \text{If invertible, } a_t &= \frac{(1-B)^2}{(1-\alpha_1 B)(1-\alpha_2 B)} z_t = \left[1 - \frac{(1-\alpha_1 B)}{1-\alpha_1 B} \right] \left[1 - \frac{(1-\alpha_2 B)}{1-\alpha_2 B} \right] \\ &= \left[1 - \frac{(1-\alpha_1 B)}{1-\alpha_1 B} \right] e_t \end{aligned}$$

$$\text{Where } e_t = \left[1 - \frac{(1-\alpha_2 B)}{1-\alpha_2 B} \right] \quad \text{when } \alpha_1 = \alpha_2, \text{ Double Exponential Smoothing to}$$

be discussed later.

Illustration $\theta_1 = 1.1, \quad \theta_2 = -.28$

$$\Rightarrow \alpha_1 = .4, \quad \alpha_2 = .7 \quad \text{Excise: plot } \pi_1, \pi_2, \dots, \pi_{20} \quad \text{show } \sum_{\ell=1}^{\infty} \pi_{\ell} = 1$$

For (p,2,q) model in general

$$\phi_p(B)(1-B)^2 z_t = \theta_q(B)a_t$$

ψ weights

$$z_t = a_t + \psi_1 a_{t-1} + \dots + \psi_{t-m} a_m + f_t(w's)$$

where $f_t(w's)$ is linear and series starts at $t=m$

$$\phi_p(B)(1-B)^2(1 + \psi_1 B + \psi_2 B^2 + \dots) = \theta_q(B)$$

Then ψ 's satisfy the difference equation eventually.

$$\phi_p(B)(1-B)^2 \psi_l = 0 \quad \psi_l = A_1 + A_2 l + c_1 \alpha_1^l + \dots + c_p \alpha_p^l$$

$$\gamma_0(t) = O(t^3)$$

π 's are obtained from

$$\theta_q(B)(1 - \pi_1 B - \pi_2 B^2 - \dots) = \phi_p(B)(1-B)^2$$

$$\text{note that } \sum_{j=1}^{\infty} \pi_j = 1$$

More generally for (p,d,q) model

$$\phi_p(B)(1-B)^d z_t = \theta_q(B)a_t \quad \text{series starts at } t=m(\text{for } d>0)$$

$$\psi \text{'s} \quad \phi_p(B)(1-B)^d \psi(B) = \theta_q(B)$$

eventually ψ 's satisfy the difference equation

$$\phi_p(B)(1-B)^d \psi_l = 0 \quad \psi_l = A_1 + A_2 l + \dots + A_d l^{d-1} + c_1 \alpha_1^l + \dots + c_p \alpha_p^l$$

$$\gamma_0(t) = O(t^{2d-1}) \quad d > 0$$

π 's are obtained from the relation

$$\theta_q(B)\pi(B) = \phi_p(B)(1-B)^d \quad \text{and} \quad \sum_{j=1}^{\infty} \pi_j = 1 \quad \text{for } d \geq 1$$

Remarks

- 1) for $d=0$, stationary $z_t = z_t$
for $d>0$, simply z_t because $B(z_t - \eta) = z_{t-1} - \eta$
- 2) $\phi_p(B)(1-B)^d = \Phi_{p+d}(B) \therefore$ we can simply use ARMA(p,q) as long as we recognize that $\Phi(B)$ may have zeros on the unit circle.(unit roots if Φ contains $(1-B)^d$)

- 3) The relations:

$$\phi_p(B)(1-B)^d \psi(B) = \theta_q(B)$$

$$\theta_q(B)\pi(B) = \phi_p(B)(1-B)^d$$

allow us to obtain ψ 's and π 's as functions of ϕ 's and θ 's. The meanings of ψ 's and π 's depend on nature of model. Thus, for stationary series

$z_t = a_t + \psi_1 a_{t-1} + \dots$. But for nonstationary series

$z_t = a_t + \psi_1 a_{t-1} + \dots + \psi_{t-m} a_m + f_t(w's)$. In one case we can assume series starts at infinite past ($m \rightarrow \infty$), but in the often m is a finite time point. Also, π 's are useful when the model is invertible, otherwise useless, as we shall see later when we discuss prediction.

- 4) $\phi(B)\pi(B) = 1$

Prediction

General theory (x, y) are R.V. having a joint pdf

Suppose we wish to estimate x by a function of y , $h(y)$

If the loss function is MSE

$$E_{x,y} \left((x - h(y))^2 \right)$$

Then the best estimate of x , $\hat{x}$, is

$$\hat{x} = E(x) \text{ the conditional expectation}$$

Why?

$$E(x - h(y))^2 = E(x - \hat{x} + \hat{x} - h(y))^2 = E(x - \hat{x})^2 + E(\hat{x} - h(y))^2 + 2E(x - \hat{x})(\hat{x} - h(y))$$

write $E = E_y E_{x|y}$ will do

Forecasting, suppose we have obs. $z_1, \dots, z_T$ What can we say about z_{T+l} , $l = 1, 2, \dots$?

$$\text{AR}(1) \quad (1 - \phi B)\dot{z}_t = a_t \Rightarrow (1 - \phi B)\dot{z}_{T+l} = a_{T+l}$$

$$l = 1 \quad \dot{z}_{T+1} = \phi \dot{z}_T + a_{T+1}$$

$$E_{z_{T+1}|Z} \dot{z}_{T+1} = \phi \dot{z}_T = \hat{\dot{z}}_T(1)$$

$$e_T(1) = \dot{z}_{T+1} - \hat{\dot{z}}_T(1) = a_{T+1} \text{ (very important to note)}$$

a_{T+1} one-step ahead fore.em

$$l = 2 \quad \hat{\dot{z}}_{T+2} = \phi \hat{\dot{z}}_{T+1} + a_{T+2}$$

$$\hat{\dot{z}}_T(2) = \phi \hat{\dot{z}}_T(1)$$

$$e_T(2) = \phi e_T(1) + a_{T+2}$$

$$\text{where } e_T(2) = \hat{\dot{z}}_{T+2} - \hat{\dot{z}}_T(2)$$

Thus, in general $l = 1, \dots$,

$$\hat{z}_T(l) = \phi \hat{z}_T(l-1)$$

$$(1 - \phi B) \hat{z}_T(l) = 0 \quad \text{Diff. equ}$$

$$\hat{z}_T(l) = \phi^l \hat{z}_T$$

$$\therefore \hat{z}_T(l) \rightarrow \mu \quad \text{mean of series}$$

$$e_T(l) - \phi e_T(l-1) = a_{T+l} \quad l = 1, 2, \dots \quad e_T(0) = 0$$

or $l = 1, 2, \dots, L$

$$\begin{bmatrix} 1 & & & & \\ -\phi & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\phi & 1 \end{bmatrix} \begin{bmatrix} e_T(1) \\ \vdots \\ e_T(L) \end{bmatrix} = \begin{bmatrix} a_{T+1} \\ \vdots \\ a_{T+L} \end{bmatrix} \quad \text{to get it dist of } e_T(1), \dots, e_T(L)$$

$$\text{In particular } e_T(l) = a_{T+l} + \psi_1 a_{T+l-1} + \dots + \psi_{l-1} a_{T+1} \quad \psi_l = \phi^l$$

$$\text{var}(e_T(l)) = \frac{1 - \phi^{2l}}{1 - \phi^2} \sigma_a^2 = \begin{cases} \sigma_a^2 & l = 1 \\ \sigma_a^2 / (1 - \phi^2) & l \rightarrow \infty \end{cases} \quad \text{var of } Z_t$$

Notation

$$E_T \quad \text{conditional expectation at time T of } E z_{T+l} = \hat{z}_T(l) \quad \text{given } Z_1 \dots Z_T$$

Generalization to higher order AR-straightforward

$$\text{e.g. AR(2)} \quad z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t \quad \text{put } t = T + l, l = 1, 2, \dots$$

$$l = 2 \quad \hat{z}_T(1) = \phi_1 z_T + \phi_2 z_{T-1} \quad e_T(1) = z_{T+1} - \hat{z}_T(1) = a_{T+1}$$

$$l = 2 \quad \hat{z}_T(2) = \phi_1 \hat{z}_T(1) + \phi_2 z_T \quad e_T(2) = z_{T+2} - \hat{z}_T(2) = a_{T+2} + \phi_1 e_T(1)$$

$$l = 3 \quad \hat{z}_T(3) = \phi_1 \hat{z}_T(2) + \phi_2 \hat{z}_T(1) \quad e_T(3) = z_{T+3} - \hat{z}_T(3) = \phi_1 e_T(2) + \phi_2 e_T(1) + a$$

$$\text{MA(1)} \quad z_t = a_t - \theta a_{t-1} \quad \text{available } z_1, \dots, z_T$$

$$z_{T+1} = a_{T+1} - \theta a_T$$

$$E_T(z_{T+1}) = ? = E_{a_{T+1}|z_{\sim}}(a_{T+1}) - \theta E_{a_T|z_{\sim}}(a_T)$$

$$\text{Now} \quad E_{\tilde{z}}(a_{T+1}) = 0 \quad \text{but} \quad E_{\tilde{z}}(a_T) = ?$$

Cont'd on AR(2)

$$\hat{z}_T(l) = \phi_1 z_T(l-1) + \phi_2 \hat{z}_T(l-2)$$

$$\text{where } \hat{z}_T(j) = z_{T+j} \quad \text{for } j \leq 0$$

$$e_T(l) = \phi_1 e_T(l-1) + \phi_2 e_T(l-2) + a_{T+l}$$

$$\text{where } e_T(j) = 0 \quad \text{for } j \leq 0$$

$$\text{Thus } (1 - \phi_1 B - \phi_2 B^2) \hat{z}_T(l) = 0 \quad \therefore \hat{z}_T \rightarrow 0 \quad \text{as } l \rightarrow \infty$$

$$(1 - \phi_1 B - \phi_2 B^2) e_T(l) = a_{T+l}$$

$$\text{For } l = 1, 2, 3 \dots L$$

$$\begin{bmatrix} 1 & & & & & \\ -\phi_1 & 1 & & & & \\ -\phi_2 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & -\phi_2 & -\phi_1 & 1 & \end{bmatrix} \begin{bmatrix} e_T(1) \\ \vdots \\ \vdots \\ \vdots \\ e_T(L) \end{bmatrix} = \begin{bmatrix} a_{T+1} \\ \vdots \\ \vdots \\ \vdots \\ a_{T+L} \end{bmatrix}$$

$$\begin{bmatrix} e_T(1) \\ \vdots \\ \vdots \\ \vdots \\ e_T(L) \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ \psi_1 & \ddots & & & \\ \psi_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \psi_{L-1} & \cdots & \psi_2 & \psi_1 & 1 \end{bmatrix} \begin{bmatrix} a_{T+1} \\ \vdots \\ \vdots \\ \vdots \\ a_{T+L} \end{bmatrix}$$

$$e_T(L) = a_{T+L} + \psi_1 a_{T+L-1} + \dots + \psi_{L-1} a_{T+1}$$

$$\text{var}(e_T(l)) = \begin{cases} \sigma_a^2 & l = 1 \\ (1 + \psi_1^2 + \dots + \psi_{l-1}^2 \sigma_a^2) & l > 1 \end{cases}$$

$$\text{var}(e_T(l)) \rightarrow \text{var}(z_l) \quad \text{as } l \rightarrow \infty$$

Thus, we have 2 routes:

1) From j-th distribution of $z_{T+1}, z_T, \dots, z_1$, get

conditional expectation of z_{T+1} given $\tilde{z}$

$$\text{cov} \begin{bmatrix} z_1 \\ \vdots \\ z_T \\ z_{T+1} \end{bmatrix} = \text{cov} \begin{pmatrix} \tilde{z} \\ z_{T+1} \end{pmatrix} = \begin{bmatrix} 1+\theta^2 & -\theta & & & \\ -\theta & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\theta \\ & & & -\theta & 1+\theta^2 \end{bmatrix} \sigma_a^2 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & 1+\theta^2 \end{bmatrix}$$

$$\therefore E(\dot{z}_{T+1} | \dot{z}) = -\Sigma_{21} \Sigma_{11}^{-1} \dot{z} \quad \text{for all values of } \theta, \sigma^2$$

2) $a_T = \dot{z}_T + \theta a_{T-1} = \dot{z}_T + \theta \dot{z}_{T-1} + \theta^2 a_{T-2} \quad \text{etc.}$

$$\therefore a_T = \dot{z}_T + \theta \dot{z}_{T-1} + \dots + \theta^{T-1} \dot{z}_1 + \theta^T a_0$$

→ Thus, if $|\theta| < 1$ and T large $\underset{\sim}{E} a_T \doteq \dot{z}_T + \theta \dot{z}_{T-1} + \dots + \theta^{T-1} z_1$ (invertible)

$$\text{and } \hat{z}_T(1) \doteq -\theta z_T - \theta^2 z_{T-1} - \dots - \theta^{T-1} z_1 \doteq \pi_1 z_T + \pi_2 z_{T-1} + \dots + \pi_T z_1$$

On the other hand if $\theta = 1$, this is not the conditional expectation
 $\Rightarrow$ Hence the π 's are only meaningful when the model is invertible

Note that whether the model is invertible or not $\underset{\sim}{E}(z_{T+1})$ always exists

In what follows, we assume invertibility unless otherwise stated

$$e_T(1) = z_{T+1} - \hat{z}_T(1) \doteq a_{T+1} \quad \text{approximately} \quad \text{var}(e_T(1)) = \sigma_a^2$$

$$l = 2, 3, \dots \quad E(\dot{z}_{T+l}) = 0 \quad \text{var}(e_T(l)) = (1 + \theta^2) \sigma_a^2$$

The above argument can now be generalized to MA(q)

e.g. q=2

$$l = 1 \quad z_{T+1} = a_{T+1} - \theta_1 a_T - \theta_2 a_{T-1}$$

$$\hat{z}_T(1) = -\theta_1 E a_T - \theta_2 E a_{T-1}$$

$$\text{when invertible} \quad a_t = (1 - \theta_1 B - \theta_2 B^2)^{-1} z_t \doteq \dot{z}_t - \pi_1 \dot{z}_{t-1} - \dots - \pi_{l-1} \dot{z}_1$$

$$\therefore e_T(1) = a_{T+1}$$

$$l = 2 \quad \dot{z}_{T+2} = a_{T+2} - \theta_1 a_{T+1} - \theta_2 a_T$$

$$\hat{z}_T(2) = -\theta_2 E(a_T) \quad e_T(2) = a_{T+2} - \theta a_{T+1}$$

$$l > 2 \quad \hat{z}_T(l) = 0 \quad e_T(l) = a_{T+l} - \theta_1 a_{T+l-1} - \theta_2 a_{T+l-2} \quad \text{etc.}$$

$$\text{var}(e_T(\ell)) = \sigma_a^2 (1 + \theta_1^2 + \theta_2^2), \quad \text{var of } z_t$$

Thus for stationary AR of MA types of model

$$l \rightarrow \infty \quad \hat{z}_T(\ell) \rightarrow 0 \quad z_T(l) \rightarrow \mu$$

$$\text{var}(e_T(l)) \rightarrow \gamma_0 \quad \text{constant}$$

Next (0,1,1) model

$$(1 - B)z_t = (1 - \theta B)a_t$$

$$\text{Assume series starts at } t=1 \quad z_1 = \omega_1 + a_1$$

Given $z_1, \dots, z_T$, forecast z_{T+l} , $l = 1, 2, \dots$

$$z_{T+l} - z_{T+l-1} = a_{T+l} - \theta a_{T+l-1}$$

$$l = 1 \quad z_{T+1} - z_T = a_{T+1} - \theta a_T$$

$$\hat{z}_T(1) = z_T - \theta E_T a_T$$

Alternatively see p.26

$$z_{T+1} = (1 + \theta)[z_T + \dots + \theta^{T-1} z_1] + \theta^T E_T \omega_1$$

for T large and $|\theta| < 1$

$$\hat{z}_T(1) \doteq (1 - \theta)[z_T + \dots + \theta^{T-1} z_1]$$

$$e_T(1) = z_{T+1} - \hat{z}_T(1) \doteq a_{T+1} \quad \text{var}(e_T(1)) = \sigma_a^2$$

$$l > 1 \quad \hat{z}_T(l) = \hat{z}_T(l-1) + E_T(a_{T+l}) - \theta E_T(a_{T+l-1}) = \hat{Z}_T(l-1)$$

$$(1 - B)\hat{z}_T(l) = 0 \quad \text{forecast a horizontal line}$$

$$e_T(l) = e_T(l-1) + a_{T+l} - \theta a_{T+l-1}$$

$$(1 - B)e_T(l) = a_{T+l} - \theta a_{T+l-1}$$

$$\begin{bmatrix} 1 & & & & \\ -1 & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & -1 & 1 \end{bmatrix} \begin{bmatrix} e_T(1) \\ \vdots \\ \vdots \\ \vdots \\ e_T(L) \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -\theta & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & -\theta & 1 \end{bmatrix} \begin{bmatrix} a_{T+1} \\ \vdots \\ \vdots \\ \vdots \\ a_{T+L} \end{bmatrix}$$

$$e_T(l) = a_{T+l} + (1 - \theta)(a_{T+l-1} + \dots + a_{T+1}) = a_{T+l} + \psi_1 a_{T+l-1} + \dots + \psi_{l-1} a_{T+1}$$

$$\therefore \text{var}(e_T(l)) = \begin{cases} \sigma_a^2 & l = 1 \\ \sigma_a^2(1 + (l-1)(1 - \theta)^2) & l > 1 \end{cases}$$

$\therefore \text{var}$ increases without bound

Note that for $\theta = 1$, exact formulae can be worked out for given distributional assumption about ω_1

→ show (0,1,1) example p182

Updating Suppose we are now at time $T+1$, so z_{T+1} becomes available,

$$z_{T+2} = z_{T+1} + a_{T+2} - \theta a_{T+1}$$

$$(1) \hat{z}_{T+1}(1) \doteq (1-\theta)z_{T+1} + \theta(1-\theta)a_T + \theta^2(1-\theta)z_{T-1} + \dots$$

$$(2) \because a_{T+1} = z_{T+1} - \hat{z}_T(1) \quad \therefore \hat{z}_{T+1}(1) = z_{T+1} - \theta(z_{T+1} - \hat{z}_T(1)) = (1-\theta)z_{T+1} + \theta\hat{z}_T(1)$$

(interpolation)

if $0 < \theta < 1$

$$(3) \hat{z}_{T+1}(1) = \hat{z}_T(1) + (1-\theta)a_{T+1} \quad \text{adaptive learning}$$

new = old + (1- θ)error

Forecasting for general ARIMA(p,d,q)

$$\text{write } \Phi_{p+d}(B)z_t = \theta_q(B)a_t$$

$$\text{where } \Phi_{p+d}(B) = \phi_p(B)(1-B)^d$$

$$z_t - \Phi z_{t-1} - \dots - \Phi_{p+d} z_{t-(p+d)} = a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q}$$

at time T , forecast $z_{T+l}, l = 1, 2, \dots$

$$z_{T+l} = \Phi_1 z_{T+l-1} + \dots + \Phi_{p+d} z_{T+l-(p+d)} + a_{T+l} - \theta_1 a_{T+l-1} - \dots - \theta_q a_{T+l-q}$$

Again

$$\text{let } \hat{z}_T(l) = E_T(z_{T+l})$$

$$E_T z_{T+l} = \Phi_1 E_T z_{T+l-1} + \dots + \Phi_{p+d} E_T z_{T+l-(p+d)} + E_T a_{T+l} - \theta_1 E_T a_{T+l-1} - \dots - \theta_q E_T a_{T+l-q}$$

for $l = 1, 2, \dots$

$$E_T(z_{T+j}) = z_{T+j} \quad \text{for } j \leq 0 \quad E_T a_{T+j} \begin{cases} = 0 & j > 0 \\ \doteq a_{T+j} & j \leq 0 \end{cases}$$

and a_{T+j} is one step ahead forecast error $a_{T+j} = z_{T+j} - \hat{z}_{T+j}$

(assume invertibility)

Thus for $l > \max(p+d, q)$

$$(1 - \Phi_1 B - \dots - \Phi_{p+d} B^{p+d}) \hat{z}_T(l) = 0$$

$$\text{or } \hat{z}_T(l) = b_0 + \dots b_{d-1} l^{d-1} + c_1 \alpha_1^l + \dots c_p \alpha_p^l \quad |\alpha_j| < 1, j = 1, 2, \dots, p$$

$$\text{for } l \text{ large} \quad \hat{z}_T(l) \rightarrow b_0 + \dots b_{d-1} l^{d-1}$$

i.e. all depends on $(1-B)^d \quad \phi_p(B)$ transient

$d = 0$ stationary $\hat{z}_T(l) \rightarrow 0$ (or μ the mean level)

$d = 1$ b_0 horizontal line

$d = 2$ $b_0 + b_1 l$ straight line

$d = 3$ $b_0 + b_1 l + b_2 l^2$ quadratic

Note that the coefficient b 's d c 's all depend on T , the origin of forecast

For forecast errors $e_T(l) = z_{T+l} - \hat{z}_T(l)$, we have that

$$e_T(l) - \Phi_1 e_T(l-1) - \dots - \Phi_{p+d} e_T(l-p-d) = a_{T+l} - \theta_1 [a_{T+l-1} - \frac{E}{T} a_{T+l-1}] - \dots - \theta_q [a_{T+l-q} - \frac{E}{T} a_{T+l-q}]$$

where $e_T(j) = 0$ if $j \leq 0$

Then for $l = 1, \dots, L$ $L > \max(p+d, q)$

$$\begin{bmatrix} 1 & & & & & \\ -\Phi_1 & \ddots & & & & \\ \vdots & \ddots & \ddots & & & \\ -\Phi_{p+d} & & \ddots & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ & & -\Phi_{p+d} & \dots & -\Phi_1 & 1 \end{bmatrix} \begin{bmatrix} e_T(1) \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ e_T(L) \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ -\theta_1 & \ddots & & & & \\ \vdots & \ddots & \ddots & & & \\ -\theta_q & & \ddots & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ & & -\theta_q & \dots & -\theta_1 & 1 \end{bmatrix} \begin{bmatrix} a_{T+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{T+L} \end{bmatrix}$$

$$e_T(l) = a_{T+1} \quad l = 1 \quad \therefore \text{var}(e_T(l)) = \begin{cases} \sigma_a^2 & l = 1 \\ \sigma_a^2 (1 + \psi_1^2 + \dots + \psi_{l-1}^2) \end{cases}$$

$$= a_{T+l} + \psi_1 a_{T+l-1} + \dots + \psi_{l-1} a_{T+1}$$

Role of $\pi(B)$ in a_T 's forecast weights of $z_T, z_{T-1}, \dots$

We see that from previous discussion

$$d = 0 \quad \psi_l \rightarrow 0 \text{ as } l \uparrow \quad \therefore \text{var}(e_T(l)) \rightarrow \text{constant}$$

$$d = 1 \quad \psi_l \rightarrow \text{constant} \text{ as } l \uparrow \quad \therefore \text{var}(e_T(l)) \rightarrow O(l)$$

$$d = 2 \quad \psi_l \rightarrow A_0 + A_1 l \text{ as } l \uparrow \quad \therefore \text{var}(e_T(l)) \rightarrow O(l^2) \quad \text{etc.}$$

Model Building –Given a observed series $(z_1, \dots, z_n)$, how do we select a member of the ARIMA(p,d,q) to represent the data at hand.

B-J approach –iterative procedure

- Tentative Spec. (identification)
- Estimation (MLE)
- Diagnostic checking (residual analysis)

Tentative Spec : commonly used techniques are SACF+SPACF

show BJ-C and BJ-A.

Usefulness A SACF

- Specify order of MA (low order ones)
- Indicate when low order MA won't work
- Indicate when differencing is call for ,(if r_K persistently high)

Usefulness of SPACF

- Specify order of AR

Draw backs of ACF+PACF

(1) Differencing judgment of complex

(2) Don't work for mixed model

Show SACF, SPACF, SEACF for BJ-C, BJ-A, generate (0,1,1)

EACF $\Phi_p(B)z_t = \theta(B)a_t$

$P = p + d$ $\Phi(B) = \phi(B)(1 - B)^d$

Basic idea : if we knew $\Phi(B)$ then $\Phi(B)z_t$ follows MA(q) and we can then exploit “cutting off” property of ACF for MA.

Consider first stationary case

ARMA(1,1) $z_{t-k}(z_t - \phi z_{t-1}) = z_{t-k}(a_t - \theta a_{t-1})$

upon normalizing

$$k=1 \quad \rho_1 - \phi\rho_0 = -\theta\sigma_a^2/\gamma_0 \quad \phi \neq \rho_1 \quad \text{unless } \theta = 0$$

$$k=2 \quad \rho_2 - \phi\rho_1 = 0 \quad \phi = \rho_2 / \rho_1$$

$$k=3 \quad \rho_3 - \phi\rho_2 = 0 \quad \phi = \rho_3 / \rho_2 \quad \text{etc.}$$

Suppose we define $\phi^{(l)}$ to be solution of $\rho_{\ell+1} - \phi^{(l)}\rho_\ell = 0$

and consider the transformed process $w_t^{(l)} = Z_t - \phi^{(l)}Z_{t-1}$

We see that for the ARMA(1,1)

$$\phi^{(1)} = \rho_2 / \rho_1 = \phi \quad w_t^{(1)} \sim MA(1)$$

$$\phi^{(2)} = \rho_3 / \rho_2 = \phi \quad w_t^{(1)} \sim MA(1)$$

$$\text{i.e. } w_t^{(l)} \sim MA(1) \quad \text{for } l = 1, 2, 3, \dots$$

Now suppose the model is ARMA(1,2)

$$z_{t-k}(z_t - \phi z_{t-1}) = z_{t-k}(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})$$

$$k=2 \quad \rho_1 - \phi \rho_0 = -\theta_2 \sigma_a^2 / \gamma_0 \neq 0$$

$$k=3 \quad \rho_2 - \phi \rho_1 = 0$$

$$k=4 \quad \rho_3 - \phi \rho_2 = 0$$

Then for $\phi^{(\ell)}$ and $w_t^{(\ell)}$ defined above, we see that

$$\left. \begin{array}{l} \phi^{(\ell)} = \phi \\ \text{and } w_t^{(\ell)} \sim MA(2) \end{array} \right\} \text{ for } \ell = 2, 3, \dots$$

More generally suppose the model is ARMA(1,q)

$$z_{t-k}(z_t - \phi z_{t-1}) = z_{t-k}(a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q})$$

$$k=q \quad \rho_q - \phi \rho_{q-1} = -\theta_q \sigma_a^2 / \gamma_0 \neq 0$$

$$k=q+1 \quad \rho_{q+1} - \phi \rho_q = 0$$

$$k=q+2 \quad \rho_{q+2} - \phi \rho_{q+1} = 0$$

Then $\phi^{(\ell)} = \phi$ and $w_t^{(\ell)} \sim MA(q)$ for $\ell = q, q+1, \dots$

We see that the ACF's of $w_t^{(\ell)}$'s do have the cutting off properties, but it would be too much to consider all of them (for $\ell = 1, 2, 3, \dots$). To simplify it, just consider one for each ℓ

Define $\rho_\ell(1) = \ell^{th}$ autocorrelation of $w_t^{(\ell)}$

If model is ARMA(1,1) $\rho_1(1) \neq 0$ $\rho_2(1) = 0$ $\rho_3(1) = 0$ etc.

If model is ARMA(1,2) $\rho_2(1) \neq 0$ $\rho_3(1) = 0$ etc.

In general for ARMA(1,q)

$$\left\{ \begin{array}{ll} \rho_\ell(1) \neq 0 & \ell = q \\ \rho_\ell(1) = 0 & \ell > q \end{array} \right. \quad 1^{st} \text{ extended ACF}$$

Exercise : show that $\rho_\ell(1)$ function of $\rho_1, \dots, \rho_{\ell+1}$

Change $w_t^{(\ell)}$ to $w_{1,t}^{(\ell)}$

For ARMA(2,q) model ,we need to get a handle on (ϕ_1, ϕ_2) of do the same trick

$$z_{t-k}(z_t - \phi_1 z_{t-1} - \phi_2 z_{t-2}) = z_{t-k}(a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q})$$

$$k=q \quad \rho_q - \phi_1 \rho_{q-1} - \phi_2 \rho_{q-2} = -\theta_q \sigma_a^2 / \gamma_0 \neq 0$$

$$k=q+1 \quad \rho_{q+1} - \phi_1 \rho_q - \phi_2 \rho_{q-1} = 0$$

$$k=q+2 \quad \rho_{q+2} - \phi_1 \rho_{q+1} - \phi_2 \rho_q = 0$$

$$k=q+3 \quad \rho_{q+3} - \phi_1 \rho_{q+2} - \phi_2 \rho_{q+1} = 0$$

If we define $\underset{\sim}{\phi}^{(q)}$ as the solution of

$$\begin{bmatrix} \rho_q & \rho_{q-1} \\ \rho_{q+1} & \rho_q \end{bmatrix} \underset{\sim}{\phi}^{(q)} = \begin{bmatrix} \rho_{q+1} \\ \rho_{q+2} \end{bmatrix} \quad \underset{\sim}{\phi}^{(q)} = \begin{bmatrix} \phi_1^q \\ \phi_2^q \end{bmatrix}$$

and define $w_{2,t}^{(q)} = z_t - \phi_1^{(q)} z_{t-1} - \phi_2^{(q)} z_{t-2}$

then, $\underset{\sim}{\phi}^{(q)} = \underset{\sim}{\phi}$ and $w_{2,t}^q \sim MA(q)$

Let $\rho_q(2)$ be the q^{th} autocorrelation of $w_{2,t}^q$, then $\rho_q(2) \neq 0$

Similarly, define $\underset{\sim}{\phi}^{(q+1)}$ as the solution of

$$\begin{bmatrix} \rho_{q+1} & \rho_q \\ \rho_{q+2} & \rho_{q+1} \end{bmatrix} \underset{\sim}{\phi}^{(q+1)} = \begin{bmatrix} \rho_{q+2} \\ \rho_{q+3} \end{bmatrix}$$

$w_{2,t}^{(q+1)} = z_t - \phi_1^{(q+1)} z_{t-1} - \phi_2^{(q+1)} z_{t-2} \sim MA(q)$ process

$\rho_{q+1}(2)$ be the $q+1^{th}$ autocorrelation of $w_{2,t}^{(q+1)}$

Thus define

$$\underset{\sim(2)}{\phi^{(\ell)}} \text{ solution of } \begin{bmatrix} \rho_{\ell} & \rho_{\ell-1} \\ \rho_{\ell+1} & \rho_{\ell} \end{bmatrix} \underset{\sim(2)}{\phi^{(\ell)}} = \begin{bmatrix} \rho_{\ell+1} \\ \rho_{\ell+2} \end{bmatrix}$$

and $w_{2,t}^{(\ell)} = z_t - \phi_1^{(\ell)} z_{t-1} - \phi_2^{(\ell)} z_{t-2}$

$\rho_{\ell}(2)$ be the ℓ^{th} autocorrelation of $w_{2,t}^{(q+1)}$

Then for ARMA(2,q)

$$\rho_{(\ell)}(2) = \begin{cases} c & \ell = q \\ 0 & \ell > q \end{cases} \quad 2^{nd} \text{ extended ACF}$$

For m^{th} extended ACF $m = 1, 2, 3, \dots$

Let

$$\underset{\sim m}{\phi^{(\ell)}} = \begin{bmatrix} \phi_{1(m)}^{(\ell)} \\ \vdots \\ \phi_{m(m)}^{(\ell)} \end{bmatrix} \text{ solution of } \begin{bmatrix} \rho_{\ell} & \rho_{\ell-1} & \cdots & \cdots & \rho_{\ell-m+1} \\ \rho_{\ell+1} & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \rho_{\ell-1} \\ \rho_{\ell-m+1} & & & & \rho_{\ell} \end{bmatrix} \underset{\sim m}{\phi^{(\ell)}} = \begin{bmatrix} \rho_{\ell+1} \\ \vdots \\ \vdots \\ \vdots \\ \rho_{\ell+m} \end{bmatrix}$$

$$w_{m,t}^{(\ell)} = z_t - \phi_{1(m)}^{(\ell)} z_{t-1} - \cdots - \phi_{m(m)}^{(\ell)} z_{t-m}$$

$\rho_{\ell}(m)$ ℓ^{th} autocorrelation of $w_{m,t}^{(\ell)}$

Then for ARMA(p,q)

$$\rho_{(\ell)}(p) = \begin{cases} c & \ell = q \\ 0 & \ell > q \end{cases}$$

Let $\rho_{(\ell)}(0)$ ordinary ACF

MA \ AR		0	1	2	3	4	
0		$\rho_{(1)}(0)$	$\rho_{(2)}(0)$	$\rho_{(3)}(0)$	$\rho_{(4)}(0)$	$\rho_{(5)}(0)$	Ord ACF
1		$\rho_{(1)}(1)$	$\rho_{(2)}(1)$	$\rho_{(3)}(1)$	$\rho_{(4)}(1)$	$\rho_{(5)}(1)$	1 st EACF
2		$\rho_{(1)}(2)$	$\rho_{(2)}(2)$	$\rho_{(3)}(2)$	$\rho_{(4)}(2)$	$\rho_{(5)}(2)$	2 nd EACF

Now in practice p =unknown so we don't know which row to consider.

It can be shown that for ARMA (p,q)

$$\rho_{(\ell)}(m) = \begin{cases} c & \text{if } m \geq p \quad \ell = q + m - p \\ 0 & \text{if } m \geq p \quad \ell > q + m - p \end{cases} \quad (*)$$

$|c| < 1$ some arbitrary constant

This gives rise to the Triangular Pattern

Why? consider the simple cases $z_t = a_t$

$$\rho_{(\ell)}(0) = 0 \quad \ell = 1, 2, \dots$$

$$\rho_1 = \rho_2 = \dots = 0$$

$\phi^{(1)}$ solution of $\rho_2 - \phi^{(1)}\rho_1 = 0$, $\phi^{(1)}$ arbitrary constant

$$w_{1t}^{(1)} = z_t - \phi^{(1)}z_{t-1} \quad \Rightarrow \quad w_{1t}^{(1)} = a_t - \phi^{(1)}a_{t-1} \quad \text{MA}(1)$$

$$\therefore \rho_1(1) \neq 0$$

$\phi^{(2)}$ solution of $\rho_3 - \phi^{(2)}\rho_2 = 0$, $\phi^{(2)}$ arbitrary constant

$$w_{1t}^{(2)} = z_t - \phi^{(2)}z_{t-1} \quad \Rightarrow \quad w_{1t}^{(2)} = a_t - \phi^{(2)}a_{t-1} \quad \text{MA}(1)$$

$$\therefore \rho_2(1) = 0 \quad \text{etc.}$$

The above gives theory for the stationary case.

Sample version $\Phi(B)z_t = \theta(B)a_t$

(1) get "consistent" estimates of $\Phi^{(\ell)}$'s

(2) transform series to $\hat{w}_t^{(\ell)}$

(3) compute autocorrelation $\hat{\rho}_\ell(m)$

Iterated Regression consider ARMA(1,q) case

$$(1) \quad \text{fit} \quad z_t = \Phi^{(0)} z_{t-1} + e_t^{(0)}$$

$$\hat{\Phi}^{(0)} \quad \text{OLS of } \Phi^{(0)}$$

$$\hat{e}_t^{(0)} = z_t - \hat{\Phi}^{(0)} z_{t-1}$$

$$(2) \quad \text{fit} \quad z_t = \Phi^{(1)} z_{t-1} + \beta_1^{(1)} \hat{e}_{t-1}^{(0)} + e_t^{(1)} \quad \text{by OLS}$$

$$\text{get } \hat{\Phi}^{(1)}, \quad \hat{w}_{1,t}^{(1)} = z_t - \hat{\Phi}^{(1)} z_{t-1} \quad \hat{\rho}_1(1) \quad \text{lag 1 SAC of } \hat{w}_{1,t}^{(1)}$$

$$\text{calculate } \hat{e}_t^{(1)} = z_t - \hat{\Phi}^{(1)} z_{t-1} - \hat{\beta}_1^{(1)} \hat{e}_{t-1}^{(0)}$$

$$(3) \quad \text{fit} \quad z_t = \Phi^{(2)} z_{t-1} + \beta_1^{(2)} \hat{e}_{t-1}^{(1)} + \beta_2^{(2)} \hat{e}_{t-2}^{(0)} + e_t^{(2)} \quad \text{by OLS}$$

$$\text{get } \hat{\Phi}^{(2)}, \quad \hat{w}_{2,t}^{(2)} = z_t - \hat{\Phi}^{(2)} z_{t-1} \quad \hat{\rho}_2(1) \quad \text{lag2 SAC}$$

$$\text{calculate } \hat{e}_t^{(2)} \text{ etc.}$$

Repeat the process to get

$$\hat{\Phi}^{(\ell)}, \hat{w}_{1,t}^{(\ell)}, \hat{\rho}_\ell(1) \quad \ell = 1, 2, \dots$$

For 2nd extended sample ACF, start with

$$\text{Fitting } z_t = \Phi_{1(2)}^{(0)} z_{t-1} + \Phi_{2(2)}^{(0)} z_{t-2} + e_t^{(0)} \quad \text{etc.}$$

In general for m^{th} extended sample ACF, start with

$$z_t = \Phi_{1(m)}^{(0)} z_{t-1} + \dots + \Phi_{m(m)}^{(0)} z_{t-m} + e_t^{(0)} \quad \text{by OLS etc.}$$

Then for stationary or nonstationary ARMA(p,q)

$$\Phi_{i(m)}^{(\ell)} \xrightarrow{p} \Phi_i \quad \text{for } m = p \quad \ell \geq q$$

$$\text{Transform to } \hat{w}_{m,t}^{(\ell)} \text{ asymptotically } \hat{\rho}_\ell(m) \xrightarrow{p} \rho_\ell(m) \quad (*)$$

Comment an effectiveness for nonstationary case

Estimation : MLE Derivation of likelihood function

$$\text{AR}(1) \quad \dot{z}_t = z_t - \mu \quad \text{stationary case} \quad z_t - \mu = \Phi(z_{t-1} - \mu) + a_t \quad -1 < \Phi < 1$$

given $z_1, \dots, z_n$

$$\ell(\mu, \Phi, \sigma_a^2 | z) \propto P(z_1, \dots, z_n)$$

$$\text{But} \quad P(z_1, \dots, z_n) = P(z_1)P(z_2, \dots, z_n | z_1) = P(z_1)P(z_2 | z_1) \cdots P(z_n | z_{n-1})$$

$$\ell(\mu, \Phi, \sigma_a^2 | z) = \ell_1(\mu, \Phi, \sigma_a^2 | z_1) \ell_*(\mu, \Phi, \sigma_a^2 | z)$$

$$\text{where} \quad \ell_1(\mu, \Phi, \sigma_a^2 | z_1) \propto P(z_1)$$

$$\ell_*(\mu, \Phi, \sigma_a^2 | z) \propto P(z_2 | z_1) \cdots P(z_n | z_{n-1})$$

$P(z_1) = ?$ When stationary is assumed (as well as normal)

$$\gamma_0 = \text{var}(z_1) = \sigma_a^2 / (1 - \Phi^2) \quad E(z_1) = \mu$$

$$\therefore P(z_1) = \frac{1}{\sqrt{2\pi}} (1 - \Phi)^{1/2} \sigma_a^{-1} e^{-\frac{(1-\Phi^2)}{2\sigma_a^2} (z-\mu)^2} = \ell(\mu, \Phi, \sigma_a^2 | z_1)$$

$$\text{For } t=2, \dots, n \quad P(z_t | z_{t-1}) = \frac{1}{\sqrt{2\pi}} \sigma_a^{-1} e^{-\frac{1}{2\sigma_a^2} a_t^2}$$

$$\text{where} \quad a_t = (z_t - \mu) - \Phi(z_{t-1} - \mu)$$

$$\therefore \ell_*(\mu, \Phi, \sigma_a^2 | z) \propto \sigma_a^{-(n-1)} e^{-\frac{1}{2\sigma_a^2} \sum_{t=2}^n a_t^2}$$

$$\ln \ell = \ln \ell_1 + \ln \ell_* \quad \ln = \log e$$

clearly for n moderately large $\ln \ell_*$ dominates

$\therefore$ Approximate likelihood just by ℓ_*

Now $a_t = (z_t - \mu) - \Phi(z_{t-1} - \mu) = z_t - c - \Phi z_{t-1} \quad t = 2, \dots, n$

where $c = \mu(1 - \Phi)$

Reparameterize $(\Phi, \mu) \rightarrow (\Phi, c)$

nonlinear model in terms of (Φ, μ)

linear model in terms of Φ, c if restriction $|\Phi| \leq 1$ is ignored

In practice we choose the latter. Thus we can use OLS to estimate c & Φ
(because we linear model form for (*))

$\therefore$ MLE of (c, Φ) obtained by $\min_{c, \Phi} S(c, \Phi)$ sum of square function

Also MLE of σ_a^2 is

$$\hat{\sigma}_a^2 = \frac{1}{n-1} \sum_{t=2}^n \hat{a}_t^2(\hat{c}, \hat{\Phi}) = \frac{1}{n-1} S(\hat{c}, \hat{\Phi})$$

where $\hat{a}_t = z_t - \hat{c} - \hat{\Phi} z_{t-1}$ residuals

Asymptotic theory of MLE $(\sqrt{n}\hat{c}, \sqrt{n}\hat{\Phi})$ Normal with finite means ,

var&cov.(1940's)

In nonstationary case When $\Phi = 1$, meaning of c and asymptotic theory very

different(to be discussed later) . However , $(\hat{c}, \hat{\Phi})$ and the same in either stationary or

nonstationary case

AR(p) $(1 - \Phi_1 B - \dots - \Phi_p B^p) \dot{z}_t = a_t$ or $(1 - \Phi_1 B - \dots - \Phi_p B^p) z_t = c + a_t$

where $c = \mu(1 - \Phi_1 - \dots - \Phi_p)$ when stationary is assumed

Adopting same argument

$$\ell(c, \Phi, \sigma_a^2 | z) \propto P(z_1, \dots, z_n) = P(z_1, \dots, z_p) P(z_{p+1}, \dots, z_n | z_1, \dots, z_p)$$

$$\propto \ell_1(c, \Phi, \sigma_a^2 | z_1, \dots, z_p) \ell_*(c, \Phi, \sigma_a^2 | z)$$

$$\propto \ell_*(c, \Phi, \sigma_a^2 | z) \quad \Phi = (\Phi_1, \dots, \Phi_p)'$$

$$\ell_*(c, \Phi, \sigma_a^2 | z) \propto \sigma_a^{-(n-p)} e^{-\frac{1}{2\sigma_a^2} S(c, \Phi)}$$

$$S(c, \Phi) = \sum_{t=p+1}^n [z_t - c - \Phi_1 z_{t-1} - \dots - \Phi_p z_{t-p}]^2 = \sum_{t=p+1}^n a_t^2$$

Use OLS to estimate (c, Φ) $\hat{a}_t = z_t - \hat{c} - \hat{\Phi}_1 z_{t-1} - \dots - \hat{\Phi}_p z_{t-p}$ residuals

$$\text{and } \sigma_a^2 = \frac{1}{n-p} S(\hat{c}, \hat{\Phi}) \quad \text{ignoring restrictions for stationary}$$

This approach again works for both stationary & nonstationary models. The key difference is that for stationary case $\sqrt{n}(\hat{c}, \hat{\Phi})$ will have asymptotic normality

In the nonstationary case, meaning of c is different & also asymptotic distribution theory very different.

Show B-J, F and B-J-C example

Exercise, work out $P(z_1, \dots, z_p)$ for $p=2$ (assuming stationary)

Note that basically, we estimate c, Φ by $\min \sum a_t^2$, one step ahead forecast error.

This will be discussed later.

$$\text{MA}(1) \quad \dot{z}_t = a_t - \theta a_{t-1} \quad t = 1, \dots, n$$

$$P(z_1, \dots, z_n) \propto \sigma_a^{-n} |\Omega|^{-\frac{n}{2}} e^{-\frac{1}{2\sigma_a^2} \dot{z}' \Omega^{-1} \dot{z}} = \ell(c, \theta, \sigma_a^2 | z)$$

$$\text{where } \Omega = \begin{bmatrix} 1+\theta^2 & -\theta & & & \\ -\theta & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\theta \\ & & & -\theta & 1+\theta^2 \end{bmatrix}_{n \times n} \quad c = \mu$$

Difficulty : computing $\dot{z}' \Omega^{-1} \dot{z}$ $|\Omega|$ (much easier)

Alternatively note that

$$\begin{bmatrix} \dot{z}_1 \\ \vdots \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ -\theta & \ddots & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & -\theta & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} c_\theta a_0 \quad c_\theta = \theta$$

$$\dot{z} = D_\theta a - \Delta c_\theta a_0$$

$$\text{Consider } \begin{bmatrix} a_0 \\ \dot{z} \\ \sim \end{bmatrix} = \begin{bmatrix} 1 & \\ -\Delta c_\theta & D_\theta \end{bmatrix} \begin{bmatrix} a_0 \\ a \\ \sim \end{bmatrix} \quad |J| = 1$$

$$P(a_0, a) \rightarrow P(a_0, z) \rightarrow P(z)$$

$$P(a_0, a) = \sigma_a^{-(n+1)} e^{-\frac{1}{2\sigma_a^2} \left[a_0^2 + \sum_{t=1}^n a_t^2 \right]} = P(a_0, z) \Rightarrow P(z) = \int P(a_0) P(z | a_0) da_0$$

to get likelihood function

$$\begin{bmatrix} a_0 \\ a \\ \sim \end{bmatrix} = \begin{bmatrix} 1 & \\ D_\theta^{-1} \Delta c_\theta & D_\theta^{-1} \end{bmatrix} \begin{bmatrix} a_0 \\ \dot{z} \\ \sim \end{bmatrix} = Y - Xa_0 \quad D_\theta^{-1} = \begin{bmatrix} 1 & & & & \\ \theta & \ddots & & & \\ \theta^2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \theta^{n-1} & \dots & \theta^2 & \theta & 1 \end{bmatrix}$$

$$\text{where } Y = \begin{bmatrix} \cdot \\ D_\theta^{-1} \dot{z} \\ \sim \end{bmatrix} \quad X = - \begin{bmatrix} 1 \\ D_\theta^{-1} \Delta c_\theta \end{bmatrix}$$

$$\therefore a_0^2 + \sum_{t=1}^n a_t^2 = (Y - Xa_0)'(Y - Xa_0)$$

(1) conditional likelihood method assume $a_0 = 0$

$$\ell(c, \theta, \sigma_a^2 | z) \propto P(z | a_0 = 0)$$

$$P(z | a_0 = 0) \propto \sigma_a^{-n} e^{-\frac{1}{2\sigma_a^2} \sum_{t=1}^n a_t^2}$$

where $a_t = z_t - c + \theta a_{t-1}$ with $a_0 = 0$

$$\sum_{t=1}^n a_t^2 = S_{(0)}(c, \theta) \quad \text{MLE of } (\theta, c) \quad \min_{\theta, c} S_{(0)}(c, \theta)$$

$$\text{and } \hat{\sigma}_a^2 = \frac{1}{n} S_{(0)}(\hat{c}, \hat{\theta})$$

(2) exact likelihood

$$P(a_0, z) \propto \sigma_a^{-(n+1)} e^{-\frac{1}{2\sigma_a^2} (Y - Xa_0)'(Y - Xa_0)}$$

$$(Y - Xa_0)'(Y - Xa_0) = (Y - X\tilde{a}_0)'(Y - X\tilde{a}_0) + (a_0 - \tilde{a}_0)^2 X'X$$

$$\therefore P(z) \propto \sigma_a^{-n} |X'X|^{\frac{1}{2}} e^{-\frac{1}{2\sigma_a^2} (Y - X\tilde{a}_0)'(Y - X\tilde{a}_0)} \quad \text{by integrating out } a_0$$

where $\tilde{a}_0 = (X'X)^{-1}X'Y$

$$\therefore |X'X| = |\Omega| \quad \tilde{Z}'\Omega^{-1}\tilde{Z} = (Y - X\tilde{a}_0)'(Y - X\tilde{a}_0)$$

Computation : to get MLE of (θ, c, σ_a^2)

(1) Conditional Likelihood

given (θ, c) $a_t(\theta, c) = z_t - c + \theta a_{t-1}(\theta, c)$ with $a_0 = 0$; Recursion

$$\text{get } \min_{\theta, c} \sum_{t=1}^n a_t^2(c, \theta) = \min_{\theta, c} S_{(0)}(c, \theta) = S_{(0)}(\hat{c}, \hat{\theta}) \text{ nonlinear}$$

$$\hat{\sigma}_a^2 = \frac{1}{n} S_{(0)}(\hat{c}, \hat{\theta})$$

(2) Exact likelihood

$$\text{given } (\theta, c) \text{ It turns out that } (X'X) = \frac{1 - \theta^{2(n+1)}}{1 - \theta^2}$$

$$\text{and } \tilde{a}_0 = \frac{-(1 - \theta^2)}{1 - \theta^{2(n+1)}} \sum_{t=1}^n \theta^t a_t(c, \theta) \text{ (estimate } \tilde{a}_0 \text{ from } a_t(c, \theta)'s)$$

$$\text{and } (Y - X\tilde{a}_0)'(Y - X\tilde{a}_0) = \sum_{t=0}^n \tilde{a}_t^2(c, \theta)$$

$$\text{where } \tilde{a}_t(c, \theta) = z_t - c + \theta \tilde{a}_{t-1}(c, \theta) \text{ with } \tilde{a}_0(c, \theta) = \tilde{a}_0$$

$$\text{Thus } \ell(c, \theta, \sigma_a^2 | z) \propto \sigma_a^2 (X'X)^{-\frac{1}{2}} e^{-\frac{1}{2\sigma_a^2} \sum_{t=0}^n \tilde{a}_t^2(c, \theta)}$$

Since $X'X$ is a function of θ only.

For given (c, θ) $\sigma_a^2 = \frac{1}{n} \sum_{t=0}^n \tilde{a}_t^2(c, \theta)$

By concentrating out σ^2

$$\ell_{\max}(c, \theta | z) \propto \left\{ \frac{1 - \theta^{2(n+1)}}{1 - \theta^2} \right\}^{-\frac{1}{2}} \left\{ \sum_{t=0}^n \tilde{a}_t^2(c, \theta) \right\}^{-\frac{n}{2}} \propto \left\{ \sum_{t=0}^n \tilde{b}_t^2(c, \theta) \right\}^{-\frac{n}{2}}$$

$$\tilde{b}_t(c, \theta) = \left\{ \frac{1 - \theta^{2(n+1)}}{1 - \theta^2} \right\}^{\frac{1}{2n}} \tilde{a}_t(c, \theta)$$

S.S. so we can use nonlinear L.S. to find $(\hat{c}, \hat{\theta})$

This approach is due to Analysis of Newbold. It is not used in SCA because it cannot be generalized to vector model.

We can write

$$\ell(c, \theta, \sigma_a^2 | z) \propto \sigma_a^{-n} e^{-\frac{1}{2\sigma_a^2} S(c, \theta, \sigma_a^2)}$$

where

$$S(c, \theta, \sigma_a^2) = \sum_{t=0}^n \tilde{a}_t^2(c, \theta) + \sigma_a^2 \ln \frac{1 - \theta^{2(n+1)}}{1 - \theta^2}$$

given σ_a^2 nonlinear L.S. to $\min_{c, \theta} S(c, \theta, \sigma_a^2)$

$$\text{given } (c, \theta) \quad \hat{\sigma}_a^2 = \frac{1}{n} \sum_{t=0}^n \tilde{a}_t^2(c, \theta)$$

It is clear that for exact likelihood, computational burden much heavier.

Show (0,1,1) example $w_t = (1 - B)z_t$ $c = 0$

Reason for considering exact likelihood is that when true value of θ is clear to 1 and small sample size. e.g. $n=10$ conditional likelihood leads to severe bias (if $\theta_0 = 1$, then true value of θ min. expected likelihood is about .7). This has a large impact on seasonal models.

For example

$$z_t = S_t + a_t \quad S_t = S_{t-12} \quad \text{deterministic}$$

$$(1 - B^{12})z_t = (1 - \theta B^{12})a_t \quad \theta = 1$$

$n=120$ (10yr) really means sample size of 10 for each month.

The above approach can be generalized to MA(q)

$$\dot{z}_t = a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}$$

$$t=1 \quad \dot{z}_1 = a_1 - \theta_1 a_0 - \cdots - \theta_q a_{1-q}$$

$$\text{let } a_* = (a_0, \cdots, a_{1-q})'$$

- Conditional likelihood: for given (c, θ)

Recursively calculate

$$a_t(c, \theta) = z_t + \theta_1 a_{t-1} + \cdots + \theta_q a_{t-q}$$

$$\text{with } a_* = (a_0, \dots, a_{1-q})' = 0$$

$$S_{(0)}(c, \theta) = \sum_{t=1}^n a_t^2(c, \theta)$$

Use nonlinear least squares to $\min_{c, \theta} S_{(0)}(c, \theta)$

$$\hat{\sigma}_a^2 = \frac{1}{n} S_{(0)}(\hat{c}, \hat{\theta})$$

- Exact likelihood for given (c, θ)

estimate a_* by $\tilde{a}_*$ from $a_t(c, \theta)$'s

recursively compute

$$\tilde{a}_t(c, \theta) = z_t + \theta_1 \tilde{a}_{t-1} + \cdots + \theta_q \tilde{a}_{t-q}$$

$$\text{with } \tilde{a}_0(c, \theta) = \tilde{a}_{0*}, \dots, \tilde{a}_{1-q}(c, \theta) = \tilde{a}_{1-q,*}$$

$$\text{let } S(c, \theta, \sigma_a^2) = \sum_{t=1-q}^n \tilde{a}_t^2(c, \theta) + \sigma_a^2 \ln |X'X|$$

Then, given σ_a^2 nonlinear L.S. to $\min_{c, \theta} S(c, \theta, \sigma_a^2)$

$$\text{given } (c, \theta) \quad \hat{\sigma}_a^2 = \frac{1}{n} \sum_{t=1-q}^n \tilde{a}_t^2(c, \theta) \text{ etc.}$$

Key feature of the above

AR(p) , start calculation with $t=p+1$

MA(q) , start with assuming $a_0 = a_{-1} = \dots = a_{1-q} = 0$

For mixed model. we combine both features.

$$z_t = c + \Phi_1 Z_{t-1} + \dots + a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q}$$

$$a_t = z_t - c - \Phi_1 z_{t-1} - \dots - \Phi_p z_{t-p} + \theta_1 a_{t-1} + \dots + \theta_q a_{t-q}$$

start calculation with $t=p+1$

(1) conditional likelihood : $a_p = \dots = a_{p+1-q} = 0$

given c, Φ, θ recursively compute

$$a_t(c, \Phi, \theta) = z_t - c - \Phi_1 z_{t-1} - \dots - \Phi_p z_{t-p} + \theta_1 a_{t-1}(c, \Phi, \theta) + \dots + \theta_q a_{t-q}(c, \Phi, \theta)$$

$$\min_{c, \Phi, \theta} S_{(0)}(c, \Phi, \theta) = \min \sum_{t=p+1}^n a_t^2(c, \Phi, \theta)$$

$$\hat{\sigma}^2 = \frac{1}{n-p} S_{(0)}(\hat{c}, \hat{\Phi}, \hat{\theta})$$

(2) Exact likelihood

get estimate $\tilde{a}_p, \dots, \tilde{a}_{p+1-q}$ from $a_t(c, \Phi, \theta)$'s

Recursively compute

$$\tilde{a}_t(c, \Phi, \theta) = z_t - c - \Phi_1 z_{t-1} - \dots - \Phi_p z_{t-p} + \theta_1 \tilde{a}_{t-1}(c, \Phi, \theta) + \dots + \theta_q a_{t-q}(c, \Phi, \theta)$$

using the initial $\tilde{a}$'s

$$\text{let } S(c, \Phi, \theta, \sigma_a^2) = \sum_{t=p+1-q}^n \tilde{a}_t^2(c, \Phi, \theta) + \sigma_a^2 \ln |X'X|$$

For given σ_a^2

$$\min_{c, \Phi, \theta} S(c, \Phi, \theta, \sigma_a^2)$$

For given c, Φ, θ

$$\sigma_a^2 = \frac{1}{n-p} \sum_{t=p+1-q}^n \tilde{a}_t^2(c, \Phi, \theta)$$

Iterate between (c, Φ, θ) and σ_a^2 until convergence

Show B-JA example

Diagnostic Checking

Examination of residuals

$$\hat{a}_t = z_t - \hat{c} - \hat{\Phi}_1 z_{t-1} - \dots - \hat{\Phi}_p z_{t-p} + \theta_1 \hat{a}_{t-1} + \dots + \theta_q \hat{a}_{t-q}$$

Roughly $\hat{a}_t$ estimate of a_t

Ideally if model correct, then asymptotically

$$r(\hat{a}_t) \xrightarrow{P} 0$$

Box & Pierce (JASA 70) portmantean feast

$$n \sum_{k=1}^m r_k^2(\hat{a}_t) \sim \chi_{m-p-q}^2$$

Ljung and Box modification (1978, Biometrika)

$$n(n+2) \sum_{\ell=1}^m \frac{1}{n-\ell} r_\ell^2(\hat{a}_t) \sim \chi_{m-p-q}^2$$

Derivation if the above depends on stationary assumption

(Yanxiu Li extended this to estimation in nonstationary case)

See BJ-F,C,A, (0,1,1) examples

$$y_t = d \sin\left(\frac{2\pi}{12}t + \omega\right) + a_t = d_1 \sin\left(\frac{2\pi}{12}t + \omega_1\right) + d_2\left(\frac{2\pi}{6}t + \omega_2\right) + a_t$$

$$\frac{2\pi}{12} \quad \frac{2\pi}{6} \quad \frac{2\pi}{4} \quad \frac{2\pi}{3} \quad \frac{2\pi}{2} \quad 2\pi$$

The first can be written as

$$d \sin\left(\frac{2\pi}{12}t + \omega\right) = d\left[\sin\frac{2\pi}{12}t \cos \omega + \cos\frac{2\pi}{12}t \sin \omega\right] = \beta_1 x_{1t} + \beta_2 x_{2t}$$

$$x_{1t} = \sin\frac{2\pi}{12}t \quad x_{2t} = \cos\frac{2\pi}{12}t \quad \beta_1 = d \cos \omega \quad \beta_2 = d \sin \omega$$

$$\therefore y_t = \beta_1 x_{1t} + \beta_2 x_{2t} + \hat{a}_t$$

$$\text{or } y_t = \eta + \beta_1 x_{1t} + \beta_2 x_{2t} + a_t$$

$$\text{or } y_t = \alpha_0 + \alpha_1 t + \beta_1 x_{1t} + \beta_2 x_{2t} + a_t \quad \text{deterministic trend of seasonal approach}$$

Another commonly used approach is thru dummy or indicator variable

B-J & Dave Beeon's contribution (1964) BJR chapter 9

Seasonal MA $MA(1) \rightarrow MA(1)_{12}$

$$Z_t = (1 - \Theta B^{12})a_t$$

$$\begin{cases} \rho_k = \frac{-\Theta}{1 - \Theta^{12}} & -\gamma_0 = (1 + \Theta^{12})\sigma_a^2 \\ \rho_k = 0 & k \neq 12 \end{cases}$$

$$MA(Q)_{12}$$

$$z_t = (1 - \Theta_1 B^{12} - \dots - \Theta_Q B^{12Q}) a_t = \Theta(B^{12}) a_t$$

$$\gamma(B) = \Theta(B^{12}) \Theta(F^{12}) \sigma_a^2$$

$$\begin{cases} \rho_k \neq 0 & k = 12, 24, \dots, 12Q \\ \rho_k = 0 & otherwise \end{cases}$$

Seasonal AR

$$(1 - \phi B^{12}) z_t = a_t$$

$$\gamma(B) = \{\phi(B^{12}) \phi(F)^{12}\}^{-1} \sigma_a^2$$

$$\rho_{12,\ell} = \phi^\ell \quad \ell = 1, 2, \dots$$

$$\rho_k = 0 \quad otherwise$$

$$\text{in general } \phi(B^{12}) = (1 - \phi B^{12} - \dots - \phi_p B^{12p}) a_t$$

and we can have seasonal differential as well

$$\phi(B^{12})(1 - B^{12})^D z_t = \Theta(B^{12}) a_t \quad (p, D, Q)_{12}$$

The above is equivalent to splitting up the series into twelve pieces each having an ARIMA(p,D,Q) model

Two question

- (1) why should seasonal dependence be the same for each pieces or each month? This will lead to what is known as periodic time series model.
- (2) The above ignore the dependence structure between adjustment months.

Multiplicative MA models

$$z_t = (1 - \Theta B^{12})e_t \quad e_t = (1 - \theta B)a_t$$

$$\Rightarrow z_t = (1 - \Theta B^{12})(1 - \theta B)a_t$$

$$\gamma(B) = \sigma^2(1 - \theta B)(1 - \theta F)(1 - \Theta B^{12})(1 - \Theta F^{12})$$

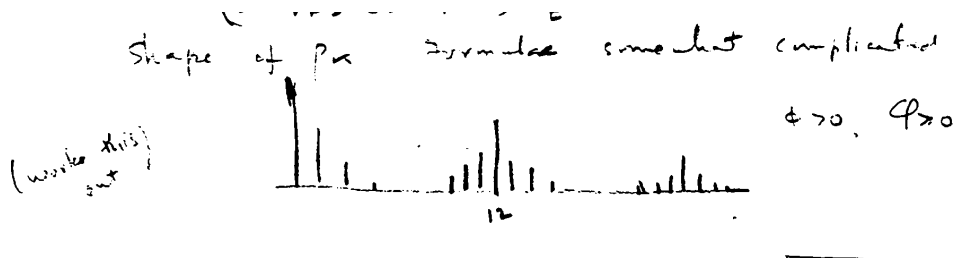
$$\gamma_0 = \sigma^2(1 + \theta^2)(1 + \Theta^2)$$

$$\rho_1 = \frac{-\theta}{1+\theta^2} \quad \rho_{12} = \frac{-\Theta}{1+\Theta^2} \quad \rho_{11} = \rho_{13} = \rho_1 \times \rho_2$$

$$\rho_k = 0 \quad \textit{otherwise} \quad 4 \text{ possible shapes}$$

Multiplicative AR models

$$(1 - \phi B)(1 - \varphi B^{12})z_t = a_t$$



Multiplicative $(p, d, q) \times (p, D, Q)_{12}$

$$\varphi(B^{12})\phi(B)(1-B)^d(1-B^{12})^D z_t = \theta(B)\Theta(B^{12})a_t$$

Stochastic structure can be constructed as was done before for nonseasonal models

Useful ones in practice

$$(1-B)(1-B^{12})z_t = (1-\theta B)(1-\Theta B^{12})a_t \quad \text{Airline model}$$

$$(1-B^{12})z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

$$(1 - \phi B)(1 - B^{12})z_t = (1 - \Theta B^{12})a_t$$

The Airline model $(0,1,1) \times (0,1,1)_{12}$

$$(1-B)(1-B^{12})z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

$$w_t = (1-B^{12})u_t \quad u_t = (1-B)z_t$$

1st difference of $z_t \rightarrow u_t$

seasonal difference of $u_t \rightarrow z_t$

Identification

SACF of z_t persistently high $\rightarrow (1-B)z_t = u_t$

SACF of u_t persistently high at lags 12,24,...

SACF of w_t that of $(0,1,1) \times (0,1,1)_{12}$

Estimating $w_t = (1-B)(1-B^{12})z_t$ lose 1,...,13

to get likelihood $w_t = a_t - \theta_1 a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}$

conditional $a_t = w_t + \theta_1 a_{t-1} + \Theta a_{t-12} - \theta\Theta a_{t-13} \quad t=14, \dots, n$

with $a_1 = a_2 = \dots = a_{13} = 0 \quad S_c(\theta, \Theta) = \sum_{t=14}^n a_t^2$

Exact likelihood see Tiao

If desired, initial estimate of $\theta \ \Theta$ from R_1 and R_{12} can be easily obtained

Diagnostic Checks

See χ^2 SACF of residuals

Forecasting

$Z_1, Z_2, \dots, Z_t$ available

$$Z_t = Z_{t-1} + Z_{t-12} - Z_{t-13} + a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta \Theta a_{t-13}$$

$$Z_{t+\ell} = Z_{t+\ell-1} + Z_{t+\ell-12} - Z_{t+\ell-13} + a_{t+\ell} - \theta a_{t+\ell-1} - \Theta a_{t+\ell-12} + \theta \Theta a_{t+\ell-13}$$

$\hat{Z}_t(1), \dots, \hat{Z}_t(13)$ depend on known Z_t 's and a_t 's.

a_t 's are function of $Z_1, \dots, Z_t$ (assuming $t \gg 1$)

$$\text{for } \ell > 13 \quad \hat{Z}_t(\ell) = \hat{Z}_t(\ell-1) + \hat{Z}_t(\ell-12) - \hat{Z}_t(\ell-13)$$

$$\text{or} \quad (1-B)(1-B^{12}) \hat{Z}_t(\ell) = 0$$

$$\ell = 12r + m, \quad m = 1, \dots, 12 \text{ (month)}$$

$$r = 0, 1, 2, \dots \text{ (year)}$$

$$\text{Then} \quad \hat{Z}_t(12r + m) = b_{0,m} + r b_1$$

$$\text{where } b_{0,m} = \hat{Z}_t(m), \quad b_1 = \hat{Z}_t(13) - \hat{Z}_t(1) \quad (\text{change as } t \text{ advance})$$

Thus we have 12 parallel straight lines both $b_{0,m}$ and b_1 depend on t -forecasting origin,

so both the sloped and seasonality are adaptive.

$$\begin{aligned} \hat{Z}_t(12r + m) &= \left(b_{0,m} - \frac{m}{12} b_1 \right) + \ell \left(\frac{b_1}{12} \right) \\ &= \left(\bar{b}_0 - \frac{\bar{m}}{12} b_1 \right) + \left[(b_{0,m} - \bar{b}_0) - \left(\frac{m - \bar{m}}{12} \right) b_1 \right] + \ell \left(\frac{b_1}{12} \right) \end{aligned}$$

$$\text{since } \ell = 12r + m, r = \frac{\ell - m}{12}, \bar{b}_0 = \frac{1}{12} (b_{0,1} + \dots + b_{0,12}), \bar{m} = 6.5$$

$$\therefore \hat{Z}_t(\ell) = \alpha_0 + S_t + \ell \frac{b_1}{12}$$

$$\text{where } \alpha_0 = \bar{b}_1 - \frac{\hat{m}}{12} b_1$$

$$S_\ell = S_{\ell-12} \text{ with } S_1 + \dots + S_{12} = 0$$

$$\text{and } S_m = b_{0,m} - \bar{b}_0 - \frac{1}{12}(m - \bar{m})b_1, \quad m = 1, \dots, 12.$$

The above can also be written as

$$\hat{Z}_t(\ell) = \alpha_0 + \sum_{j=1}^k d_j \sin\left(\frac{2\pi j}{12} \ell + w_j\right) + \left(\frac{b_1}{12}\right) \ell \quad \text{verify} \leftarrow \text{Exercise}$$

The important thing to note is that the seasonal coefficient and slope are also changing as origin of forecast is advanced.

Note:

$$(i) \quad (1 - B)(1 - B^{12})Z_t = (1 - \theta B)(1 - \theta B^{12})a_t$$

$$(1 - B)^2(1 + B + \dots + B^{11})Z_t = a_t$$

$$u_t = (1 + B + \dots + B^{11})Z_t \Rightarrow (1 - B)^2 u_t = a_t$$

moving filter

$$(ii) \quad (1 - B^{12}) = (1 - B^6)(1 - B^6)$$

$$(1 - B^6) = (1 - B^3)(1 + B^3) = (1 - B)(1 + B + B^2)(1 + B)(1 - B + B^2)$$

$$(1 + B^6) = (1 + B^2)(1 - B^2 + B^4) = (1 + B^2)(1 - \sqrt{3}B + B^2)(1 + \sqrt{3}B + B^2)$$

12 zeros on the unit circle

$$(1-B) (1+B) (1+B^2) (1+B+B^2) (1-B+B^2) (1-\sqrt{3}B+B^2) (1+\sqrt{3}B+B^2)$$

Find the correspondence between the zeros and the

EX: if $(1-\sqrt{3}B+B^2)y_t = 0$

$$y_t = A_1 e^{-i\theta} + A_2 e^{i\theta} \quad \theta = \frac{\pi}{6}, \text{ see next page}$$

The ψ weights

$$e_t(l) = \begin{cases} (1 - \psi_1^2 - \dots - \psi_{l-1}^2) \sigma_a^2, & l > 1 \\ \sigma_a^2, & l = 1 \end{cases}$$

$$(1-B)(1-B^{12})\psi(B) = (1-\theta B)(1-\Theta B^{12})$$

The π weights

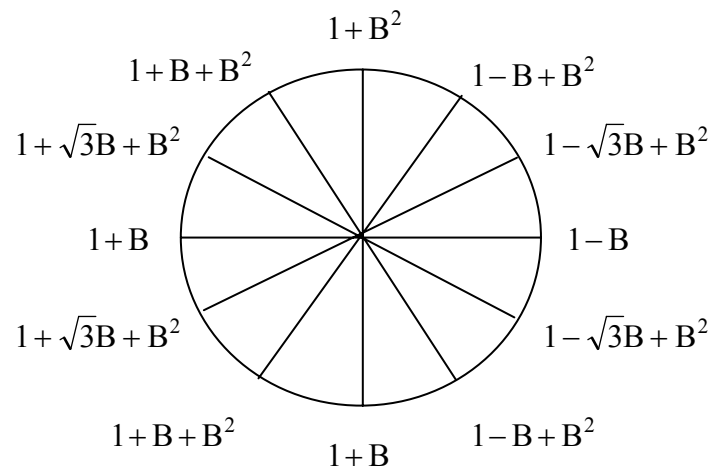
$$\pi(B)(1-\theta B)(1-\Theta B^{12}) = (1-B)(1-B^{12})$$

$$\begin{aligned} a_{t+1} &= \frac{(1-B)(1-B^{12})}{(1-\theta B)(1-\Theta B^{12})} Z_t \\ &= \left[1 - \frac{(1-\theta)B}{1-\theta B} \right] \left[1 - \frac{(1-\Theta)B^{12}}{1-\Theta B^{12}} \right] Z_{t+1} = Z_{t+1} - \hat{Z}_t(1) \\ &= \left[1 - \frac{(1-\theta)B}{1-\theta B} - \frac{(1-\Theta)B^{12}}{1-\Theta B^{12}} \left(1 - \frac{(1-\theta)B}{1-\theta B} \right) \right] Z_{t+1} \end{aligned}$$

$$\begin{aligned} \therefore \hat{Z}_t(1) &= \frac{(1-\theta)}{1-\theta B} Z_t + \frac{(1-\theta)}{1-\Theta B^{12}} \left[1 - \frac{(1-\theta)B}{1-\theta B} \right] Z_{t+1-12} \\ &= \frac{(1-\theta)}{1-\theta B} Z_t + \frac{(1-\theta)}{1-\Theta B^{12}} u_{t+1-12} \end{aligned}$$

interpretation

12 zeros on unit circle correspond to $1 - B^{12}$



$$x^2 - \sqrt{3}x + 1 = 0 \quad x = \frac{\sqrt{3} \pm i}{2} = \frac{\sqrt{3}}{2} \pm \frac{i}{2} \quad \theta = 30^\circ, 270^\circ$$

$$x^2 + \sqrt{3}x + 1 = 0 \quad x = \frac{-\sqrt{3} \pm i}{2} \quad \theta = 150^\circ, 210^\circ$$

$$x^2 - x + 1 = 0 \quad x = \frac{1 \pm \sqrt{3}i}{2} \quad \theta = 60^\circ, 300^\circ$$

$$x^2 + x + 1 = 0 \quad x = \frac{-1 \pm \sqrt{3}i}{2} \quad \theta = 120^\circ, 240^\circ$$

$$x^2 + 1 = 0 \quad x = \frac{\pm 2i}{2} = \pm i \quad \theta = 90^\circ, 270^\circ$$

General rule for tentative specification

Seasonal diff: SACF high at lags 12, 24, ...

Seasonal MA: SACF cut off at lags 12 or 24

Seasonal AR: SACF tails off at lags 12, 24, ...

Multiplicative model:

Note shape at low lags.

Symmetric cuts off or tails off around 12, 24, 36, ...

Over Differencing problem

Frequently actual time series contain a deterministic seasonal component

$$Z_t = S_t + N_t \quad \leftarrow ARIMA$$

where $S_t = S_{t-12}$

$$(1 - B^{12})Z_t = (1 - B^{12})N_t = (1 - \Theta B^{12}), \quad \Theta = 1$$

unless n is large (typically we only have at most 10 years of data, conditional likelihood approach yield $\hat{\Theta}$ biased downward substantially).

Also mention periodic problem especially nonhomogeneous variables.

Discrete linear

Dynamic Models

$$y_t = v_0 x_t + v_1 x_{t-1} + v_2 x_{t-2} + \dots$$

$$= (v_0 + v_1 B + v_2 B + \dots) x_t$$

v_j : effect of x_{t-j} on Y_t impulse response

$v(B)$: transfer function

clearly behavior if y_t depend on

- (1) input sequence $\{x_t\}$
- (2) $v(B)$

Note if $v(B)$ converges for $|B| \leq 1$

$$g = v_0 + v_1 + v_2 + \dots \quad \text{steady state gain (food intake example)}$$

parsimonious models for $v(B)$

$$v(B) = \frac{w(B)}{\delta(B)} x_{t-b}, \quad x_{t-b} = B^b x_t$$

b: delay (integer)

$$w(B) = w_0 - w_1 B - \dots - w_s B^s$$

$$\delta(B) = \delta_0 - \delta_1 B - \dots - \delta_r B^r \quad \leftarrow \begin{array}{ll} \text{all zeros} & \text{outside V.C., stable; } g \text{ exists} \\ \text{on V.C.} & \text{not stable} \\ \text{inside V.C.} & \text{explosive} \end{array}$$

Relationship to ARMA model:

$$y_t = v(B)x_t \quad \rightarrow \quad Z_t = \frac{\theta(B)}{\Phi(B)} a_t$$

In practice $\{x_t\}$

(1) deterministic

indicator type

nonindicator type

(2) stochastic – BJR Chapter 11, transfer function modeling or (B521)

We shall concentrate on indicator type for

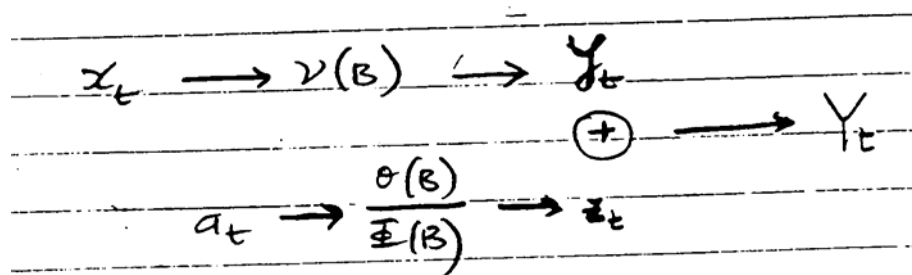
intervention analysis

outlier of structural change detection

In reality

$$Y_t = y_t + Z_t, \quad \{y_t\}, \{Z_t\} \text{ independent}$$

$$y_t = v(B)x_t \quad Z_t = \frac{\theta(B)}{\Phi(B)}a_t$$



In practice time series are often subject to exogenous change

(1) time of occurrence known; estimate of effect with known model – intervention analysis

(2) time of occurrence unknown but model known – detection of outlier and structural change

(3) all unknown.

$$Y_t = y_t + z_t$$

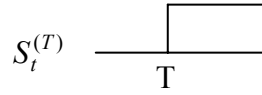
$$y_t = v(B)x_t$$

$$x_t = \begin{cases} P_t^{(T)} \\ S_t^{(T)} \end{cases}$$

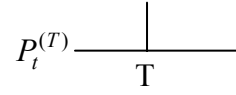
$$P_t^{(T)} = \begin{cases} 1, & t = T \\ 0, & t \neq T \end{cases} \text{ pulse, } S_t^{(T)} = \begin{cases} 1, & t \geq T \\ 0, & t < T \end{cases} \text{ step}$$

$$\text{and } (1-B)S_t^{(T)} = P_t^{(T)} \text{ or } S_t^{(T)} = \frac{1}{1-B}P_t^{(T)}$$

see figure B (p.24) for various useful intervention models



Input
Output



$$(1) \quad \frac{w(B)}{\delta(B)} B^b S_t^{(T)}$$

$$wBS_t^{(T)}$$

$$\frac{w(B)}{\delta(B)} B^b P_t^{(T)}$$

$$\frac{wB}{1-\delta B} P_t^{(T)}$$

$$(2) \quad \frac{w}{1-\delta B} BS_t^{(T)}$$

$$\left(\frac{\alpha_1}{1-\delta B} + \frac{\alpha_2}{1-B} \right) P_t^{(T)}$$

$$= \frac{w_0 - w_1 B}{(1-\delta B)(1-B)} BP_t^{(T)}$$

$$(3) \quad \frac{w}{1-B} BS_t^{(T)}$$

$$\left(\alpha_0 + \frac{\alpha_1}{1-\delta B} + \frac{\alpha_2}{1-B} \right) BP_t^{(T)}$$

Extension to more than one intervention obvious.

Example 1: Ozone Jan 1955 – Dec 1972

Output #8

- (1) Rule 63 Jan '60 change composition of gasoline – level shift.
- (2) Modification of engines in new cars since '66 – expect down trend – summer v.s. winter trend.

$$Y_t = \alpha_1 \xi_{1t} + \alpha_2 \frac{1}{1-B^{12}} \xi_{2t} + \alpha_3 \frac{1}{1-B^{12}} \xi_{3t} + z_t$$

$$Z_t = \frac{(1-\theta B)(1-\Theta B^{12})}{1-B^{12}} a_t$$

model for Z_t given in p.227 as M1, $Z_t \leftrightarrow x$

$$Y_1(\text{Binary}) \rightarrow \xi_{1t} = s_t^{(T)} = \begin{cases} 1, & t \geq T = \text{Jan } 60 \\ 0, & \text{otherwise} \end{cases}$$

$$Y_2(\text{Binary}) \rightarrow \xi_{2t} = \begin{cases} 1, & \text{summer months beginning 66 June – Oct.} \\ 0, & \text{otherwise} \end{cases}$$

$$Y_3(\text{Binary}) \rightarrow \xi_{3t} = \begin{cases} 1, & \text{winter months beginning 66 Nov. – May.} \\ 0, & \text{otherwise} \end{cases}$$

Estimation

$$(1-B^{12})Y_t = \alpha_1(1-B^{12})\xi_{1t} + \alpha_2\xi_{2t} + \alpha_3\xi_{3t} + (1-\theta B)(1-\Theta B^{12})$$

see output #8 p.234

$c_1 = \alpha_1, \quad c_2 = \alpha_2, \quad c_3 = \alpha_3 \quad \text{etc.}$

See change in behavior of residuals between $t = 60-80$ of since $t = 140$.

Example 2

Crest toothpaste market share of crest weekly.

January 58 to June 64, 276 obs.

August 60, ADA endorsement of Crest

$t = 134$, Crest aggressively advertise this endorsement effect expected to begin at $t = 135$. See output #9, p.246.

(1) build an ARMA model for data since $t = 138$

$$(1 - B)Z_t = (1 - \theta B)a_t$$

(2) Apply same model to all data with intervention

$$y_t = \frac{w}{1 - \delta B} S_t^{(T)} + \frac{1 - \theta B}{1 - B} a_t, \quad T = 135$$

$$\hat{w} = 0.0877, \quad \hat{\delta} = 0.5636$$

$$\text{eventual gain } g = \frac{w}{1 - \delta} = \frac{0.0877}{0.4364} = 20\%$$

(3) Output also have another formulation

2 level shifts at $t = 135$ and $t = 136$.

example 3: Monthly electricity demand (logged) p.136

Key features: Growth tapering off

Seasonality

Oil crisis after September 73'

Econ. Downturn in 75'

Using data throw 9/73'

$$y_t = \frac{\omega_0}{1-B^{12}} \left(1 + \frac{\omega_1}{1-\delta B^{12}} \right) x_t + \frac{(1-\theta_1 B)(1-\theta_{12} B^{12})}{1-B^{12}} a_t$$

$$x_t = \begin{cases} 1, & t \geq 13 \\ 0, & t < 13 \end{cases}$$

verify $\omega_0(1+\omega_1)$ initial growth rate (2nd year V.S. 1st)

$\omega_0(1+\omega_1+\omega_1\delta)$ growth rate next year (3rd year V.S. 2nd)

$\omega_0(1+\omega_1+\omega_1\delta+\omega_1\delta^2)$ next

$\vdots$

$\omega_0 \left(1 + \frac{\omega_1}{1-\delta} \right)$ eventual growth rate.

ω_0	ω_1	δ	θ_1	θ_{12}	σ^2
.104	-.18	.69	-.22	.60	.020
(.02)	(.12)	(.17)			

➔ $\omega_0(1+\omega_1)=8.5\%$

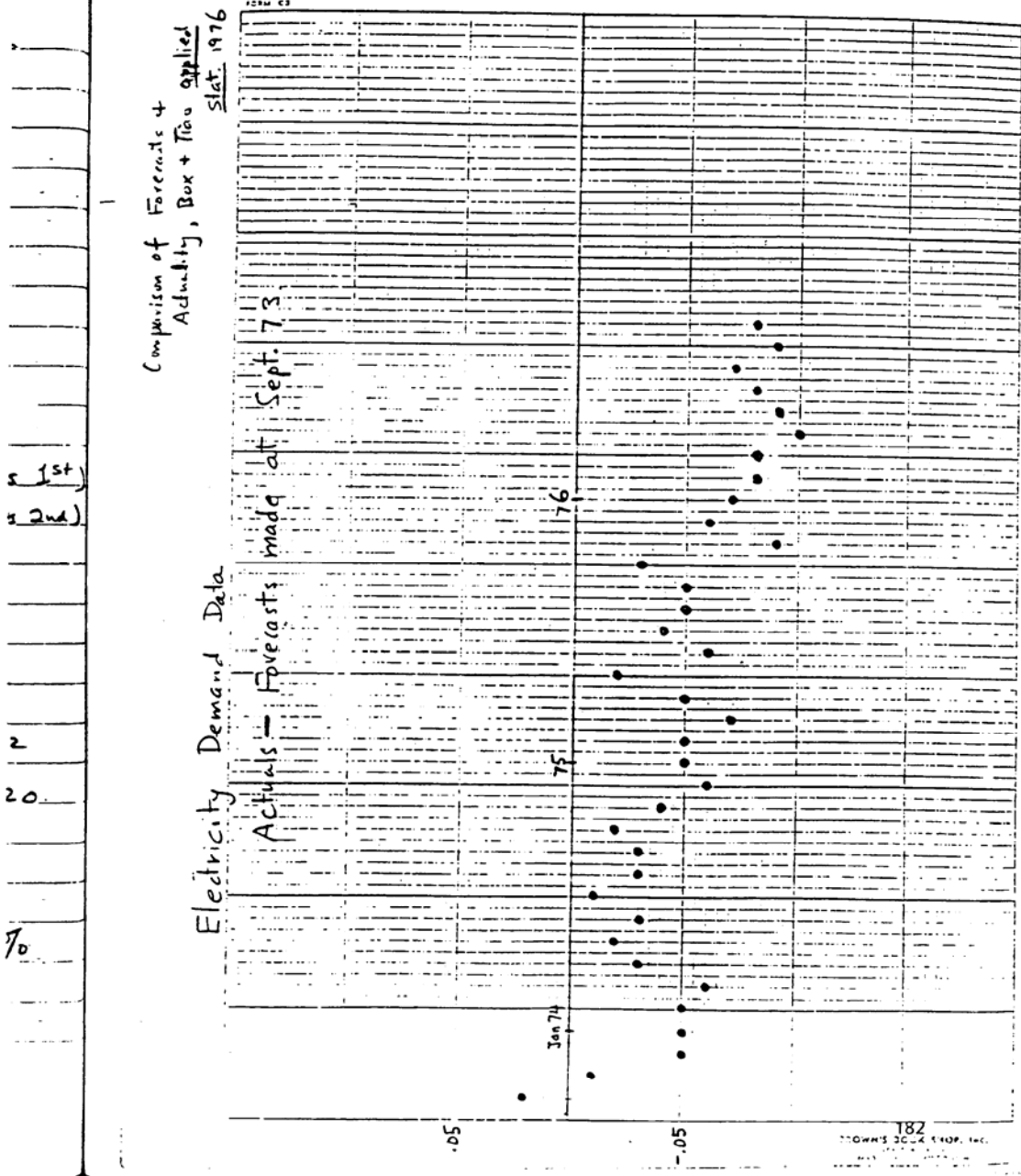
$\omega_0(1+\omega_1+\delta\omega_1)=7\%$

$\omega_0 \left(1 + \frac{\omega_1}{1-\delta} \right) = 4.4\%$

Now from comparison of forecast-actual

Level shift 12/73'
 11/75'

-70- 7/1- #5399 p.52-58



$$y_t = \beta_1 \xi_{1t} + \beta_2 \xi_{2t} + \frac{\omega_0}{1-B^{12}} \left(1 + \frac{\omega_1}{1-\delta B^{12}} \right) x_t + \frac{(1-\theta_1 B)(1-\theta_{12} B^{12})}{1-B^{12}} a_t$$

$$\xi_{1t} = \begin{cases} 0, & t < \text{Dec. 73'} \\ 1, & t \geq \text{Dec. 73'} \end{cases} \quad \xi_{2,t} = \begin{cases} 0, & t < \text{Nov. 75'} \\ 1, & t \geq \text{Nov. 75'} \end{cases}$$

β_1	β_2	ω_0	ω_1	δ	σ^2
-.03	-.031	.096	-.14	.75	.018
(.009)	(.009)	(.02)	(.08)	(.13)	

level change in 73' = 3%

level change in 75' = 3%

$\omega_0(1+\omega_1)$ initial growth rate = 8.2%

$\omega_0 \left(1 + \frac{\omega_1}{1-\delta} \right)$ eventual growth rate = 4.2%

These & examples in papers show a variety of applications of intervention analysis.

(1) time of occurrence known.

(2) model of expected effect assumed known.

Estimation of Time Series in the presence of outliers, *Technometrics*(1988), 30, 193-204, I. Chang, G.C. Tiao and C. Chen.

Outliers

- Unusual observations spurious observation

Must be in relation to a model

e.g. large observation outside 3 s.d. of normal would not be all that unusual if we had a long tail distribution.

- Outlier in time series different in many ways to outlier in an ordinary regression model.

$$y_i = x_i \beta + \varepsilon_i$$

outlier at $P_t^{(b)}$ say typically means that

$$y_i = x_i \beta + \varepsilon_i + w I_i(b), \quad I_i(b) = \begin{cases} 1, & i = b \\ 0, & \text{otherwise} \end{cases}$$

so we have an large y_i or large residuals $\hat{\varepsilon}_i$ for $i = b$

masking effect—because of estimate of variance of ε_i i.e. $\hat{\sigma}^2$

contaminated by large $\hat{\varepsilon}_i$

But in time series

$$y_t = \phi y_{t-1} + a_t$$

suppose y_T is particularly large then not only at time T we will have a large residual, but also at time T+1.

$\therefore$ the problem is more complex.

- 2 approaches
 - (1) M-estimator (Robust estimate) Huber type
 - (2) Wisc. Type. I. Chang's thesis (detection route $\leftrightarrow$ estimation)

Outlier as non-repetitive intervention at unknown time t

Basically changes in the level structure of series (not variability)

4 different types

(1) IO (Fox, JRSS 72)

(2) AO

(3) LS

(4) TC

$$y_t = \frac{\theta(B)}{\Phi(B)} a_t = \Psi(B) a_t \quad \text{no outlier, homogeneous}$$

$$\text{IO: } y_t = \Psi(B) (a_t + \omega P_t^{(b)}) \quad \text{large innovation at } t=b$$

$$\text{AO: } y_t = \Psi(B) a_t + \omega P_t^{(b)} \quad \text{gross error}$$

$$\text{LS: } y_t = \Psi(B) a_t + \omega \frac{1}{1-B} P_t^{(b)} \quad \text{level shift}$$

$$\text{TC: } y_t = \Psi(B) a_t + \omega \frac{1}{1-\delta B} P_t^{(b)} \quad \text{temporary (transit change)}$$

Of course outliers could be due to other types of intervention, but mixture of these four should cover a lot of ground.

Effect on estimate.

AO v.s. IO

Effect on residuals.

Detection of outliers

(I) Suppose $\Psi(B)$ known, b known, type known, $\pi(B)y_t = e_t$ invertible.

$$\text{IO} \quad e_t = a_t + \omega P_t^{(b)}$$

$$\text{AO} \quad e_t = a_t + \omega \pi(B) P_t^{(b)} \quad \pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \dots$$

$$\text{LS} \quad e_t = a_t + \omega \alpha(B) P_t^{(b)} \quad \alpha(B) = \frac{\pi(B)}{1-B} = 1 - \alpha_1 B - \alpha_2 B^2$$

Clearly $\hat{\omega}_{I,b} = e_b$ $\text{var}(\hat{\omega}_{I,b}) = \sigma^2$

For AO

t	$P_t^{(b)}$	$\pi(B)P_t^{(b)}$
1	0	0
2	$\vdots$	$\vdots$
3	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$
$b-1$	0	0
b	1	1
$b+1$	0	$-\pi_1$
$b+2$	0	$-\pi_2$
$\vdots$	$\vdots$	$\vdots$

$$e_t = a_t + \omega x_t, \quad \hat{\omega} = \frac{\sum x_t e_t}{\sum x_t^2}$$

$$\begin{aligned} \hat{\omega}_{A,b} &= (1 + \pi_1^2 + \dots)^{-1} (e_b - \pi_1 e_{b+1} - \pi_2 e_{b+2} - \dots) \\ &= (1 + \pi_1^2 + \dots)^{-1} \pi(F) e_b \end{aligned}$$

$$\text{var}(\hat{\omega}_{A,b}) = \frac{\sigma^2}{(1 + \pi_1^2 + \dots)} = \hat{\sigma}_{\hat{\omega}_A}^2$$

For LS is a similar argument

$$\hat{\omega}_{L,b} = (1 + \alpha_1^2 + \alpha_2^2 + \dots)^{-1} \alpha(F) e_b$$

$$\text{var}(\hat{\omega}_{L,b}) = \frac{\sigma^2}{(1 + \alpha_1^2 + \alpha_2^2 + \dots)} = \sigma_{\hat{\omega}_L}^2$$

Why easy to show that $\text{var}(\hat{\omega}_I) \geq \text{var}(\hat{\omega}_A) \geq \text{var}(\hat{\omega}_b)$

$$\lambda_{I,b} = \frac{e_b}{\sigma} \quad \lambda_{1,b}$$

$$\lambda_{A,b} = \frac{\hat{\omega}_{A,b}}{\sigma_{\hat{\omega}_A}} \quad \lambda_{2,b}$$

$$\lambda_{L,b} = \frac{\hat{\omega}_{L,b}}{\sigma_{\hat{\omega}_L}} \quad \lambda_{3,b}$$

similarly for TC

(II) Suppose $\Psi(B)$ known, b known, type unknown.

$$\lambda_b = \max\{|\lambda_{1b}|, |\lambda_{2b}|, |\lambda_{3b}|\}$$

Then if $\lambda_b > c$, we have an outlier of one of the 3 types.

e.g. if $\lambda_b > c$ and $\lambda_b = |\lambda_{1b}|$, we have an IO type.

(could use LR test route, see Tech Paper)

Distribution is general very complex, since we are dealing with distribution of maximize of highly correlated random variable.

(III) Suppose $\Psi(B)$ unknown (but order of model known), bundle Iterative Estimation and Detection Produced

(i) Estimates $\hat{\phi}'_s$, $\hat{\theta}'_s$ and $\hat{\sigma}^2$ are obtained assuming no outliers, get $\hat{e}_t$ and

$$\text{note } \hat{\sigma}^2 = \frac{1}{n} \sum \hat{e}_t^2.$$

(ii) Calculate $\hat{\lambda}_t = \max\{|\hat{\lambda}_{1t}|, |\hat{\lambda}_{2t}|, |\hat{\lambda}_{3t}|\}$. (overtyp)

$$\text{Calculate } \eta_b = \max_t \hat{\lambda}_t$$

If $\eta_b > c$ (3 or 3.5 typically in practice), then we have an outlier, and also we can determine its type.

(a) if $\eta_b = |\lambda_{1b}| \Rightarrow \text{IO}$

$$\text{adjust } \tilde{e}_t = \hat{e}_t - \hat{\omega}_{I,b} = 0$$

(b) if $\eta_b = |\lambda_{2b}| \Rightarrow \text{AO}$

$$\tilde{e}_t = \hat{e}_t - \hat{\omega}_{A,b} \hat{\pi}(B) P_t^{(b)}$$

(c) if $\eta_b = |\lambda_{3b}| \Rightarrow \text{LS}$

$$\tilde{e}_t = \hat{e}_t - \hat{\omega}_{L,b} \hat{\alpha}(B) P_t^{(b)}$$

From adjusted residuals, calculate new $\hat{\sigma}^2$.

- (iii) Repeat (iv) to see if we have more outliers, this is to reduce masking effect due to large $\hat{e}_t$
- (iv) Suppose after a few iterations we find outliers at $b_1, b_2, \dots, b_k$, and we tentatively know their types. Then we form the intervention models

$$y_t = \sum_{i=1}^k w_i L_i(B) P_t^{(b)} + \frac{\theta(B)}{\Phi(B)} a_t$$

$$L_i = 1, \quad \text{AO}$$

$$= \frac{\theta(B)}{\Phi(B)}, \quad \text{IO}$$

$$= \frac{1}{1-B}, \quad \text{LS}$$

We estimate w 's, ϕ 's, θ 's together, get new residuals of σ^2 , and then so to

(ii).

Model respecification, we can also let

$$\tilde{y}_t = y_t - \sum_{i=1}^k \hat{w}_i L_i(B) P_t^{(b)}$$

and then attempt to respecify the model

EX (1) crest data

(2) variety stone data

(Hillmer Bell, Tiao, other examples in the papers, also Disc. #10. output)

Read #1241(Leut Tiao JASA,1976)

unobserved Component Models (Signal extraction) Haneyt Todd(1983) JBES

$$z_t = S_t + N_t$$

signal noise (movies average filter)

Questions:

- (i) if we know models for S_t , N_t
what's model for z_t ?
- (ii) given $\{z_t\}$ how to estimate S_t ?
- (iii) given $\{z_t\}$ & its model, how to specify S_t and estimate it (decomposition

problem). Identification

(I) Suppose S_t follows $(1-\phi B)S_t = b_t$ and $N_t = e_t$, b_t, e_t are iid normal with variance

σ_b^2 and σ_e^2 , b_t and e_t are independent.

Spectrum or Covariance General function of z_t is

$$\frac{\sigma_b^2}{(1-\phi B)(1-\phi F)} + \sigma_e^2 = \gamma_z(B) \quad B = e^{-i2\pi f} \text{ spectrum}$$

$$\gamma_z(B) = \gamma_s(B) + \gamma_N(B)$$

$$\gamma_z(B) = \frac{\sigma_b^2 + \sigma_e^2(1-\phi B)(1-\phi F)}{(1-\phi B)(1-\phi F)} = \sigma_a^2 \frac{(1-\theta B)(1-\theta F)}{(1-\phi B)(1-\phi F)} \quad \text{ARMA}(1,1)$$

$$\sigma_a^2(1+\theta^2) = \sigma_e^2(1+\phi^2) + \sigma_b^2 \quad -\sigma_a^2\theta = -\sigma_e^2\phi$$

$$\therefore \text{ given } (\phi, \sigma_e^2, \sigma_b^2) \text{ solve for } (\sigma_a^2, \theta)$$

(ii) Suppose we have $z_1, \dots, z_n$; how to estimate $S_1, \dots, S_n$

$$P\left(\begin{matrix} z \\ \sim \end{matrix}, \begin{matrix} S \\ \sim \end{matrix}\right) = P\left(\begin{matrix} z \\ \sim \end{matrix} \middle| \begin{matrix} S \\ \sim \end{matrix}\right) P\left(\begin{matrix} S \\ \sim \end{matrix}\right)$$

Clearly $N_t = z_t - S_t = e_t$

$$\therefore P\left(\underset{\sim}{z} \middle| \underset{\sim}{S}\right) \propto \sigma_e^{-n} e^{-\frac{1}{2\sigma_e^2} \underset{\sim}{N}' \underset{\sim}{N}} \quad \underset{\sim}{N} = (\underset{\sim}{z} - \underset{\sim}{S})$$

$$\propto \sigma_e^{-n} e^{-\frac{1}{2\sigma_e^2} \sum_{t=1}^n (z_t - S_t)^2} \propto \sigma_e^{-n} e^{-\frac{1}{2\sigma_e^2} (\underset{\sim}{z} - \underset{\sim}{S})' (\underset{\sim}{z} - \underset{\sim}{S})}$$

$P(\underset{\sim}{S})$ since S_t is AR(1) so that

$$P(\underset{\sim}{S}) \propto (1 - \phi^2)^{\frac{1}{2}} \sigma_b^{-n} e^{-\frac{1}{2\sigma_b^2} \left[(1 - \phi^2) S_1^2 + \sum_{t=2}^n (S_t - \phi S_{t-1})^2 \right]}$$

$$\propto \sigma_b^{-n} |R|^{\frac{1}{2}} e^{-\frac{1}{2\sigma_b^2} \underset{\sim}{S}' R^{-1} \underset{\sim}{S}} \quad R = (1 - \phi^2)^{-1} \begin{bmatrix} 1 & \phi & \dots & \phi^{n-1} \\ \phi & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \phi \\ \phi^{n-1} & \dots & \phi & 1 \end{bmatrix}$$

Now to estimate $\underset{\sim}{S}$, we clearly should use $P(\underset{\sim}{S} \middle| \underset{\sim}{z})$ the conditioned distⁿ of $\underset{\sim}{S}$

given $\underset{\sim}{z}$

$$\begin{aligned} & \frac{1}{\sigma_e^2} (\underset{\sim}{S} - \underset{\sim}{z})' (\underset{\sim}{S} - \underset{\sim}{z}) + \frac{1}{\sigma_b^2} \underset{\sim}{S}' R^{-1} \underset{\sim}{S} = (\underset{\sim}{S} - \underset{\sim}{z})' A (\underset{\sim}{S} - \underset{\sim}{z}) + \underset{\sim}{S}' B \underset{\sim}{S} \\ & = (\underset{\sim}{S} - \hat{\underset{\sim}{S}})' (A + B) (\underset{\sim}{S} - \hat{\underset{\sim}{S}}) + \underset{\sim}{z}' A (A + B)^{-1} B \underset{\sim}{z} \end{aligned} \quad \text{useful identify}$$

$$\text{when } \hat{\underset{\sim}{S}} = (A + B)^{-1} A \underset{\sim}{z} \quad A = \frac{1}{\sigma_e^2} T \quad B = \frac{1}{\sigma_b^2} R^{-1}$$

$$\therefore (A + B) \hat{\underset{\sim}{S}} = A \underset{\sim}{z}$$

$$\text{or } \left(\frac{1}{\sigma_e^2} I + \frac{1}{\sigma_b^2} R^{-1} \right) \hat{\underset{\sim}{S}} = \frac{1}{\sigma_e^2} \underset{\sim}{z}$$

$$(I + \alpha R^{-1}) \hat{\underset{\sim}{S}} = \underset{\sim}{z} \quad \alpha = \frac{\sigma_e^2}{\sigma_b^2}$$

$$\mathbf{I} + \alpha R^{-1} = \mathbf{I} + \alpha \begin{bmatrix} 1 & -\phi & & & \\ -\phi & 1+\phi^2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1+\phi^2 & -\phi \\ & & & -\phi & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\alpha & -\alpha\phi & & & \\ -\alpha\phi & 1+\alpha(1+\phi^2) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 1+\alpha(1+\phi^2) & -\alpha\phi \\ & & & -\alpha\phi & 1+\alpha \end{bmatrix}$$

$\therefore$ in the middle

$$z_t = \{1 + \alpha(1 + \phi^2)\} \hat{S}_t - \alpha\phi(\hat{S}_{t-1} + \hat{S}_{t+1})$$

$$\{1 + \alpha(1 + \phi^2) - \alpha\phi(B + F)\} \hat{S}_t = z_t$$

$$\{1 + \alpha(1 - \phi B)(1 - \phi F)\}^{-1} z_t = \hat{S}_t$$

$$\frac{\sigma_b^2}{\sigma_b^2 + \sigma_e^2(1 - \phi B)(1 - \phi F)} z_t = \hat{S}_t \quad \text{or} \quad \frac{\sigma_b^2}{\sigma_a^2(1 - \theta B)(1 - \theta F)} z_t = \hat{S}_t$$

$$\text{Since } \sigma_b^2 = \sigma_a^2(1 - \theta B)(1 - \theta F) - \sigma_e^2(1 - \phi B)(1 - \phi F)$$

$$\hat{S}_t = \left\{ 1 - \frac{\sigma_e^2(1 - \phi B)(1 - \phi F)}{\sigma_a^2(1 - \theta B)(1 - \theta F)} \right\} z_t = \left\{ 1 - \left(\frac{\sigma_e^2}{\sigma_a^2} \right) \frac{(1 - \phi B)(1 - \phi F)}{(1 - \theta B)(1 - \theta F)} \right\} z_t$$

Note that $(\sigma_a^2, \theta, \phi)$ are parameters of the overall model for z_t

$$\hat{S}_t = \{\omega_0 + \omega_1(B + F) + \omega_2(B^2 + F^2) + \omega_3(B^3 + F^3) + \dots\} z_t \quad \text{symmetric filter}$$

e.g. $\phi = 1$ (Cleveland-Tiao, 1976, JASA)

$$\omega_0 = 1 - \frac{\sigma_e^2}{\sigma_a^2} \left(\frac{2}{1 + \theta} \right) \quad \omega_j = \frac{\sigma_e^2}{\sigma_a^2} \frac{1 - \theta}{1 + \theta} \theta^{j-1}, \omega_0 + 2 \sum_{j=1}^{\infty} \omega_j = 1$$

$$\frac{(1-\phi B)(1-\phi F)}{(1-\theta B)(1-\theta F)} = \frac{(1-\phi B)(B-\phi)}{(1-\theta B)(B-\theta)} = \alpha_1 + \frac{\alpha_2}{1-\theta B} + \frac{\alpha_3}{B-\theta}$$

$$\text{For } \alpha_3: \quad \alpha_3 = \frac{(1-\phi\theta)(\theta-\phi)}{1-\theta^2}$$

multiply both sides by $B-\theta$ then let $B \rightarrow \theta$

$$\text{For } \alpha_2: \quad \alpha_2 = \frac{\left(1-\frac{\phi}{\theta}\right)\left(\frac{1}{\theta}-\phi\right)}{\frac{1}{\theta}-\theta} = \frac{(\theta-\phi)(1-\phi\theta)}{\theta(1-\theta^2)}$$

multiply both sides by $1-\theta B$, then let $B \rightarrow \frac{1}{\theta}$

$$\text{For } \alpha_1: \quad \text{let } B = \phi \quad \alpha_1 + \frac{(\theta-\phi)(1-\phi\theta)}{(1-\theta^2)} \left[\frac{1}{\theta(1-\phi\theta)} + \frac{1}{\phi-\theta} \right] = 0$$

$$\alpha_1 = -\frac{1}{1-\theta^2} \left[\frac{\theta-\phi}{\theta} - (1-\phi\theta) \right] = \frac{\phi}{\theta}$$

$$\frac{(1-\phi B)(1-\phi F)}{(1-\theta B)(1-\theta F)} = \frac{\phi}{\theta} + \frac{(\theta-\phi)(1-\phi\theta)}{1-\theta^2} \left[\frac{1}{\theta} \frac{1}{1-\theta B} + \frac{F}{1-\theta F} \right]$$

$$= 1 + \frac{(\phi-\theta)^2}{1-\theta^2} + \frac{(\theta-\phi)(1-\phi\theta)}{1-\theta^2} \left[\frac{B}{1-\theta B} + \frac{F}{1-\theta F} \right]$$

$$\therefore \omega_0 = 1 - \frac{\sigma_e^2}{\sigma_a^2} \left(1 + \frac{(\phi-\theta)^2}{1-\theta^2} \right)$$

$$\omega_j = \frac{\sigma_e^2}{\sigma_a^2} \frac{(\phi-\theta)(1-\phi\theta)}{1-\theta^2} \theta^{j-1}$$

Also in general $p\left(\begin{smallmatrix} s \\ \sim \end{smallmatrix} \middle| \begin{smallmatrix} z \\ \sim \end{smallmatrix}\right)$ jointly normal

$$E\left(\begin{smallmatrix} s \\ \sim \end{smallmatrix} \middle| \begin{smallmatrix} z \\ \sim \end{smallmatrix}\right) = \hat{s} = (A + B)^{-1} A z$$

$$\text{cov}\left(\begin{smallmatrix} s \\ \sim \end{smallmatrix} \middle| \begin{smallmatrix} z \\ \sim \end{smallmatrix}\right) = (A + B)^{-1} = \sigma_e^2 (\mathbf{I} + \alpha R^{-1})^{-1}$$

For the widdle

$$s_t - \hat{s}_t = s_t - \frac{\sigma_b^2}{\sigma_a^2 \theta(B) \theta(F)} (s_t + N_t) = \left\{ 1 - \frac{\sigma_b^2}{\sigma_a^2 \theta(B) \theta(F)} \right\} s_t - \frac{\sigma_b^2}{\sigma_a^2 \theta(B) \theta(F)} N_t$$

$$\begin{aligned} r_{s_t - \hat{s}_t}(B) &= \left\{ 1 - \frac{\sigma_b^2}{\sigma_a^2 \theta(B) \theta(F)} \right\}^2 \frac{\sigma_b^2}{(1 - \phi B)(1 - \phi F)} + \frac{(\sigma_b^2)^2 \sigma_e^3}{[\sigma_a^2 \theta(B) \theta(F)]^2} \\ &= \frac{\sigma_e^4 \sigma_b^2 (1 - \phi B)(1 - \phi F) + \sigma_e^2 \sigma_b^2}{[\sigma_a^2 \theta(B) \theta(F)]^2} = \frac{\sigma_e^2 \sigma_b^2}{\sigma_a^2 \theta(B) \theta(F)} \\ &= \sigma_e^2 \left[1 - \frac{\sigma_e^2 \phi(B) \phi(F)}{\sigma_a^2 \theta(B) \theta(F)} \right] \end{aligned}$$

From which we can work out variance and covariance Etc.

(iii) We have $(1 - \phi B)s_t = (1 - \eta B)b_t \quad N_t = e_t \Leftrightarrow (1 - \phi B)z_t = (1 - \theta B)a_t$

$$\sigma_b^2 (1 - \eta B)(1 - \eta F) + \sigma_e^2 (1 - \phi B)(1 - \phi F) = \sigma_a^2 (1 - \theta B)(1 - \theta F)$$

$$\sigma_e^2 \leq \sigma_a^2 \frac{(1 - \theta B)(1 - \theta F)}{(1 - \phi B)(1 - \phi F)}$$

$$\therefore 0 \leq \sigma_e^2 \leq \sigma_a^2 \min_{\lambda} \frac{(1 - \theta B)(1 - \theta F)}{(1 - \phi B)(1 - \phi F)} = \sigma_e^2 \quad B = e^{-i\lambda}$$

This gives rise to a range of decomposition Canonical Decomposition.

$$\min_{\lambda} \frac{(1-\theta B)(1-\theta F)}{(1-\phi B)(1-\phi F)} = \min_{\lambda} \frac{1+\theta^2-2\theta \cos \lambda}{1+\phi^2-2\phi \cos \lambda} \quad 0 \leq \lambda \leq \pi$$

$$= \min_{\lambda} g(\lambda)$$

$$\frac{dg(\lambda)}{d\lambda} = \frac{(1+\phi^2-2\phi \cos \lambda)2\theta \sin \lambda - (1+\theta^2-2\theta \cos \lambda)2\phi \sin \lambda}{(1+\phi^2-2\phi \cos \lambda)^2}$$

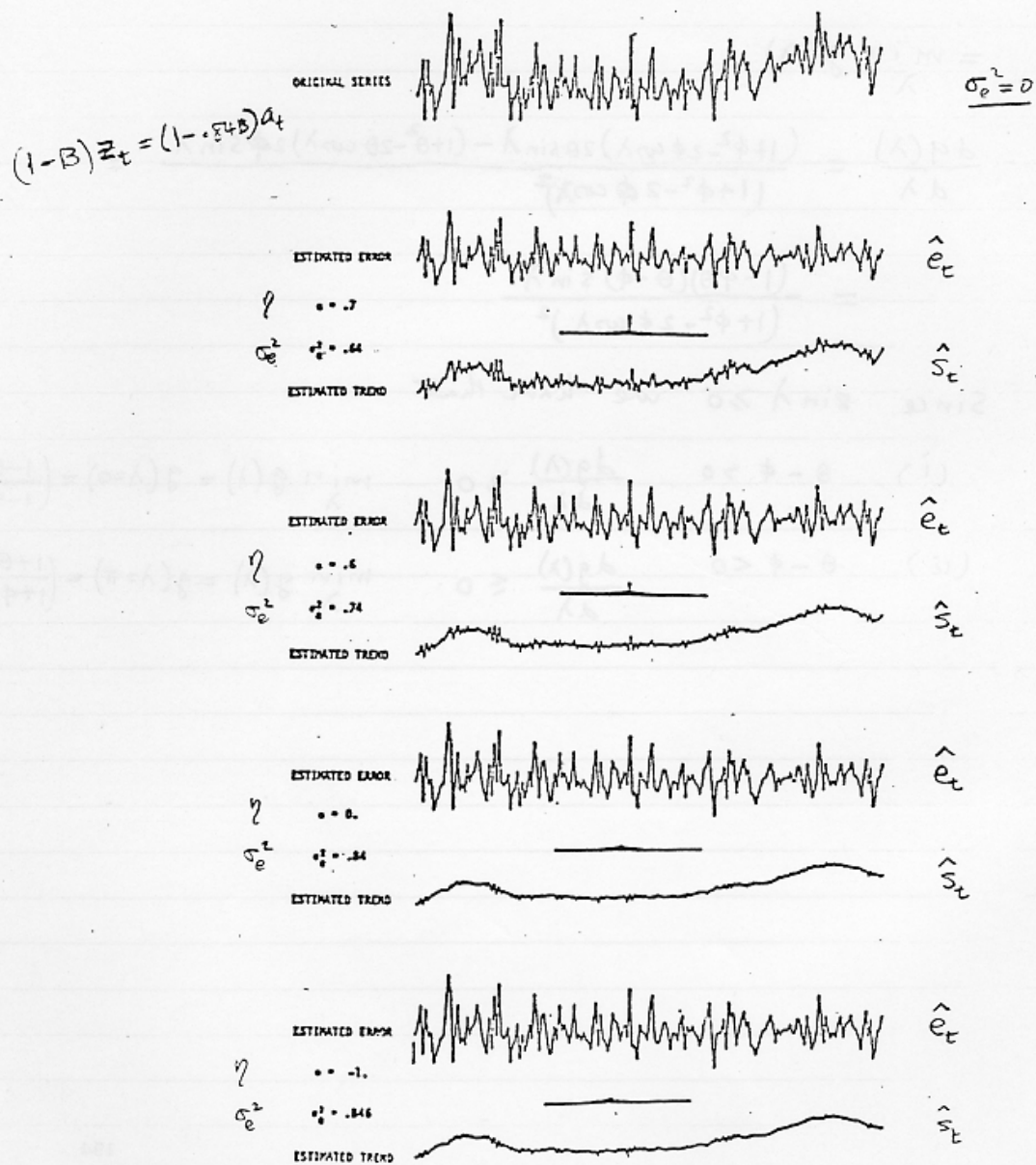
$$= \frac{-(1-\phi\theta)(\theta-\phi)\sin \lambda}{(1+\phi^2-2\phi \cos \lambda)^2}$$

Since $\sin \lambda \geq 0$ we have that

$$(i) \quad \theta - \phi > 0 \quad \frac{dg(\lambda)}{d\lambda} \geq 0 \quad \min_{\lambda} g(\lambda) = g(\lambda = 0) = \left(\frac{1-\theta}{1-\phi} \right)^2$$

$$(ii) \quad \theta - \phi < 0 \quad \frac{dg(\lambda)}{d\lambda} \leq 0 \quad \min_{\lambda} g(\lambda) = g(\lambda = \pi) = \left(\frac{1+\theta}{1+\phi} \right)^2$$

Figure 2. SOME POSSIBLE DECOMPOSITIONS INTO TRENDS AND ERROR, CONSUMER PRICE INDEX EXAMPLE



Here $\phi = 1$

$$\min_{\lambda} \frac{(1-\phi B)(1-\phi F)}{(1-\lambda B)(1-\lambda F)} = \frac{(1+\theta)^2}{4} \text{ at } \lambda = \pi.$$

more generally

$$\begin{aligned}
z_t &= s_t + N_t \\
\phi(B)s_t &= \eta(B)b_t \quad N_t = e_t \\
r_z(B) &= r_s(B) + \sigma_e^2 \quad \sigma_a^2 \frac{\theta(B)\theta(F)}{\phi(B)\phi(F)} = \frac{\eta(B)\eta(F)}{\phi(B)\phi(F)} \sigma_b^2 + \sigma_e^2 \\
\therefore \sigma_a^2 \theta(B)\theta(F) &= \sigma_b^2 \eta(B)\eta(F) + \sigma_e^2 \phi(B)\phi(F) \quad (A)
\end{aligned}$$

$$\hat{s}_t = \left(1 - \frac{\sigma_e^2 \phi(B)\phi(F)}{\sigma_a^2 \theta(B)\theta(F)} \right) z_t \quad r_{s-\hat{s}}(B) = \sigma_e^2 \left(1 - \frac{\sigma_e^2}{\sigma_a^2} \pi(B)\pi(F) \right)$$

and

$$\sigma_e^2 \phi(B)\phi(F) \leq \sigma_a^2 \theta(B)\theta(F)$$

$$\text{or } 0 \leq \sigma_e^2 \leq \min_{\lambda} \sigma_a^2 \frac{\theta(B)\theta(F)}{\phi(B)\phi(F)} \quad B = e^{-i\lambda}$$

once we know σ_e^2 then we can solve for σ_b^2 , $\eta(B)$ from (A)

Canonical Decomposition (Tiao/Hillmer) $\max \sigma_e^2 = \overline{\sigma}_e^2$

(1978 Biom)

Some asymptotic distribution theory (for stationary case)

$$\text{AR}(1) \quad z_t = \phi z_{t-1} + a_t \quad t = 1, \dots, n$$

$$\hat{\phi} = \frac{\sum_{t=2}^n z_t z_{t-1}}{\sum_{t=2}^n z_{t-1}^2}$$

$$\text{Now} \quad \hat{\phi} - \phi = \frac{\sum z_t z_{t-1} - \sum z_{t-1}^2 \phi}{\sum z_{t-1}^2} = \frac{\sum z_{t-1} (z_t - \phi z_{t-1})}{\sum z_{t-1}^2} = \frac{\sum z_{t-1} a_t}{\sum z_{t-1}^2}$$

$$= \frac{\frac{1}{n} \sum a_t^2}{\frac{1}{n} \sum z_{t-1}^2} \bullet \frac{\frac{1}{n} \sum z_{t-1} a_t}{\frac{1}{n} \sum a_t^2}$$

$$\sqrt{n}(\hat{\phi} - \phi) = \frac{\frac{1}{n} \sum a_t^2}{\frac{1}{n} \sum z_{t-1}^2} \frac{\frac{1}{\sqrt{n}} \sum z_{t-1} a_t}{\frac{1}{n} \sum a_t^2}$$

$$= w_n \frac{\frac{1}{\sqrt{n}} \sum_{t=2}^n a_t (a_{t-1} + \phi a_{t-2} + \phi^2 a_{t-3} + \dots)}{\frac{1}{n} \sum a_t^2}$$

$$w_n \rightarrow (1 - \phi^2)$$

$$\sqrt{n}(\hat{\phi} - \phi) \doteq (1 - \phi^2) \sqrt{n}(r_1^* + \phi r_2^* + \phi^2 r_3^* + \dots)$$

$$\text{assume:} \quad \sqrt{n} \begin{bmatrix} r_1^* \\ r_2^* \\ \vdots \\ r_l^* \end{bmatrix} \sim N(0, \mathbf{I}) \quad (\text{will be proved})$$

$$\sqrt{n}(\hat{\phi} - \phi) \rightarrow N(0, 1 - \phi^2)$$

$$\text{AR}(2) \quad z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t$$

$$\begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \end{bmatrix} = \begin{bmatrix} \sum z_{t-1}^2 & \sum z_{t-2} z_{t-1} \\ \sum z_{t-1} z_{t-2} & \sum z_{t-2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum z_{t-1} z_t \\ \sum z_{t-2} z_t \end{bmatrix}$$

$$\begin{bmatrix} \hat{\phi}_1 - \phi_1 \\ \hat{\phi}_2 - \phi_2 \end{bmatrix} = \begin{bmatrix} \sum z_{t-1}^2 & \sum z_{t-2} z_{t-1} \\ \sum z_{t-1} z_{t-2} & \sum z_{t-2}^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum z_{t-1} a_t \\ \sum z_{t-2} a_t \end{bmatrix} - \begin{bmatrix} \sum z_{t-1}^2 & \sum z_{t-1} z_{t-2} \\ \sum z_{t-1} z_{t-2} & \sum z_{t-2}^2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$= \begin{bmatrix} \sum z_{t-1}^2 & \sum z_{t-1} z_{t-2} \\ \sum z_{t-1} z_{t-2} & \sum z_{t-2}^2 \end{bmatrix}^{-1} \begin{pmatrix} \sum z_{t-1} a_t \\ \sum z_{t-2} a_t \end{pmatrix}$$

$$\sqrt{n} \begin{pmatrix} \hat{\phi}_1 - \phi_1 \\ \hat{\phi}_2 - \phi_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{n} \sum z_{t-1}^2 & \frac{1}{n} \sum z_{t-1} z_{t-2} \\ \frac{1}{n} \sum z_{t-1} z_{t-2} & \frac{1}{n} \sum z_{t-2}^2 \end{bmatrix} \frac{1}{\sqrt{n}} \begin{bmatrix} \sum z_{t-1} a_t \\ \sum z_{t-2} a_t \end{bmatrix}$$

$$\sqrt{n} \begin{pmatrix} \hat{\phi}_1 - \phi_1 \\ \hat{\phi}_2 - \phi_2 \end{pmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix} \begin{pmatrix} \frac{1}{n} \sum a_t^2 \\ \frac{1}{n} \sum a_t^2 \end{pmatrix} \frac{1}{\sqrt{n}} \begin{bmatrix} \sum z_{t-1} a_t / \frac{1}{n} \sum a_t^2 \\ \sum z_{t-2} a_t / \frac{1}{n} \sum a_t^2 \end{bmatrix}$$

We have the vector as

$$\sqrt{n} \begin{bmatrix} \sum a_t (a_{t-1} + \psi_1 a_{t-2} + \psi_2 a_{t-3} + \dots) \\ \sum a_t (a_{t-2} + \psi_1 a_{t-3} + \psi_2 a_{t-4} + \dots) \end{bmatrix} / \sum a_t^2$$

$$= \sqrt{n} \begin{bmatrix} r_1^* + \psi_1 r_2^* + \psi_2 r_3^* + \dots \\ r_2^* + \psi_1 r_3^* + \psi_2 r_4^* + \dots \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{say}$$

$$\text{var}(x_1) = (1 + \psi_1^2 + \psi_2^2 + \dots) = \gamma_0 / \sigma_a^2$$

$$\text{var}(x_2) = (1 + \psi_1^2 + \psi_2^2 + \dots) = \gamma_0 / \sigma_a^2$$

$$\text{cov}(x_1, x_2) = (\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 + \dots) = \gamma_1 / \sigma_a^2$$

$$\begin{aligned} \therefore \sqrt{n} \begin{pmatrix} \hat{\phi} - \phi \\ \sim \quad \sim \end{pmatrix} &\sim N \left(0, \begin{bmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{bmatrix}^{-1} \sigma_a^2 \right) \\ &\sim N \left(0, \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}^{-1} \gamma_0^{-1} \sigma_a^2 \right) \end{aligned}$$

$$\begin{aligned} \text{cov} \left(\sqrt{n} \begin{pmatrix} \hat{\phi} - \phi \\ \sim \quad \sim \end{pmatrix} \right) &= \frac{1}{1 - \rho_1^2} \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & 1 \end{bmatrix} \frac{\sigma_a^2}{\gamma_0} \\ &= \frac{1 - \rho_1\phi_1 - \rho_2\phi_2}{1 - \rho_1^2} \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & 1 \end{bmatrix} \quad \text{since } \rho_1 = \frac{\phi_1}{1 - \phi_2}, \rho_2 = \phi_2 + \frac{\phi_1^2}{1 - \phi_2} \\ &= \begin{bmatrix} 1 - \phi_2^2 & -\phi_1(1 + \phi_2) \\ -\phi_1(1 + \phi_2) & 1 - \phi_2^2 \end{bmatrix} \quad \left(\begin{array}{l} \text{note if } \phi_2 = 0 \\ \text{var}(\hat{\phi}_2) = \frac{1}{n} \text{ partial} \end{array} \right) \end{aligned}$$

$$\text{corr}(\sqrt{n}\hat{\phi}_1, \sqrt{n}\hat{\phi}_2) = \frac{-\phi_1(1 + \phi_2)}{1 - \phi_2^2}$$

In general AR(p) stationary

$$\sqrt{n} \begin{pmatrix} \hat{\phi} - \phi \\ \sim \quad \sim \end{pmatrix} \sim N(0, \Gamma_p^{-1} \sigma_a^2) \quad \Gamma_p = \begin{bmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{p-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \gamma_1 \\ & & & \gamma_0 \end{bmatrix} \text{ symmetric}$$

MA(1)

$$\begin{aligned}
\pi_1(z) &= \frac{\sum z_t z_{t-1}}{\sum z_{t-1}^2} \\
&= \frac{\sum (a_t - \theta a_{t-1})(a_{t-1} - \theta a_{t-2})}{\sum z_{t-1}^2} \\
&= \frac{\sum a_t a_{t-1} + \theta^2 \sum a_{t-1} a_{t-2} - \theta \sum a_{t-1}^2 - \theta \sum a_t a_{t-2}}{\sum z_{t-1}^2} \\
&= \frac{(1 + \theta^2) \sum a_t a_{t-1} - \theta \sum a_{t-1}^2 - \theta \sum a_t a_{t-2}}{\sum z_{t-1}^2} \\
&= \frac{\frac{1}{n} \sum a_{t-1}^2}{\frac{1}{n} \sum z_{t-1}^2} \left[(1 + \theta^2) r_1^* - \theta r_2^* - \theta \right] \\
&= [r_1 + \rho_1 r_2^* + \rho_1]
\end{aligned}$$

$$\sqrt{n}(r_1(z) - \rho_1) = \sqrt{n}[r_1^* + \rho_1 r_2^*]$$

$$\text{var} \sqrt{n}(r_1(z) - \rho_1) = 1 + \rho_1^2$$

Thus, the key to many of the distribution is the asymptotic distribution of

$$\sqrt{n}(r_1^*, r_2^*, \dots)$$

$$\sqrt{nr_1^*} = \sqrt{n} \frac{\sum a_t a_{t-1}}{\sum a_t^2} = \frac{\frac{1}{\sqrt{n}} \sum a_t a_{t-1}}{\frac{1}{n} \sum a_t^2} = \frac{\frac{1}{\sqrt{n}} \sum a_t a_{t-1}}{\sigma_a^2}$$

asymptotic distribution of $\frac{1}{\sqrt{n}} \sum a_t a_{t-1}$

$$\begin{array}{cccc} a_1 a_2 & a_2 a_3 & a_3 a_4 & a_4 a_5 \cdots \\ u_1 & u_2 & u_3 & u_4 \cdots \end{array}$$

$\cdots u_{t-2} \ u_{t-1} \ u_t \ u_{t+1} \cdots$ not independent

but we have that $(\cdots u_{t-3} u_{t-2}) \ (u_t u_{t+1} \cdots)$ independent, call 1-dependent process.

Assume u_t stationary

Theorem If $E(u_t) = 0 \quad \text{var}(u_t) = \sigma_u^2 < \infty$

then $\frac{1}{\sqrt{n}} \sum_{t=1}^n u_t \rightarrow N(0, v)$

$$v = E(u_1^2) + 2E(u_1 u_2) = \sigma_a^4 \quad \text{since } E(u_1 u_2) = 0$$

$$\underbrace{u_1, \dots, u_{k-1}; u_k}_{Y_1}; \underbrace{u_{k+1}, \dots, u_{2k-1}; u_{2k}}_{Y_2}; \underbrace{u_{2k+1}, \dots, u_{3k-1}; u_{3k}}_{Y_3}; u_{3k+1} \cdots$$

we see that $Y_1, Y_2, Y_3, \dots$ are iid random variable.

The proof consist of choosing k in such a way that

$$\frac{1}{\sqrt{n}} \sum_{u=1}^n u_t = \frac{1}{\sqrt{n}} \sum Y's \quad \text{then apply standards CLT to the r.h.s.}$$

$$\rightarrow K > 2, \quad n = Kq + r \quad 0 \leq r < K \quad q = \left\lfloor \frac{n}{K} \right\rfloor$$

$$\begin{aligned} \bar{Y}_{Kn} &= \frac{1}{\sqrt{n}} [Y_1 + Y_2 + \cdots + Y_q] \\ &= \frac{1}{\sqrt{n}} [(u_1 + \cdots + u_{K-1}) + (u_{K+1} + \cdots + u_{2K-1}) + \cdots + (u_{(q-1)K+1} + \cdots + u_{qK-1})] \end{aligned}$$

$$\bar{X}_{Kn} = \frac{1}{\sqrt{n}} [u_K + u_{2K} + \cdots + u_{qK}]$$

$$R_{Kn} = \frac{1}{\sqrt{n}} [u_{2K+1} + \cdots + u_n]$$

$$\frac{1}{\sqrt{n}} \sum_{t=1}^n u_t = \bar{Y}_{Kn} + \bar{X}_{Kn} + R_{Kn}$$

$$\text{var}(Y_1) = (K-1)\sigma^4 \quad \text{var}\left(\frac{Y_1}{\sqrt{K-1}}\right) = \sigma^4$$

$$\bar{Y}_{Kn} = \frac{\sqrt{K-1}}{\sqrt{n}} \left[\frac{Y_1}{\sqrt{K-1}} + \frac{Y_2}{\sqrt{K-1}} + \cdots + \frac{Y_q}{\sqrt{K-1}} \right]$$

$$= \frac{1}{\sqrt{q}} [\alpha_1 + \alpha_2 + \cdots + \alpha_q] \xrightarrow{\text{CLT}} N(0, \sigma^4) \quad \text{as } q \rightarrow \infty$$

$$\bar{X}_{Kn} = \frac{1}{\sqrt{n}} (u_K + u_{2K} + \cdots + u_{qK}) \quad E(\bar{X}_{Kn}) = 0$$

$$\text{var}(\bar{X}_{Kn}) = \frac{1}{n} q \sigma^4 \leq \frac{1}{K} \sigma^4 \rightarrow 0 \quad \text{as } K \rightarrow \infty$$

$$\therefore \bar{X}_{Kn} \xrightarrow{P} 0$$

For R_{Kn} $E(R_{Kn}) = 0$

$$\text{var}(R_{Kn}) = \frac{r}{n} \sigma^4 \rightarrow 0 \quad \text{since } r < K$$

In the change we are using the fact

$$n \rightarrow \infty$$

$$K \rightarrow \infty$$

$$q \rightarrow \infty$$

$$q = \left\lceil \frac{n}{K} \right\rceil \quad \text{so we need to choose } K \text{ such that all the above hold.}$$

$$\text{e.g. } K = n^\delta \quad |\delta| < 1 \text{ will do.}$$

Similar treatment could be apply to r_2^*

$$\frac{1}{\sqrt{n}} \sum_{t=3}^n a_t a_{t-2}$$

$$\begin{array}{ccccccccc} a_1 a_3 & a_2 a_4 & a_3 a_5 & a_4 a_6 & a_5 a_7 & a_6 a_8 & \cdots \\ u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & \cdots \end{array}$$

Here $(\cdots, u_{t-4}, u_{t-3})$ $(u_t, u_{t+1}, u_{t+2}, \cdots)$ independent. Call 2-dependent.

$$\bar{Y}_{Kn} = \frac{1}{\sqrt{n}} [(u_1 + \cdots + u_{K-2}) + (u_{K+1} + \cdots + u_{sK-2}) + \cdots]$$

$$\bar{X}_{Kn} = \frac{1}{\sqrt{n}} [(u_{K-1} + u_K) + (u_{2K-1} + u_{2K}) + \cdots]$$

$$\bar{R}_{Kn} = \frac{1}{\sqrt{n}} [\text{remainder}]$$

General Thm

$$\cdots u_1, \cdots, u_t, \cdots$$

is an m-dependent stationary process (m: finite integer)

i.e.

$$(\cdots u_{t-m-2}, u_{t-m-1}) \quad (u_t, u_{t+1}, u_{t+2}, \cdots) \text{ independent}$$

$$\text{with } E(u_t) = 0 \quad \text{var}(u_t) = \sigma^2 < \infty$$

$$\text{Then } \frac{1}{\sqrt{n}}(u_1 + \cdots + u_n) \xrightarrow{L} N(0, v)$$

$$v = \text{var}(u_1) + 2 \text{cov}(u_1, u_2) + \cdots + 2 \text{cov}(u_1, u_{m+1})$$

see Fuller or Anderson (p.427)

Diagnostic checking

Asymptotic distⁿ of Box, Pierce, Ljung Q statistic

stationary ARMA(p,q)

$$\hat{a}_t = z_t - \hat{\phi}_1 z_{t-1} - \cdots - \hat{\phi}_p z_{t-p} + \hat{\theta}_1 \hat{a}_{t-1} - \cdots - \hat{\theta}_q \hat{a}_{t-q}$$

$$r_l(\hat{a}_t)$$

Presumably $\hat{a}_t$ estimate a_t

$$r_l(a_t) = \frac{\sum a_t a_{t-K}}{\sum a_t^2} \quad (r_l^* \text{ in last lecture})$$

$$r_l(a_t) \xrightarrow{P} 0$$

$$\text{var}(r_l) = \frac{1}{n} \quad \text{also } r_l, r_{K+l} \text{ uncorrelated}$$

$$\text{cov}(r) = \frac{1}{n} \mathbf{I} \quad (\text{Bartlett Formula})$$

Also a_t normal $\Rightarrow \sqrt{n} r \sim N(0, \mathbf{I})$ as shown earlier

$$n \sum_{l=1}^m r_l^2 \sim \chi_m^2 \quad \text{overall Portmanteau test}$$

(on the null hypothesis that a_t is white noise.)

However,

$\hat{a}_t$ is not a_t

in obtaining $\hat{a}_t$ we used parameter estimates.

Question: how does this affect the null distⁿ?

Box & Pierce JASA (1970), 65, 1509-26.

Consider the case true model is AR(1) $|\phi| < 1$

$$\hat{a}_t = z_t - \hat{\phi}z_{t-1} \quad \hat{\phi} = \frac{\sum z_t z_{t-1}}{\sum z_{t-1}^2} = r_1(z)$$

$$r_l(\hat{a}_t) = \frac{\sum \hat{a}_t \hat{a}_{t-l}}{\sum \hat{a}_t^2}$$

$$\text{Now } \hat{a}_t = z_t - \phi z_{t-1} - (\hat{\phi} - \phi)z_{t-1} = a_t - \delta z_{t-1} \quad \delta = (\hat{\phi} - \phi)$$

$$\begin{aligned} \therefore \sum \hat{a}_t^2 &= \sum a_t^2 - 2\delta \sum a_t z_{t-1} + \delta^2 \sum z_{t-1}^2 \\ &= \sum a_t^2 - 2\delta \sum (z_t - \phi z_{t-1})z_{t-1} + \delta^2 \sum z_{t-1}^2 \\ &= \sum a_t^2 - 2\delta \left[\sum z_t z_{t-1} - \phi \sum z_{t-1}^2 \right] + \delta^2 \sum z_{t-1}^2 \\ &= \sum a_t^2 - 2\delta^2 \sum z_{t-1}^2 + \delta^2 \sum z_{t-1}^2 = \sum a_t^2 - \delta^2 \sum z_{t-1}^2 \\ &= \left(\sum a_t^2 \right) \left[1 - \delta^2 \frac{\sum z_{t-1}^2}{\sum a_t^2} \right] \end{aligned}$$

Similarly

$$\begin{aligned} \hat{a}_t \hat{a}_{t-l} &= (a_t - \delta z_{t-1})(a_{t-l} - \delta z_{t-l-1}) \\ &= a_t a_{t-l} - \delta(a_{t-l} z_{t-1} + a_t z_{t-l-1}) + \delta^2 z_{t-1} z_{t-l-1} \end{aligned}$$

$$\sum \hat{a}_t \hat{a}_{t-l} = \sum a_t a_{t-l} - \delta W + \delta^2 \sum z_{t-1} z_{t-l-1}$$

where $W = \sum (a_{t-l}z_{t-1} + a_t z_{t-1-l})$

Thus

$$\begin{aligned} r_l(\hat{a}_t) &= \frac{\sum a_t a_{t-l} - \delta W + \delta^2 \sum z_{t-1} z_{t-1-l}}{\left(\sum a_t^2 \right) \left(1 - \delta^2 \frac{\sum z_{t-1}^2}{\sum a_t^2} \right)} \\ &= \frac{r_l - \delta \frac{W}{\sum a_t^2} + \delta^2 \left(\frac{\sum z_{t-1} z_{t-1-l}}{\sum a_t^2} \right)}{1 - \delta^2 \left(\frac{\sum z_{t-1}^2}{\sum a_t^2} \right)} \end{aligned}$$

Now

$$\frac{\sum z_{t-1}^2}{\sum a_t^2} \xrightarrow{P} \frac{\sigma_a^2 / (1 - \phi^2)}{\sigma_a^2} = \frac{1}{1 - \phi^2}$$

$$\frac{\sum z_{t-1} z_{t-1-l}}{\sum a_t^2} \xrightarrow{P} \frac{\phi^l}{1 - \phi^2}$$

For $\frac{W}{\sum a_t^2}$

$$\frac{\sum a_{t-l} z_{t-1}}{\sum a_t^2} = \frac{\sum (z_{t-l} z_{t-1} - \phi z_{t-l-1} z_{t-1})}{\sum a_t^2} \xrightarrow{P} \frac{\phi^{l-1} - \phi^{l+1}}{1 - \phi^2} = \phi^{l-1}$$

$$\frac{\sum a_t z_{t-1-l}}{\sum a_t^2} = \frac{\sum (z_t - \phi z_{t-1}) z_{t-1-l}}{\sum a_t^2} \xrightarrow{P} \frac{\phi^{l+1} - \phi^{l+1}}{1 - \phi^2} = 0$$

$$\frac{W}{\sum a_t^2} \xrightarrow{P} \phi^{l-1}$$

Now, what about δ

$$\delta = \frac{\sum z_t z_{t-1}}{\sum z_{t-1}^2} - \phi = \frac{\sum (z_t - \phi z_{t-1}) z_{t-1}}{\sum z_{t-1}^2} = \frac{\sum a_t z_{t-1}}{\sum z_{t-1}^2}$$

$$(i) \quad \delta = r_1(z_t) - \phi \xrightarrow{P} \rho_1 - \phi = 0$$

$$(ii) \quad \frac{\sum a_t z_{t-1}}{\sum z_{t-1}^2} = \left(\frac{\sum a_t^2}{\sum z_{t-1}^2} \right) \frac{\sum a_t (a_{t-1} + \phi a_{t-2} + \phi^2 a_{t-3} + \dots)}{\sum a_t^2}$$

$$= X(r_1 + \phi r_2 + \phi^2 r_3 + \dots) \quad X \xrightarrow{P} (1 - \phi^2)$$

$$\therefore \sqrt{n}r_l(\hat{a}_t) = \frac{\sqrt{n}r_l - WX\sqrt{n}(r_1 + \phi r_2 + \phi^2 r_3 + \dots) + (\sqrt{n}\delta) \delta \left(\frac{\sum z_{t-1} z_{t-1-l}}{\sum a_t^2} \right)}{1 - \delta^2 \left(\frac{\sum z_{t-1}^2}{\sum a_t^2} \right)}$$

$$\therefore \sqrt{n}r_l(\hat{a}_t) \doteq \sqrt{n}r_l - \phi^{l-1}(1 - \phi^2) (\sqrt{n}r_1 + \phi \sqrt{n}r_2 + \phi^2 \sqrt{n}r_3 + \dots)$$

$$\text{Remember now that } \text{cov}(\sqrt{n} \tilde{r}) = I \quad \tilde{x} = \sqrt{n} \tilde{r}$$

$$\sqrt{n}r_l(\hat{a}_t) \doteq x_l - \phi^{l-1}(1 - \phi^2) (x_1 + \phi x_2 + \phi^2 x_3 + \dots + \phi^{l-1} x_l + \phi^l x_{l+1} + \dots)$$

$$\text{var}(\sqrt{n}r_l(\hat{a}_t)) = 1 + \phi^{2(l-1)}(1 - \phi^2)^2 \frac{1}{1 - \phi^2} - 2(1 - \phi^2) \phi^{2(l-1)}$$

$$= 1 - \phi^{2(l-1)}(1 - \phi^2)$$

$$l = 1 \quad \text{var}(\sqrt{n}r_1(\hat{a}_t)) = 1 - (1 - \phi^2) = \phi^2$$

$$l = 2 \quad \text{var}(\sqrt{n}r_2(\hat{a}_t)) = 1 - \phi^2(1 - \phi^2) = 1 - \phi^2 + \phi^4$$

$$\text{basic } y_l = \sqrt{nr_l}(\hat{a}_l) = x_l - \phi^{l-1}(1 - \phi^2)(x_1 + \phi x_2 + \phi^2 x_3 + \dots)$$

$$\tilde{x} \sim N(0, \mathbf{I})$$

$$\therefore \text{var}(y_l) = 1 + \phi^{2(l-1)}(1 - \phi^2) - 2(1 - \phi^2)\phi^{2(l-1)}$$

$$= 1 - (1 - \phi^2)\phi^{2(l-1)}$$

$$\text{Consider } y_1, \dots, y_m \quad m \geq 2$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_m \end{bmatrix} - (1 - \phi^2) \begin{bmatrix} 1 \\ \phi \\ \vdots \\ \phi^{m-1} \end{bmatrix} \begin{bmatrix} 1 & \phi & \dots & \phi^{m-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_m \end{bmatrix} + (1 - \phi^2) \begin{bmatrix} 1 \\ \phi \\ \vdots \\ \phi^m \end{bmatrix} \phi^m (x_{m+1} + \phi x_{m+2} + \dots)$$

$$\text{Let } \beta = (1 - \phi^2)^{\frac{1}{2}} \begin{bmatrix} 1 \\ \phi \\ \vdots \\ \phi^{m-1} \end{bmatrix} \quad u = (1 - \phi^2)^{\frac{1}{2}} (x_{m+1} + \phi x_{m+2} + \dots)$$

$$\tilde{y} = (I - \beta\beta')\tilde{x} - \beta\phi^m u \quad \tilde{x}, u \text{ indep. iid } N(0, \sigma^2)$$

$$= [(I - \beta\beta') - \phi^m \beta] \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}$$

$$\tilde{y}' \tilde{y} = \begin{bmatrix} \tilde{x}' & u \end{bmatrix} \begin{bmatrix} \mathbf{I} - \beta\beta' \\ -\phi^m \beta' \end{bmatrix} \begin{bmatrix} \mathbf{I} - \beta\beta' & -\phi^m \beta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \end{bmatrix}$$

$$\text{tr} \begin{bmatrix} & \end{bmatrix} = \text{tr}(\mathbf{I} - \beta\beta')(\mathbf{I} - \beta\beta') + \phi^{2m} \beta\beta'$$

$$= m + 2\beta'\beta + (\beta'\beta)^2 + \phi^{2m}\beta'\beta$$

$$\text{now } \beta'\beta = 1 - \phi^{2m}$$

$$= m - 2(1 - \phi^{2m}) + (1 - \phi^{2m})^2 + \phi^{2m}(1 - \phi^{2m})$$

$$= m - 2 + 2\phi^{2m} + 1 - 2\phi^{2m} + \phi^{4m} + \phi^{2m} - \phi^{4m}$$

$$= m - 1 + \phi^{2m}$$

$$= m - 1 \text{ if } m \text{ is moderately large and } \phi \text{ not close to } 1.$$

$$\text{Also, we can just ignore } \beta\phi^m u \quad \therefore y = (T - \beta\beta')x$$

$$\text{and } (I - \beta\beta')(I - \beta\beta') = I - \beta\beta' \text{ of rank } m - 1$$

$$n \sum_{l=1}^m r_l^2(\hat{a}_l)^2 = y' y \sim x_{m-1}^2$$

More generally

$$n \sum_{l=1}^m r_l^2(\hat{a}_l) \sim x_{m-(p+q)}^2 \quad \text{Box-Pierce(1970)}$$

later modification Ljung/Box 1978 Biom. Davies Triggy and Newbold Biom(197?)

$$n(n+2) \sum_{l=1}^m \frac{1}{(n-l)} r_l^2(\hat{a}_l) \sim x_v^2, \quad v = m - (p+q)$$

Note:

(i) when study residual ACF

variance of lower lags can be very much smaller than $\frac{1}{n}$

(ii) The above development depend on assumption and stationarity (ψ, ϕ not close to 1)

Interesting research problem when some of the zeros of $\phi(B)$ on the unit circle (or even just close).

Nonstationary situation $\phi = 1$

$$\hat{\phi} - \phi = \frac{\sum z_{t-1} a_t}{\sum z_{t-1}^2}$$

no longer true that $\frac{1}{n} \sum z_{t-1}^2 \xrightarrow{P} c$

or that $\frac{1}{n} \sum z_{t-1} a_t \xrightarrow{P} 0$

It can however be proved that $\hat{\phi} \rightarrow \phi$ consistent section 8.5 Fuller or Dickey-Fuller paper.

Unit root estimate's consistency properties and asymptotic distⁿ

$$\text{AR}(1) \quad \sqrt{n}(\hat{\phi} - \phi) = \frac{\frac{1}{\sqrt{n}} \sum a_t z_{t-1}}{\frac{1}{n} \sum z_{t-1}^2} \xrightarrow{L} N(0, 1 - \phi^2)$$

$$2 \cdot \text{Key facts (1)} \quad \frac{1}{n} \sum z_{t-1}^2 \xrightarrow{P} \frac{\sigma_a^2}{1 - \phi^2}$$

$$(2) \quad \frac{1}{\sqrt{n}} \sum a_t z_{t-1} \rightarrow N\left(0, \frac{\sigma_a^2}{1 - \phi^2}\right) \quad \sum a_t z_{t-1} = \sum a_t (a_{t-1} + \phi a_{t-2} + \phi^2 a_{t-3} + \dots)$$

From now on assume $\sigma_a^2 = 1$

When $\phi = 1$ the above are no longer valid.

To see this in an elementary way (Ahtola/Tiao, Biom 84; #6174 p.86-). For (2),

Consider the derivative of log likelihood (score function)

Assume series starts at $t = 1$

$$z_1 = a_1 \\ z_t = \phi z_{t-1} + a_t \quad t = 2, \dots, n \quad a_t = z_t - \phi z_{t-1}$$

$$l(\phi|z) \propto e^{-\frac{1}{2} \left[a_1^2 + \sum_{t=2}^n a_t^2 \right]}$$

$$\ln l(\phi|z) = -\frac{1}{2} \left[a_1^2 + \sum_{t=2}^n a_t^2 \right]$$

$$\text{score} = \frac{\partial \ln l}{\partial \phi} = \sum_{t=2}^n a_t z_{t-1}$$

Now $z_2 = a_2 + \phi a_1$

$$z_3 = a_3 + \phi a_2 + \phi^2 a_1$$

$\vdots$

$$z_n = a_n + \phi a_{n-1} + \dots + \phi^{n-1} a_1$$

We now write $\sum_{t=2}^n a_t z_{t-1}$ in the form $\underset{\sim}{a}' \underset{\sim}{H} \underset{\sim}{a}$

$$\begin{bmatrix} 0 \\ z_1 \\ z_2 \\ \vdots \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ 1 & \ddots & & & & \vdots \\ \phi & 1 & & & & \vdots \\ \phi^2 & \phi & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \phi^{n-2} & \dots & \dots & \phi^2 & \phi & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{bmatrix} = \underset{\sim}{H} \underset{\sim}{a}$$

$$\therefore \sum_{t=2}^n a_t z_{t-1} = \underset{\sim}{a}' \underset{\sim}{H} \underset{\sim}{a} = \underset{\sim}{a}' \underset{\sim}{A} \underset{\sim}{a}$$

$$\underset{\sim}{A} = \frac{1}{2} \begin{bmatrix} 0 & 1 & \phi & \dots & \phi^{n-2} \\ 1 & \ddots & & \ddots & \vdots \\ \phi & \ddots & & & \phi \\ \vdots & & \ddots & & 1 \\ \phi^{n-2} & & \phi & 1 & 0 \end{bmatrix} \quad \text{symmetric}$$

$$\exists P \quad \text{orthogonal} \ni P'AP = \Lambda \quad \text{diagonal} \quad PP' = P'P \\ A = P\Lambda P'$$

$$\therefore \sum_{t=2}^n a_t z_{t-1} = \underset{\sim}{a}' P \Lambda P' \underset{\sim}{a} \\ = \underset{\sim}{x}' \Lambda \underset{\sim}{x} = \sum_{j=1}^n x_j^2 \Lambda_j$$

$$\text{Now } \underset{\sim}{a} \sim N(0, \mathbf{I}) \quad \underset{\sim}{x} = P' \underset{\sim}{a} \quad P \text{ orthogonal}$$

$$\Rightarrow \underset{\sim}{x} \sim N(0, \mathbf{I}) \quad x_j^2 \sim \chi_1^2 \quad j=1, \dots, n \quad \text{independent}$$

Thus $\sum_{t=2}^n a_t z_{t-1}$ = linear combination of n independent χ_1^2 variable

Key: value of Λ_j 's

Suppose $\phi = 1$

$$|\mathbf{I}_n - A_{n \times k} B_{k \times n}| = |\mathbf{I}_k - BA|$$

$$A = \frac{1}{2}[\mathbf{1}\mathbf{1}' - \mathbf{I}]$$

$$A - \lambda \mathbf{I} = \frac{1}{2}[\mathbf{1}\mathbf{1}' - \mathbf{I}] - \lambda \mathbf{I} = \frac{1}{2}[\mathbf{1}\mathbf{1}' - (1 + 2\lambda)\mathbf{I}]$$

$$= \frac{1}{2}(\mathbf{I} - (1 + 2\lambda)^{-1} \mathbf{1}\mathbf{1}')(-1)(1 + 2\lambda)$$

$$|A - \lambda \mathbf{I}| \propto (1 + 2\lambda)^n |\mathbf{I} - (1 + 2\lambda)^{-1} \mathbf{1}\mathbf{1}'| = (1 + 2\lambda)^{n-1} (1 + 2\lambda - n)$$

$$\therefore \text{eigenvalues are } \lambda = \frac{n-1}{2}, \underbrace{-\frac{1}{2}, \dots, -\frac{1}{2}}_{(n-1) \text{ of them.}}$$

$$\therefore \underset{\sim}{a}' \underset{\sim}{A} \underset{\sim}{a} = -\frac{1}{2} \sum_{j=1}^{n-1} x_j^2 + \frac{n-1}{2} x_n^2$$

$$= -\frac{1}{2} x_{n-1}^2 + \frac{n-1}{2} x_1^2 \quad x_{n-1}^2 \text{ and } x_1^2 \text{ are independent}$$

$$\frac{1}{\sqrt{n}} \sum_{t=2}^n a_t z_{t-1} = -\frac{1}{2\sqrt{n}} (x_{n-1}^2 - (n-1)) + \frac{(n-1)}{2\sqrt{n}} (x_1^2 - 1)$$

$$\doteq -\frac{1}{2} - \frac{x_{n-1}^2 - (n-1)}{\sqrt{n}} + \frac{\sqrt{n}}{2} (x_1^2 - 1)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ N(0,1) & \text{blow up} & \end{array}$$

$\therefore$ CLT does not work here.

The key problem here is that the λ_j 's are not "even" in its distⁿ.

See p.88 Table 1.

On the other hand, if we renormalize it by $\frac{1}{n}$

$$\frac{1}{n} \sum_{t=2}^n a_t z_{t-1} \doteq -\frac{1}{2\sqrt{n}} \left(\frac{x_{n-1}^2 - (n-1)}{\sqrt{n}} \right) + \frac{n-1}{2n} (x_1^2 - 1) \rightarrow (x_1^2 - 1) + O_p \left(n^{-\frac{1}{2}} \right)$$

limiting distⁿ is not normal !

Dickey-Fuller 1979 JASA paper examine dist of

$$n(\hat{\phi} - 1) \xrightarrow{L} \text{random walk (1958, White)}$$

We here discuss the related convergence properties of $\hat{\phi}$

Order $a_n = o(g_n) \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{g_n} = 0$

in magnitude

$$a_n = O(g_n) \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{g_n} = O(1)$$

$$\exists M \text{ such that } \left| \frac{a_n}{g_n} \right| < M \text{ for all } n.$$

in probability

$$\text{plim} X_n = X \Rightarrow \lim_{n \rightarrow \infty} \Pr\{|X_n - X| > \varepsilon\} \rightarrow 0$$

$$X_n = o_p(g_n) \Rightarrow \text{plim}_{n \rightarrow \infty} \frac{X_n}{g_n} \rightarrow 0$$

$$X_n = O_p(g_n) \Rightarrow \text{for } \varepsilon > 0 \exists M_\varepsilon$$

$$\Pr\{|X_n| > g_n M_\varepsilon\} \leq \varepsilon \text{ bounded in prob.}$$

Tchebychef inequality for $\varepsilon > 0$

$$\Pr\{|X_n| > \varepsilon\} \leq \frac{E(X_n)^2}{\varepsilon^2}$$

with $EX_n = 0$

$$\Pr\{|X_n| > \varepsilon\} \leq \frac{\sigma_n^2}{\varepsilon^2} \text{ or put } \varepsilon = \sigma_n \varepsilon_*$$

$$\Pr\{|X_n| > \sigma_n \varepsilon_*\} \leq \frac{1}{\varepsilon_*^2}$$

Stationary case $|\phi| < 1$

$$\hat{\phi} - \phi = \frac{\frac{1}{\sqrt{n}} \frac{1}{\sqrt{n}} \sum a_t z_{t-1}}{\frac{1}{n} \sum z_{t-1}^2}$$

$$\text{numerator} \quad \frac{1}{\sqrt{n}} \sum a_t z_{t-1} \xrightarrow{L} N\left(0, \frac{1}{1-\phi^2}\right)$$

$$\therefore \frac{1}{n} \sum a_t z_{t-1} = O_p\left(n^{-\frac{1}{2}}\right) \quad \text{convergence rate is } n^{-\frac{1}{2}}$$

denominator

$$\sum z_{t-1}^2 \doteq \sum z_t^2 \quad E\left(\sum z_t^2\right) = \frac{n}{1-\phi^2} = O(n)$$

$$\text{var}\left(\sum z_{t-1}^2\right) = O(n) \quad \text{var}\left(\frac{1}{n} \sum z_{t-1}^2\right) = O(n^{-1})$$

by Tchebychef inequality

$$\frac{1}{n} \sum z_t^2 \xrightarrow{P} \frac{1}{1-\phi^2} + O_p\left(n^{-\frac{1}{2}}\right)$$

$$\therefore [\hat{\phi} - \phi] = O_p\left(n^{-\frac{1}{2}}\right)$$

what happens when $\phi = 1$?

$$\frac{1}{n} \sum a_t z_{t-1} \rightarrow \chi_1^2 - 1 + O_p\left(n^{-\frac{1}{2}}\right)$$

Question is

$$\frac{1}{n} \sum_{t=1}^n z_{t-1}^2 \doteq \frac{1}{n} \sum_{t=1}^n z_t^2$$

$$\sum z_t^2 = \underset{\sim}{a}' \underset{\sim}{\Psi} \underset{\sim}{\Psi}' \underset{\sim}{a}$$

$$z_1 = a_t + a_{t-1} + \dots + a_1 \quad \underset{\sim}{\Psi}' \underset{\sim}{a} = \begin{bmatrix} 1 & & & & \\ 1 & \ddots & & & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 1 & & & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ \vdots \\ \vdots \\ a_n \end{bmatrix}$$

$$\begin{aligned}
E \sum_{t=1}^n z_t^2 &= \text{tr} \psi \psi' \\
&= \text{tr} \begin{bmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ n-1 & n-1 & \cdots & \cdots & 2 & 1 \\ n-2 & n-2 & \cdots & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 & \cdots & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix} = \frac{(n+1)n}{2} = O(n^2)
\end{aligned}$$

$$\begin{aligned}
\text{var} \sum z_t^2 &= c_0 \times \text{tr}(\psi \psi')^2 \\
&= c_0 \times \text{sum of squares of elements of } (\psi \psi') \\
&= c_0 \times \theta \left[1 + 2^2 + \cdots + n^2 \right] + 2 \left[1 + (1 + 2^2) + (1 + 2^2 + 3^2) + \cdots + (1 + 2^2 + \cdots + (n-1)^2) \right] \\
&= O(n^4) \quad \text{very different from stat. case.}
\end{aligned}$$

$$\text{Apply Tchebycheff} \quad \sum z_t^2 = O_p(n^2)$$

It can be shown that $\left(\sum z_t^2 \right)^{-1} = O_p(n^{-2})$ see Tiao/Tsay 1983 (Ann stat. 856-71)

$$\therefore \sum a_t z_{t-1} = O_p(n)$$

$$\therefore \frac{\sum a_t z_{t-1}}{\sum z_{t-1}^2} = O_p(n^{-2}) O_p(n) = O_p(n^{-1})$$

Convergence rate is n^{-1} ! Faster than stationary case.

Kalman Filter Obs eqn $y_t = f_t \theta_t + v_t$
 State eqn $\theta_t = g_t \theta_{t-1} + w_t$ (f_t, g_t) known constant
 $\{v_o\}\{w_t\}\theta_0$ iid (y_t, θ_t) random variable

Aim (i) Find dist^n of $y_t | y_{t-1}, \dots, y_1$ likelihood function

(ii) Find dist^n of $\theta_t | y_t, \dots, y_1$ current level est.

(1) From $p(\theta_0, v_1, w_1) = p(\theta_0)p(y_1|\theta_1)p(\theta_1|\theta_0)$

Then $p(y_1, \theta_1) = p(y_1|\theta_1)p(\theta_1)$ where $p(\theta_1) = \int p(\theta_1|\theta_0)p(\theta_0)\phi\theta_0$

$$= p(y_1)p(\theta_1|y_1)$$

$$p(y_1) = \int p(y_1|\theta_1)p(\theta_1)d\theta_1$$

$$p(\theta_1|y_1) = p(y_1|\theta_1)p(\theta_1)/p(y_1)$$

(2) From $p(\theta_0, v_1, w_1, v_2, w_2) = p(\theta_0)p(y_1|\theta_1)p(\theta_1|\theta_0)p(y_2|\theta_2)p(\theta_2|\theta_1)$

$$\rightarrow p(\theta_1|y_1)p(y_2|\theta_2)p(\theta_2|\theta_1)$$

Then $p(y_2, \theta_2|y_1) = p(y_2|\theta_2)p(\theta_2|y_1)$ where $p(\theta_2|y_1) = \int p(\theta_2|\theta_1)p(\theta_1|y_1)dE$

$$= p(y_2|y_1)p(\theta_2|y_2, y_1)$$

$$p(y_2|y_1) = \int p(y_2|\theta_2)p(\theta_2|y_1)d\theta_2$$

$$p(\theta_2|y_2, y_1) = p(y_2|\theta_2)p(\theta_2|y_1)/p(y_2|y_1)$$

(3) In general, from $p(\theta_0, v_1, w_1, v_2, w_2, \dots, v_{t-1}, w_{t-1}, v_t, w_t)$

$$= p(\theta_0)p(y_1|\theta_1)p(\theta_1|\theta_0) \dots p(y_{t-1}|\theta_{t-1})p(\theta_{t-1}|\theta_{t-2})p(y_t|\theta_t)p(\theta_t|\theta_{t-1})$$

$$\rightarrow p(\theta_{t-1}|y_{t-1}, \dots, y_1)p(y_t|\theta_t)p(\theta_t|\theta_{t-1})$$

Then $p(y_t, \theta_t | y_{t-1}, \dots, y_1) = p(y_t|\theta_t)p(\theta_t | y_{t-1}, \dots, y_1)$

where $p(\theta_t | y_{t-1}, \dots, y_1) = \int p(\theta_t|\theta_{t-1})p(\theta_{t-1}|y_{t-1}, \dots, y_1)d\theta_{t-1}$

$$p(y_t, \theta_t | y_{t-1}, \dots, y_1) = p(y_t | y_{t-1}, \dots, y_1)$$

$$p(y_t | y_{t-1}, \dots, y_1) = \int p(y_t | \theta_t) p(\theta_t | y_{t-1}, \dots, y_1) d\theta_t$$

$$p(\theta_t | y_t, \dots, y_1) = p(y_t | \theta_t) p(\theta_t | y_{t-1}, \dots, y_1) / p(y_t | y_{t-1}, \dots, y_1)$$

Key: at a given time t , we have a trivariate distⁿ

$$y_t | \theta_t, \theta_{t-1} | \theta_{t-1}, \theta_{t-1} | y_{t-1}, \dots, y_1$$

the above recursion works for all distⁿ an integration problem computationally.
Normality assumption: Suppose $\theta_0, v_t, w_t; t = 1, 2, \dots$ all normal
Then, every step, everything is normal.

In particular at time t given $y_1, y_2, \dots, y_{t-1}$, $y_t | \theta_t, \theta_{t-1} | y_{t-1}, \dots, y_1$ will be normal so

that $(y_t, \theta_t, \theta_{t-1} | y_{t-1}, \dots, y_1)$ will be trivariate normal. Using properties of normal distⁿ,

one get the following recursion formula for the mean t variance of the distⁿ of

$$y_t | y_{t-1}, \dots, y_1 \text{ or } \theta_t | y_t, \dots, y_1$$

Recursion (all from properties of multivariate normal)

$$\text{Let } \hat{\theta}_t = E(\theta_t | y_t, y_{t-1}, \dots, y_1)$$

$$\sigma_t^2 = \text{var}(\theta_t | y_t, \dots, y_1)$$

$$\hat{y}_t = E(y_t | y_{t-1}, \dots, y_1)$$

$$\Omega_t = \text{var}(y_t | y_{t-1}, \dots, y_1)$$

then

$$\hat{\theta}_t = g_t \hat{\theta}_t + f_t R_t \Omega_t^{-1} (y_t - \hat{y}_t)$$

$$\sigma_t^2 = R_t - f_t^2 R_t^2 \Omega_t^{-1}$$

$$\hat{y}_t = f_t g_t \hat{\theta}_{t-1}$$

$$\Omega_t = V_t + f_t^2 R_t \quad V_t = \text{var}(v_t)$$

$$R_t = W_t + g_t^2 \sigma_{t-1}^2 \quad W_t = \text{var}(w_t)$$

Multivariate Extension

$$\begin{matrix} y_t & = & F_t & \theta_t & + & v_t \\ \tilde{K} \times 1 & & K \times r & r \times 1 & & K \times 1 \end{matrix} \quad v_t \sim N \left(\begin{matrix} 0 \\ \tilde{V}_t \end{matrix} \right)$$

$$\begin{matrix} \theta_t & = & G_t & \theta_t & + & w_t \\ \tilde{K} \times 1 & & r \times r & r \times 1 & & K \times 1 \end{matrix} \quad w_t \sim N \left(\begin{matrix} 0 \\ \tilde{W}_t \end{matrix} \right)$$

Then

$$\hat{\theta}_t = G_t \hat{\theta}_{t-1} + R_t F_t' \Omega_t^{-1} (y_t - \hat{y}_t)$$

$$\Sigma_t = R_t - R_t F_t' \Omega_t^{-1} F_t R_t$$

$$\hat{y}_t = F_t G_t \hat{\theta}_{t-1}$$

$$\Omega_t = V_t + F_t R_t F_t'$$

$$R_t = W_t + G_t \Sigma_{t-1} G_t'$$

connections with ARMA model

$$\text{AR}(1) \quad y_t = \phi y_{t-1} + a_t$$

$$\begin{cases} y_t = \theta_t \\ \theta_t = \phi \theta_{t-1} + a_t \end{cases}$$

$$\text{AR}(2) \quad y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t$$

$$\begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ a_t \end{bmatrix}$$

thus let

$$\underset{\sim}{\theta}_t = \begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix}$$

$$\begin{cases} y_t = [0 \quad 1] \underset{\sim}{\theta}_t \\ \underset{\sim}{\theta}_t = \underset{\sim}{\Phi} \underset{\sim}{\theta}_{t-1} + \underset{\sim}{\omega}_t \end{cases}$$

$$\underset{\sim}{\Phi} = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix} \quad \underset{\sim}{\omega}_t = \begin{bmatrix} 0 \\ a_t \end{bmatrix}$$

$$\text{MA}(1) \quad y_t = a_t - \alpha a_{t-1}$$

$$\begin{bmatrix} a_{t+1} \\ a_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -\alpha & 0 \end{bmatrix} \begin{bmatrix} a_t \\ a_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} a_{t+1} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} y_t = (0,0,1) \underset{\sim}{\theta}_t \\ \underset{\sim}{\theta}_t = \underset{\sim}{G} \underset{\sim}{\theta}_{t-1} + \underset{\sim}{\omega}_t \end{cases}$$

$$\text{ARMA}(1,1) \quad y_t = \phi y_{t-1} + a_t - \alpha a_{t-1}$$

$$y_t = (0,0,1) \underset{\sim}{\theta}_t$$

$$\underset{\sim}{\theta}_t = \underset{\sim}{G} \underset{\sim}{\theta}_{t-1} + \underset{\sim}{\omega}_t$$

but

$$\underset{\sim}{G} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -\alpha & 0 \end{bmatrix}$$

ARCH model (Variance Changes)

Simplest case $a_t \sim iidN(0, \sigma_a^2)$ σ_a^2 constant over time.

Variance Changes $\sigma_a^2 \rightarrow \sigma_t^2$ depend on t.

Fixed changes intervention.

Stochastic changes

$$\sigma_t^2 = \sigma_{t-1}^2 \quad \text{with prob. } \lambda$$

$$\sigma_t^2 = \beta \sigma_{t-1}^2 \quad \text{with prob. } 1 - \lambda$$

β can be r.v. following some distⁿ.

e.g. x^2 etc.

Another type proposed by Engle (1982, #9539) is to let $e_t = \sigma_t a_t \rightarrow N(0,1)$

And relate σ_t^2 to previous e_t^2 's

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 \quad (A)$$

more generally $\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \dots + \alpha_r e_{t-r}^2$

For the simple case (A)

$$p(e_t | e_{t-1}, \dots) = p(e_t | e_{t-1}) \alpha \sigma_t e^{-\frac{1}{2\sigma_t^2} e_t^2}$$

Here $E(e_t) = 0$ $E(e_t^2 | e_{t-1}) = \sigma_t^2$

and $E(e_t^2) = E(\sigma_t^2) = \alpha_0 + \alpha_1 E(e_{t-1}^2)$

$$\therefore E(e_t^2) = \frac{\alpha_0}{1 - \alpha_1} \quad \text{provide } \alpha_0 > 0, 0 < \alpha_1 < 1$$

Also

$$E(e_t^4) = 3 \left(\frac{\alpha_0}{1 - \alpha_1} \right)^2 \frac{1 - \alpha_1^2}{1 - 3\alpha_1^2}$$

$$\text{Kurtosis} = \frac{E(e_t^4)}{[E(e_t^2)]^2} - 3$$

$$= 3 \frac{1 - \alpha_1^2}{1 - 3\alpha_1^2} > 3 \quad (\text{provide } 1 - 3\alpha_1^2 > 0)$$

$\therefore$ It is heavy tailed.

Likelihood function

$$\text{Set } P(e_1, \dots, e_n) = p(e_1)p(e_2|e_1) \cdots p(e_n|e_{n-1})$$

Function of (α_0, α_1) easy to

GARCH generalization

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

ARCH is MA GARCH is ARMA

ACF of e_t^2 for ARCH

$$\text{Since } e_t = \sigma_t a_t \quad e_t^2 = \sigma_t^2 a_t^2 = (\alpha_0 + \alpha_1 e_{t-1}^2) a_t^2$$

$$\therefore E(e_t^2) = \alpha_0 + \alpha_1 E(e_{t-1}^2) = \frac{\alpha_0}{1 - \alpha_1}$$

$$\text{Var}(e_t^2) = E(e_t^4) - [E(e_t^2)]^2 = 3 \left(\frac{\alpha_0}{1 - \alpha_1} \right)^2 \frac{1 - \alpha_1^2}{1 - 3\alpha_1^2} - \left(\frac{\alpha_0}{1 - \alpha_1} \right)^2$$

$$= \left(\frac{\alpha_0}{1 - \alpha_1} \right)^2 \left[\frac{3 - 3\alpha_1^2}{1 - 3\alpha_1^2} - 1 \right] = \left(\frac{\alpha_0}{1 - \alpha_1} \right)^2 \frac{2}{1 - 3\alpha_1^2}$$

$$E(e_t^2 e_{t-1}^2) = E e_{t-1}^2 (\alpha_0 + \alpha_1 e_{t-1}^2) \alpha_t^2$$

$$= \alpha_0 E(e_{t-1}^2) + \alpha_1 E(e_{t-1}^4) = \frac{\alpha_0^2}{1-\alpha_1} + \alpha_1 3 \left(\frac{\alpha_0}{1-\alpha_1} \right)^2 \frac{1-\alpha_1^2}{1-3\alpha_1^2}$$

$$\text{cov}(e_t^2 e_{t-1}^2) = \frac{\alpha_0^2}{1-\alpha_1} + \alpha_1 \left(\frac{\alpha_0}{1-\alpha_1} \right)^2 \frac{1-\alpha_1^2}{1-3\alpha_1^2} - \left(\frac{\alpha_0}{1-\alpha_1} \right)^2$$

$$E(e_t^2 e_{t-2}^2) = E e_{t-2}^2 (\alpha_0 + \alpha_1 e_{t-1}^2) = \frac{\alpha_0^2}{1-\alpha_1} + \alpha_1 E(e_{t-1}^2 e_{t-2}^2)$$

recursion so we can see the ACF will gradually declining to zero like AR or ARMA models.

Kalman filter

Obs. eqn $y_t = f_t \theta_t + v_t$

State space model

State $\theta_t = g_t \theta_{t-1} + w_t$

Unobserved component model

 $\{f_t, g_t\}$ constants $\{v_t\} \{w_t\}$ θ_0 iidAim: Find (i) distn of $y_t | y_{t-1}, \dots, y_1$ likelihood function(ii) distn of $\theta_t | y_t, \dots, y_1$ current level est. $t = 2, 3, \dots$ unobserved component est.for $t = 1$ $p(y_1)$ or $p(\theta_1 | y_1)$

$$P(\theta_0, w_1, v_1) = p(\theta_0) p(\theta_1 | \theta_0) p(y_1 | \theta_1) \rightarrow p(\theta_1) p(y_1 | \theta_1) = p(y_1) p(\theta_1 | y_1)$$

$$P(\theta_0, w_1, v_1; w_2, v_2) = p(\theta_0, w_1, v_1) p(\theta_2 | \theta_1) p(y_2 | \theta_2)$$

$$\rightarrow p(\theta_1 | y_1) p(\theta_2 | \theta_1) p(y_2 | \theta_2) \rightarrow p(\theta_2 | y_1) p(y_2 | y_1)$$

$$= p(y_2 | y_1) p(\theta_2 | y_2, y_1)$$

Thus, at a given time t, we have a trivariate distribution

$$p(\theta_{t-1} | y_1, \dots, y_{t-1}) p(\theta_t | \theta_{t-1}) p(y_t | \theta_t)$$

$$\rightarrow p(\theta_t | y_1, \dots, y_{t-1}) p(y_t | \theta_t) = p(y_t | y_{t-1}, \dots, y_1) p(\theta_t | y_t, y_{t-1}, \dots, y_1)$$

using properties of normal distⁿ, one gets the recursive as follows.

(if so assumed)

$$\hat{\theta}_{t-1} = E(\theta_{t-1} | y_{t-1}, \dots, y_1) = E(\theta_{t-1} | y_{t-1})$$

$$\sigma_{t-1}^2 = \text{var}(\theta_{t-1} | y_{t-1}, \dots, y_1) = \text{var}(\theta_{t-1} | y_{t-1})$$

$$p(\theta_{t-1} | y_{t-1}) \propto \sigma_{t-1} e^{-\frac{1}{2} \frac{(\theta_{t-1} - \hat{\theta}_{t-1})^2}{\sigma_{t-1}^2}}$$

$$p(\theta_t | \theta_{t-1}) \propto \sigma_{w_t} e^{-\frac{1}{2} \frac{(\theta_t - g_t \theta_{t-1})^2}{\sigma_{w_t}^2}} \quad \sigma_{w_t}^2 = w_t$$

$$p(y_t | \theta_t) \propto \sigma_{v_t} e^{-\frac{1}{2} \frac{(y_t - f_t \theta_t)^2}{\sigma_{v_t}^2}} \quad \sigma_{v_t}^2 = v_t$$

$$\text{Thus, } E(\theta_t | y_{t-1}) = g_t \hat{\theta}_{t-1} \quad E(\theta_t | y_{t-1}) = E\{E(\theta_t | \theta_{t-1})\} | y_{t-1}$$

$$\text{var}(\theta_t | y_{t-1}) = \sigma_{w_t}^2 + g_t^2 \sigma_{t-1}^2 = R_t \quad \theta_t - g_t \hat{\theta}_{t-1} = \theta_t - g_t \theta_{t-1} + g_t (\theta_{t-1} - \hat{\theta}_{t-1})$$

$$E(y_t | y_{t-1}) = g_t f_t \hat{\theta}_{t-1} = \hat{y}_t$$

$$\text{var}(y_t | y_{t-1}) = \sigma_{v_t}^2 + f_t^2 R_t = \Omega_t$$

$$\text{cov}(\theta_t, y_t | y_{t-1}) = f_t R_t$$

$$= \begin{bmatrix} R_t & f_t R_t \\ f_t R_t & \Omega_t \end{bmatrix} = \text{cov} \left(\begin{bmatrix} \theta_t \\ y_t \end{bmatrix} \middle| y_{t-1} \right)$$

$$\therefore \hat{\theta}_t = g_t \hat{\theta}_{t-1} + f_t R_t \Omega_t^{-1} (y_t - \hat{y}_{t-1})$$

$$\sigma_t^2 = R_t - f_t^2 R_t^2 \Omega_t^{-1}$$

Multivariate extension: see notes.

Connection with ARMA

$$\text{AR}(1) \quad y_t = \phi y_{t-1} + a_t$$

$$\begin{cases} y_t = \theta_t \\ \theta_t = \phi \theta_{t-1} + a_t \end{cases}$$

$$\text{AR}(2) \quad y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t$$

$$\begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \phi_2 & \phi_1 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ a_t \end{bmatrix}$$

$$\underset{\sim}{\theta}_t = \begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix}$$

$$\Rightarrow \begin{cases} \theta_t = \Phi \theta_{t-1} + w_t \\ \underset{\sim}{y}_t = [0, \underset{\sim}{1}] \underset{\sim}{\theta}_t \end{cases}$$

$$\text{MA}(1) \quad y_t = a_t = \alpha a_{t-1}$$

$$\begin{bmatrix} a_{t-1} \\ a_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -\alpha & 0 \end{bmatrix} \begin{bmatrix} a_t \\ a_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} a_{t+1} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} \underset{\sim}{\theta}_t = G \underset{\sim}{\theta}_{t-1} + w_t \\ y_t = [0 \quad 0 \quad 1] \underset{\sim}{\theta}_t \end{cases}$$

$$\text{ARMA}(1,1) \quad y_t - \phi y_{t-1} = a_t - \alpha a_{t-1}$$

$$\begin{bmatrix} a_{t-1} \\ a_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -\alpha & \phi \end{bmatrix} \begin{bmatrix} a_t \\ a_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} a_{t+1} \\ 0 \\ 0 \end{bmatrix} \quad \underset{\sim}{\theta}_t = G \underset{\sim}{\theta}_{t-1} + w_t$$

$$y_t = [0 \quad 0 \quad 1] \underset{\sim}{\theta}_t$$

Model for variance change From ARMA

$$a_t \quad iid \quad N(0, \sigma^2)$$

$$\sigma_a^2 \rightarrow \sigma_t^2$$

ARCH model Autoregression Conditional Heteroskedastic Model

$$(1982 \quad Engle) \quad e_t = y_t - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p}$$

$$\text{let } e_t = \sigma_t a_t \quad a_t \quad iid \quad N(0,1) \quad E(e_t | e_{t-1}) = 0 \quad E(e_t^2 | e_{t-1}) = \sigma_t^2$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2$$

more generally

$$\sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \dots + \alpha_r e_{t-r}^2$$

$$P(e_t | e_{t-1}) \propto \sigma_{t-1}^{-1} e^{-\frac{1}{2\sigma_t^2}}$$

$$\text{Hence} \quad E(e_t) = 0 \quad E(e_t^2) = E(\sigma_t^2) = \alpha_0 + \alpha_1 E(e_{t-1}^2)$$

$$\therefore E(e_t^2) = \frac{\alpha_0}{1 - \alpha_1} \quad \text{provided } \alpha_0 > 0 \quad 0 < \alpha_1 < 1$$

likelihood function

$$l(\alpha_0, \alpha_1 | e) \propto \prod_{t=1}^n \sigma_t^{-1} e^{-\frac{1}{2\sigma_t^2}} \quad \text{easy to derive}$$

$$\text{GARCH} \quad \sigma_t^2 = \alpha_0 + \alpha_1 e_{t-1}^2 + \beta \sigma_{t-1}^2$$

Other variance change model

$$(1) \quad \sigma_t^2 = \beta_t^2 \sigma_{t-1}^2 \quad 1 - \lambda$$

$$= \sigma_{t-1}^2 \quad \lambda$$

$$\beta_t^2 \sim X_v^2 \quad \text{Bernoulli Random level shift in } \sigma_t^2$$

$$(2) \quad g(\sigma_t^2) = \alpha_0 + w_t P_t$$

$$P_t = \begin{cases} 1 & \lambda \\ 0 & 1 - \lambda \end{cases} \quad w_t \sim \text{dist}^n$$

Random isolated spikes $\Rightarrow$ Random intervention

(3) Also generalize to random intervention (Li Hang)

Markov van

$$\sigma_t = \begin{cases} \sigma_{t-1} & s_t = 0 \\ \beta \sigma_{t-1} & s_t = 1 \end{cases}$$

$$\Pr\{s_t = 1 | s_{t-1} = 0\} = \varepsilon_1 \quad \Pr\{s_t = 0 | s_{t-1} = 1\} = \varepsilon_2$$

$$\hat{\phi} - \phi = \frac{\sum z_{t-1} a_t}{\sum z_{t-1}^2}$$

$\because \sqrt{n}(\hat{\phi} - \phi)$ is not normal

$$\sum_{t=1}^n z_{t-1} a_t \quad \text{assume } z_0 = 0$$

$$= \frac{1}{\sqrt{n}} \sum_{t=1}^n z_{t-1} a_t \rightarrow N\left(0, \frac{1}{1-\phi^2}\right) \quad \text{stationary}$$

Now $\sum_{t=1}^n z_{t-1} a_t \quad \text{with } z_0 = 0$

$$z_1 = a_1$$

$$z_2 = a_2 + \phi a_1$$

$$z_3 = a_3 + \phi a_2 + \phi^2 a_1$$

$$\vdots$$

$$z_{n-1} = a_{n-1} + \phi a_{n-2} + \cdots + \phi^{n-2} a_1$$

$$\begin{bmatrix} 0 \\ z_1 \\ \vdots \\ \vdots \\ z_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & 0 \\ \phi & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \phi^{n-2} & \cdots & \phi & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{n-1} \end{bmatrix}$$

$$n \times 1$$

$$n \times n$$

$$n \times 1$$

$$\begin{aligned}
& \sum_{t=1}^n a_t z_{t-1} \\
&= (a_1 \cdots a_n) \begin{bmatrix} 0 & 1 & \phi & \cdots & \phi^{n-2} \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \phi & \ddots & \ddots & \ddots & \phi \\ \vdots & \ddots & \ddots & \ddots & 1 \\ \phi^{n-2} & \cdots & \cdots & 1 & 0 \end{bmatrix} a_{\sim} \\
&= \frac{1}{2} a'_{\sim} C a_{\sim}
\end{aligned}$$

$$\begin{aligned}
C &= \begin{bmatrix} 0 & 1 & \phi & \cdots & \phi^{n-2} \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \phi & \ddots & \ddots & \ddots & \phi \\ \vdots & \ddots & \ddots & \ddots & 1 \\ \phi^{n-2} & \cdots & \phi & 1 & 0 \end{bmatrix} \\
\frac{1}{2} a'_{\sim} C a_{\sim} &= \frac{1}{2} a'_{\sim} P \Lambda P' a_{\sim} \\
&\quad P \text{ orthogonal matrix} \\
&\quad \Lambda \text{ diagonal-eigenvalues of } C
\end{aligned}$$

Special case

$$\phi = 1$$

$$C = \mathbf{1}\mathbf{1}' - \mathbf{I}$$

$$\begin{aligned}
C - \lambda \mathbf{I} &= \left| \mathbf{1}\mathbf{1}' - (1 + \lambda) \mathbf{I} \right| \alpha (1 + \lambda)^n \left| \mathbf{I} - \frac{1}{1 + \lambda} \mathbf{1}\mathbf{1}' \right| \\
&= (1 + \lambda)^n \left(1 - \frac{n}{1 + \lambda} \right) = (1 + \lambda)^{n-1} (\lambda + 1 - n) \\
&\quad \lambda = -1 \quad (n-1) \text{ multiplicative} \\
&\quad \lambda = n - 1
\end{aligned}$$

$$\frac{1}{2} a'_{\sim} C a_{\sim} = -\frac{1}{2} \sum_{j=1}^{n-1} x_j^2 + \frac{n-1}{2} x_n^2$$

$$x'_s \text{ iid } N(0, \sigma^2)$$

Thus

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{t=1}^n z_{t-1} a_t &= -\frac{1}{2\sqrt{n}} \sum_{j=1}^{n-1} x_j^2 + \frac{n-1}{2\sqrt{n}} x_n^2 \quad \text{assume } \sigma_n^2 = 1 \\ &= \frac{-1}{2\sqrt{n}} \chi^2(n-1) + \frac{1}{2} \frac{n-1}{\sqrt{n}} \chi^2(1) \\ &= \frac{-1}{2\sqrt{n}} \underbrace{[\chi^2(n-1) - (n-1)]}_{\text{CLT}} + \frac{(n-1)}{2\sqrt{n}} (\chi^2(1) - 1) \\ &= \frac{-1}{2\sqrt{n}} [\chi^2(n-1) - (n-1)] = -\frac{1}{2} u + \frac{1}{2} \sqrt{n} (\chi^2(1) - 1) \\ \frac{1}{n} \sum z_{t-1} a_t &\xrightarrow{L} \frac{1}{2} (\chi^2(1) - 1) \end{aligned}$$

Reason for break down of normality is the eigenvalues

$$-\frac{1}{2} \quad (n-1) \text{ of them}$$

$$\frac{n-1}{2} \quad \text{largest one}$$

$\therefore$ CLT does not apply to produce normality using this approach, easy to study finite n behavior as a function of time ϕ .

#5180

#EXAMPLE I. A GENERATED MA(1) EXAMPLE

The following data are generated from the MA(1) model

$$Z_t = 5 + a_t - .7a_{t-1}$$

where the a_t 's are iid $N(0,1.)$.

THE PC SCA STATISTICAL SYSTEM (Release 4.2)
COPYRIGHT 1985-1993, SCIENTIFIC COMPUTING ASSOCIATES. ALL RIGHTS RESERVED
SCA PRODUCT IDENTIFICATION: UTS, EXPERT, QPI & GSA
SCA SOFTWARE IDENTIFICATION: TIAO (93082419)
RELEASE DATE: 9/ 1/92
SIZE OF WORKSPACE IS 99000 SINGLE PRECISION WORDS
DATE -- 1/ 4/95 TIME -- 15:52:48
--

tsmodel m1.model z=c+(1-th*b)noise.simulate

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE TYPE OF ORIGINAL DIFFERENCING
VARIABLE OR CENTERED

Z	RANDOM	ORIGINAL	NONE				

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR ORDER	CONS- TRAIT	VALUE	STD ERROR	T VALUE
1	C	CNST	1	0	NONE	.0000	
2	TH	Z	MA	1	1	NONE	.1000

--
c=5

-- #Here we set the values of c and th
th=.7
--

simulate model m1.noise n(0,1.).nobs 100

THE UNIVARIATE TIME SERIES Z IS SIMULATED USING MODEL M1

C-1

```
--
print z
```

Z	IS	A	100	BY	1	VARIABLE		
5.212		5.734		3.822		6.633	4.258	4.355
4.337		5.473		4.736		3.827	8.644	1.854
5.318		5.293		5.331		4.462	5.437	6.309
3.566		6.139		6.391		4.263	4.031	5.474
5.701		2.572		4.280		5.903	5.964	5.126
5.475		3.507		5.914		6.951	4.435	5.604
3.568		5.885		3.148		5.054	5.783	6.228
5.139		5.947		2.685		7.020	3.625	5.206
5.149		3.134		5.666		3.711	5.812	5.673
6.174		5.230		5.815		5.465	2.898	6.592
4.472		6.062		5.690		1.988	5.424	5.713
4.681		5.938		5.208		4.737	4.615	5.337
2.866		5.839		4.945		4.019	6.561	4.140
3.983		5.618		4.356		6.710	3.335	5.379
2.209		6.389						6.604

```
--
desc z
```

VARIABLE	NAME	IS	Z
NUMBER OF OBSERVATIONS			100
NUMBER OF MISSING VALUES			0

	STATISTIC	STD. ERROR	STATISTIC/S.E.
MEAN	5.0317	.1236	40.7257
VARIANCE	1.5265		
STD DEVIATION	1.2355		
C. V.	.2455		
SKEWNESS	-.5064	.2414	
KURTOSIS	.3302	.4783	
QUARTILE			
MINIMUM	1.8542		
1ST QUARTILE	4.2797		
MEDIAN	5.3245		
3RD QUARTILE	5.8149		

MAXIMUM 8.6437

RANGE

MAX - MIN 6.7895

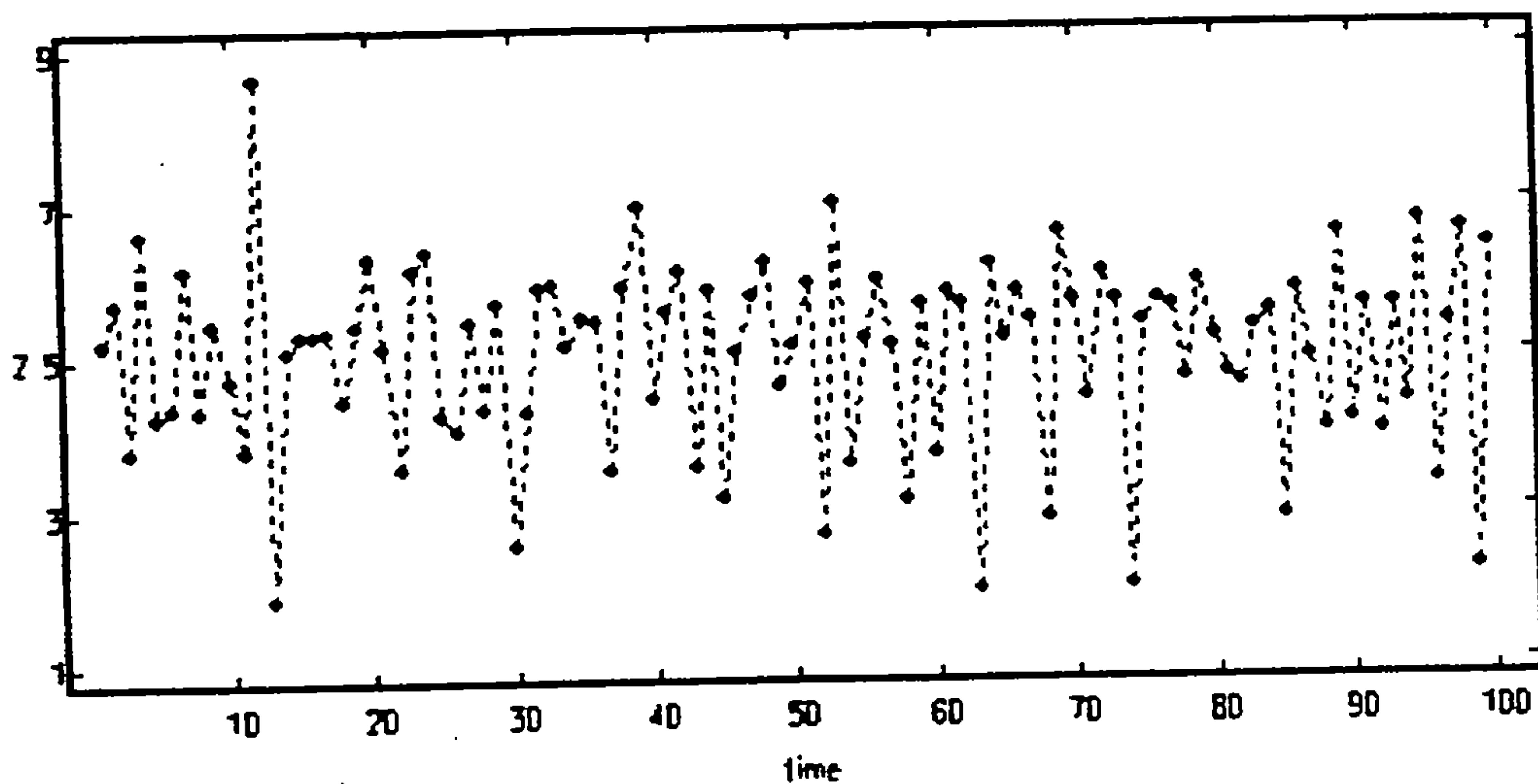
Q3 - Q1 1.5353

--

graph z

** THE FILE ALREADY EXISTS AND NOW IS OPENED. THE FILE NAME IS AAA.FMT
THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE 7

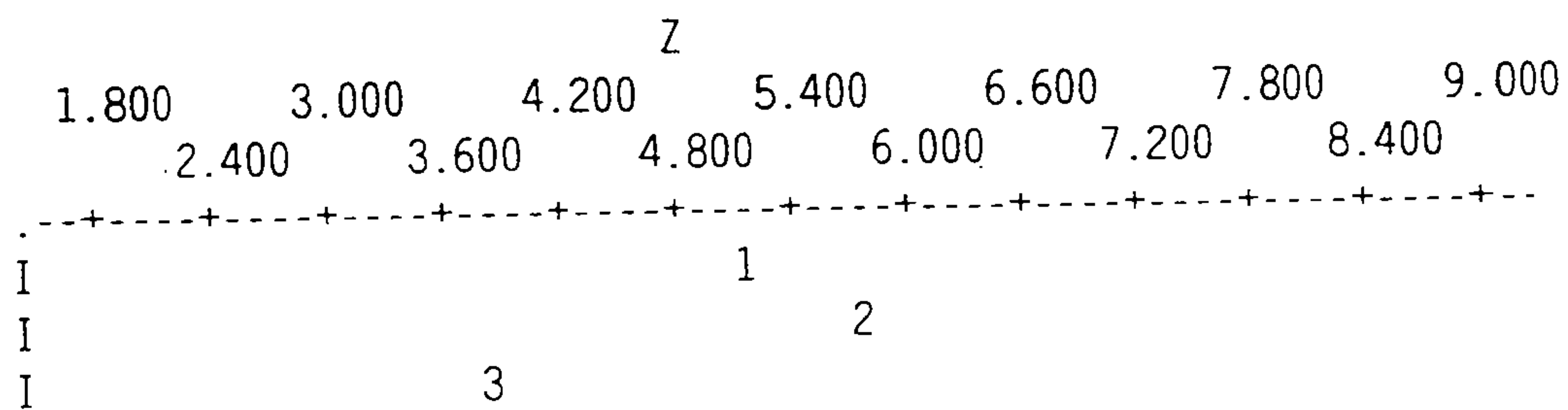
*#Here we go to the SCA GRAPHICS. After a number of steps, we obtain the
following figure.*



#We can also get a character plot of the time series Z

--

tplot z



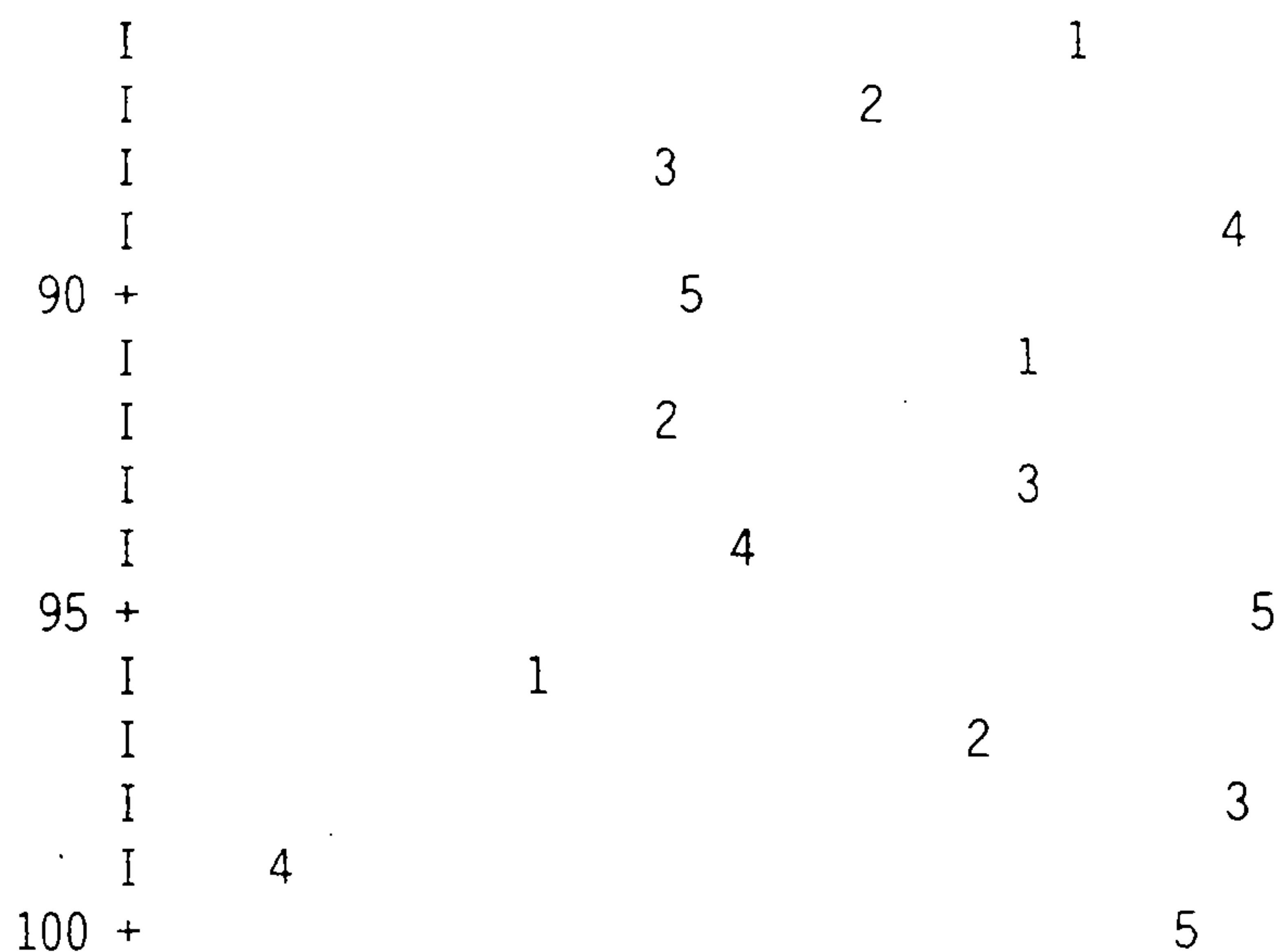
C-4

	I				4
5	+		5		
	I		1		
	I			2	
	I		3		
	I			4	
10	+		5		
	I	1			
	I				2
	I	3			
	I		4		
15	+		5		
	I		1		
	I		2		
	I	3			
	I		4		
20	+			5	
	I		1		
	I	2			
	I			3	
	I			4	
25	+		5		
	I	1			
	I		2		
	I	3			
	I		4		
30	+	5			
	I	1			
	I		2		
	I		3		
	I		4		
35	+		5		
	I		1		
	I	2			
	I			3	
	I			4	
40	+		5		
	I		1		
	I			2	
	I	3			
	I			4	

C-5

45 +		5							
I				1					
I					2				
I						3			
I				4					
50 +				5					
I						1			
I	2								
I							3		
I			4						
55 +				5					
I						1			
I				2					
I		3							
I					4				
60 +			5						
I						1			
I						2			
I	3								
I							4		
65 +				5					
I						1			
I						2			
I		3							
I							4		
70 +						5			
I				1					
I							2		
I						3			
I	4								
75 +						5			
I							1		
I							2		
I				3					
I							4		
80 +				5					
I						1			
I						2			
I							3		
I							4		
85 +		5							

C-6



#In what follows we give three methods for tentative specification of a model for this data set.

--

acf z.maxlag 12

TIME PERIOD ANALYZED	1 TO 100
NAME OF THE SERIES	Z
EFFECTIVE NUMBER OF OBSERVATIONS	100
STANDARD DEVIATION OF THE SERIES	1.2293
MEAN OF THE (DIFFERENCED) SERIES	5.0317
STANDARD DEVIATION OF THE MEAN	.1229
T-VALUE OF MEAN (AGAINST ZERO)	40.9309

AUTOCORRELATIONS

1- 12	-.47	-.05	.16	-.17	.11	-.04	-.08	.08	.06	-.23	.17	-.04
ST.E.	.10	.12	.12	.12	.12	.13	.13	.13	.13	.13	.13	.13
Q	22.5	22.7	25.6	28.6	29.9	30.1	30.7	31.5	31.9	37.8	41.0	41.2

-1.0	-.8	-.6	-.4	-.2	.0	.2	.4	.6	.8	1.0
+-----+	+-----+	+-----+	+-----+	+-----+	+-----+	+-----+	+-----+	+-----+	+-----+	+-----+

					I	
1	-.47		XXXXXXXX	XXXXXI		+
2	-.05			+ XI		+
3	.16			+ IXXXX		+
4	-.17			+ XXXXI		+
5	.11			+ IXXX		+
6	-.04			+ XI		+
7	-.08			+ XXI		+
8	.08			+ IXX		+
9	.06			+ IX		+
10	-.23		XXXXXX	XI		+
11	.17			+ IXXXX		+
12	-.04			+ XI		+

--

PARTIAL AUTOCORRELATIONS

-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
+-----+-----+-----+-----+-----+-----+-----+-----+-----+-----+

— —

eacf z

TIME PERIOD ANALYZED	1 TO 100
NAME OF THE SERIES	Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .	100
STANDARD DEVIATION OF THE SERIES . . .	1.2293
MEAN OF THE (DIFFERENCED) SERIES . . .	5.0317
STANDARD DEVIATION OF THE MEAN	.1229
T-VALUE OF MEAN (AGAINST ZERO)	40.9309

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	-.47	-.05	.16	-.17	.11	-.04	-.08	.08	.06	-.23	.17	-.04	-.10
(P= 1)	-.52	-.20	.08	-.06	.08	-.03	-.10	.07	.03	-.22	.13	-.04	-.12
(P= 2)	-.10	-.34	-.18	.01	-.01	-.09	-.01	-.02	-.02	-.21	-.10	-.10	.02
(P= 3)	-.20	-.38	-.21	-.03	.00	-.07	-.05	-.02	-.03	-.18	.08	-.12	.04
(P= 4)	-.00	-.18	-.14	.07	-.08	-.06	-.01	.02	-.01	-.15	-.09	-.15	.04
(P= 5)	-.00	-.01	-.40	-.16	.09	-.05	-.01	.02	-.02	-.14	.06	-.01	-.06
(P= 6)	-.24	.08	-.15	-.27	.04	.04	.15	-.00	-.05	-.13	.06	-.02	-.04

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 1)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 2)	0	X	0	0	0	0	0	0	0	0	0	0	0
(P= 3)	0	X	0	0	0	0	0	0	0	0	0	0	0
(P= 4)	0	0	0	0	0	0	0	0	0	0	0	0	0
(P= 5)	0	0	X	0	0	0	0	0	0	0	0	0	0
(P= 6)	X	0	0	0	0	0	0	0	0	0	0	0	0

--

save z.file'c:\b325\gma1.dat'.format 'f12.4'

** THE SPECIFIED FILE DOES NOT EXIST. A NEW FILE IS OPENED.

THE FILE NAME IS C:\B325\GMA1.DAT

THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE 7

--

stop

#EXAMPLE II. A GENERATED AR(1) EXAMPLE

Here we generate 100 observations from the following AR(1) model.

$$Z_t - 5 = .7(Z_{t-1} - 5) + a_t$$

or

$$Z_t = 1.5 + .7Z_{t-1} + a_t$$

where a_t 's are iid $N(0,1)$.

```
--
tsmodel m1.model (1-ph*b)z=c+noise.simulate
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE		TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING				
Z		RANDOM	ORIGINAL	NONE				
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	.0000		
2 PH	Z	AR	1	1	NONE	.1000		

```
--
c=1.5
--
ph=.7
--
simulate model m1.noise n(0,1.).nobs 100
```

THE UNIVARIATE TIME SERIES Z IS SIMULATED USING MODEL M1

```
--
print z
```

Z	IS	A	100	BY	1	VARIABLE		
5.449		6.198		5.279		6.436	6.132	5.237
5.550		5.778		5.555		4.223	7.284	5.433
4.540		4.847		5.343		5.017	5.292	6.710
6.088		6.485		7.938		7.648	6.299	5.994

5.313	2.976	1.292	1.704	3.166	4.174	5.255
6.236	5.112	5.466	7.548	7.773	8.238	9.288
7.985	7.962	5.833	4.768	5.050	6.412	6.573
6.649	7.485	5.356	6.301	5.272	4.949	5.796
6.288	4.547	4.401	3.094	3.438	4.419	1.901
2.120	2.716	4.029	5.225	3.689	4.646	5.850
5.836	6.815	7.822	5.049	4.109	4.442	5.265
5.326	6.264	6.818	6.663	6.053	5.996	6.428
4.377	4.267	4.224	3.292	4.549	4.346	4.920
4.191	4.525	4.087	5.665	4.713	4.652	6.257
4.140	4.569					

--

desc z

VARIABLE	NAME	IS	Z
NUMBER OF OBSERVATIONS			100
NUMBER OF MISSING VALUES			0

	STATISTIC	STD. ERROR	STATISTIC/S.E.
MEAN	5.3572	.1470	36.4516
VARIANCE	2.1600		
STD DEVIATION	1.4697		
C. V.	.2743		
SKEWNESS	-.1521	.2414	
KURTOSIS	.3249	.4783	

QUARTILE

MINIMUM	1.2922
1ST QUARTILE	4.4418
MEDIAN	5.3027
3RD QUARTILE	6.2644
MAXIMUM	9.2883

RANGE

MAX - MIN	7.9961
Q3 - Q1	1.8226

--

#Again we go to SCA GRAPHICS to get the Z series plotted.

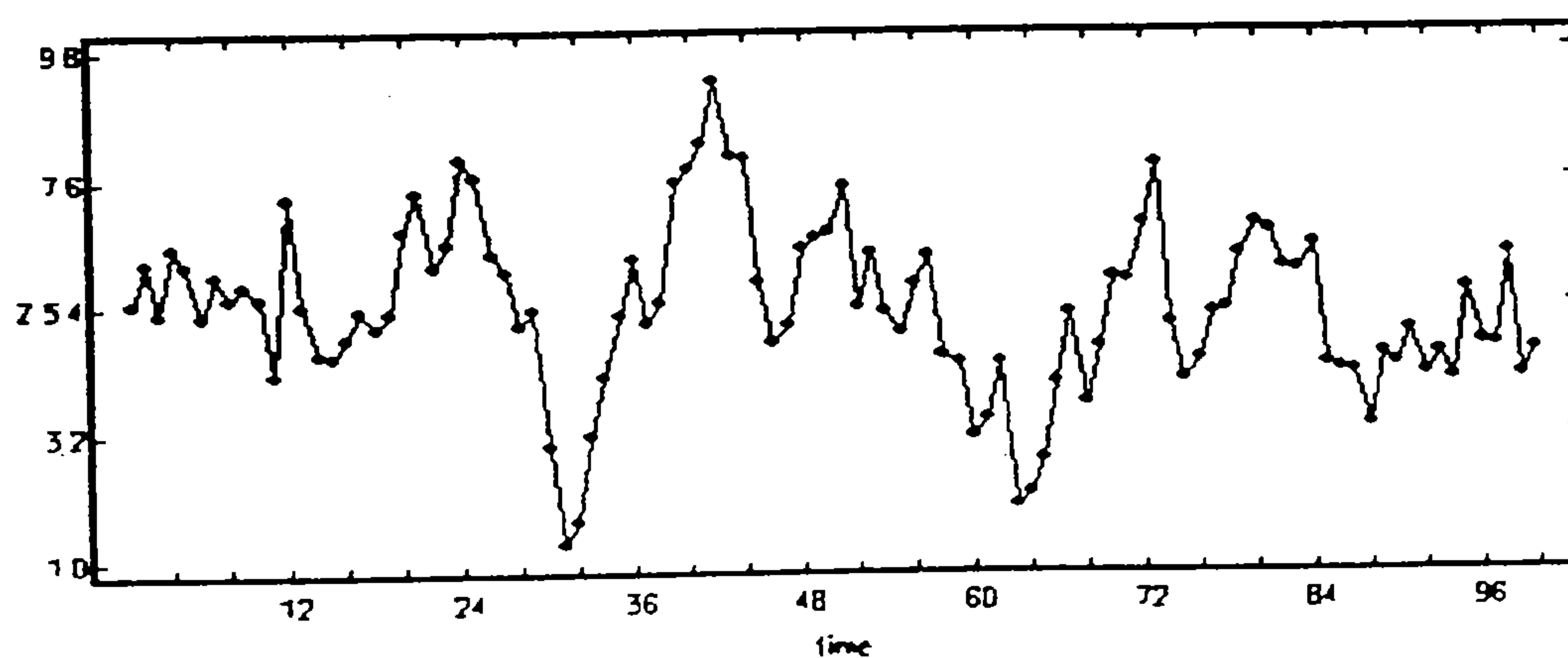
graph z

*** THE FILE ALREADY EXISTS AND NOW IS OPENED. THE FILE NAME IS AAA.FMT
THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE 7*

#Compared with the generated MA(1)series,we see that this series exhibits more momentum over time.

*#The following are the three method to help specify a model for this series:
acf,pacf and eacf.*

acf z *#Here without specifying the maximum number of lags we get
36 autocorrelations.*



C-14

TIME PERIOD ANALYZED 1 TO 100
 NAME OF THE SERIES Z
 EFFECTIVE NUMBER OF OBSERVATIONS . . . 100
 STANDARD DEVIATION OF THE SERIES . . . 1.4623
 MEAN OF THE (DIFFERENCED) SERIES . . . 5.3572
 STANDARD DEVIATION OF THE MEAN 1462
 T-VALUE OF MEAN (AGAINST ZERO) 36.6352

AUTOCORRELATIONS

1- 12	.72	.51	.37	.17	.06	-.06	-.16	-.19	-.26	-.31	-.24	-.21
ST.E.	.10	.14	.16	.17	.17	.17	.17	.17	.17	.18	.18	.19
Q	53.1	80.5	94.6	97.8	98.2	98.6	101	105	113	124	131	136
13- 24	-.16	-.06	-.02	-.02	-.03	-.10	-.15	-.16	-.15	-.07	-.06	-.06
ST.E.	.19	.19	.19	.19	.19	.19	.19	.19	.19	.19	.19	.19
Q	139	139	139	139	139	140	143	146	149	150	151	151
25- 36	-.05	-.03	.07	.18	.24	.29	.32	.28	.22	.12	.05	.05
ST.E.	.20	.20	.20	.20	.20	.20	.20	.21	.21	.22	.22	.22
Q	151	152	152	157	165	178	193	205	212	214	214	215

-1.0 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1.0
 +-----+-----+-----+-----+-----+-----+-----+-----+-----+

			I
1	.72	+	IXXXX+XXXXXXXXXXXXXXX
2	.51	+	IXXXXXX+XXXXXX
3	.37	+	IXXXXXXX+X
4	.17	+	IXXXX +
5	.06	+	IXX +
6	-.06	+	XXI +
7	-.16	+	XXXXI +
8	-.19	+	XXXXXI +
9	-.26	+	XXXXXXI +
10	-.31	+	XXXXXXXXXI +
11	-.24	+	XXXXXXI +
12	-.21	+	XXXXXI +
13	-.16	+	XXXXI +
14	-.06	+	XI +
15	-.02	+	I +
16	-.02	+	XI +
17	-.03	+	XI +

C-15

18	-.10	+	XXI	+
19	-.15	+	XXXXI	+
20	-.16	+	XXXXI	+
21	-.15	+	XXXXI	+
22	-.07	+	XXI	+
23	-.06	+	XXI	+
24	-.06	+	XXI	+
25	-.05	+	XI	+
26	-.03	+	XI	+
27	.07	+	IXX	+
28	.18	+	IXXXX	+
29	.24	+	IXXXXXXX	+
30	.29	+	IXXXXXXXX	+
31	.32	+	IXXXXXXXX	+
32	.28	+	IXXXXXXX	+
33	.22	+	IXXXXX	+
34	.12	+	IXXX	+
35	.05	+	IX	+
36	.05	+	IX	+

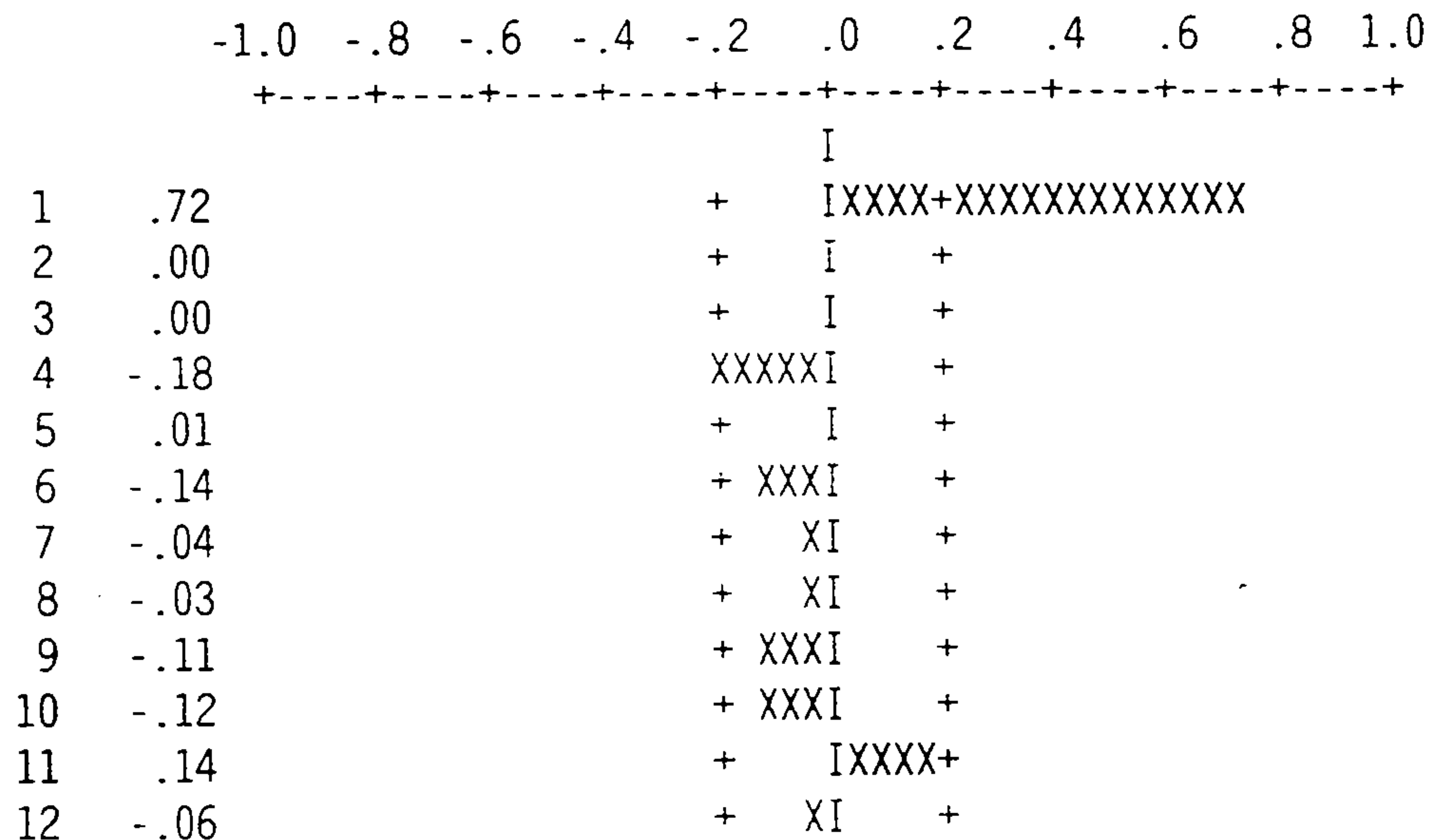
--

pacf z.maxlag 12

TIME PERIOD ANALYZED 1 TO 100
 NAME OF THE SERIES Z
 EFFECTIVE NUMBER OF OBSERVATIONS . . . 100
 STANDARD DEVIATION OF THE SERIES . . . 1.4623
 MEAN OF THE (DIFFERENCED) SERIES . . . 5.3572
 STANDARD DEVIATION OF THE MEAN 1462
 T-VALUE OF MEAN (AGAINST ZERO) 36.6352

PARTIAL AUTOCORRELATIONS

1- 12	.72	-.00	-.00	-.18	.01	-.14	-.04	-.03	-.11	-.12	.14	-.06
ST.E.	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10



--

eacf z

TIME PERIOD ANALYZED 1 TO 100
 NAME OF THE SERIES Z
 EFFECTIVE NUMBER OF OBSERVATIONS . . . 100
 STANDARD DEVIATION OF THE SERIES . . . 1.4623
 MEAN OF THE (DIFFERENCED) SERIES . . . 5.3572
 STANDARD DEVIATION OF THE MEAN 1462
 T-VALUE OF MEAN (AGAINST ZERO) 36.6352

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.72	.51	.37	.17	.06	-.06	-.16	-.19	-.26	-.31	-.24	-.21	-.16
(P= 1)	.00	.01	.22	.06	.07	-.07	-.10	.01	-.03	-.24	.06	-.06	-.13
(P= 2)	-.03	-.00	.23	-.04	.15	-.02	-.12	-.05	-.00	-.25	-.10	-.03	-.13
(P= 3)	-.06	-.03	.12	-.06	.14	-.01	-.12	.01	-.02	-.25	.05	-.13	-.10
(P= 4)	.08	-.48	-.20	.06	.03	-.08	.02	.00	.04	-.15	-.02	-.16	-.14
(P= 5)	.10	-.45	-.25	.19	.03	-.02	.01	.05	.06	-.16	-.00	-.11	.04
(P= 6)	-.24	-.05	-.37	-.13	-.05	.00	-.01	.07	.08	-.13	.02	-.04	-.08

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	0	0	0	0	0	0	0	0	0	0
(P= 1)	0	0	0	0	0	0	0	0	0	X	0	0	0
(P= 2)	0	0	X	0	0	0	0	0	0	X	0	0	0
(P= 3)	0	0	0	0	0	0	0	0	0	X	0	0	0
(P= 4)	0	X	0	0	0	0	0	0	0	0	0	0	0
(P= 5)	0	X	0	0	0	0	0	0	0	0	0	0	0
(P= 6)	X	0	X	0	0	0	0	0	0	0	0	0	0

--

save z.file'c:\b325\gar1.dat'.format 'f12.4'

** THE SPECIFIED FILE DOES NOT EXIST. A NEW FILE IS OPENED.
THE FILE NAME IS C:\B325\GAR1.DAT
THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE 7

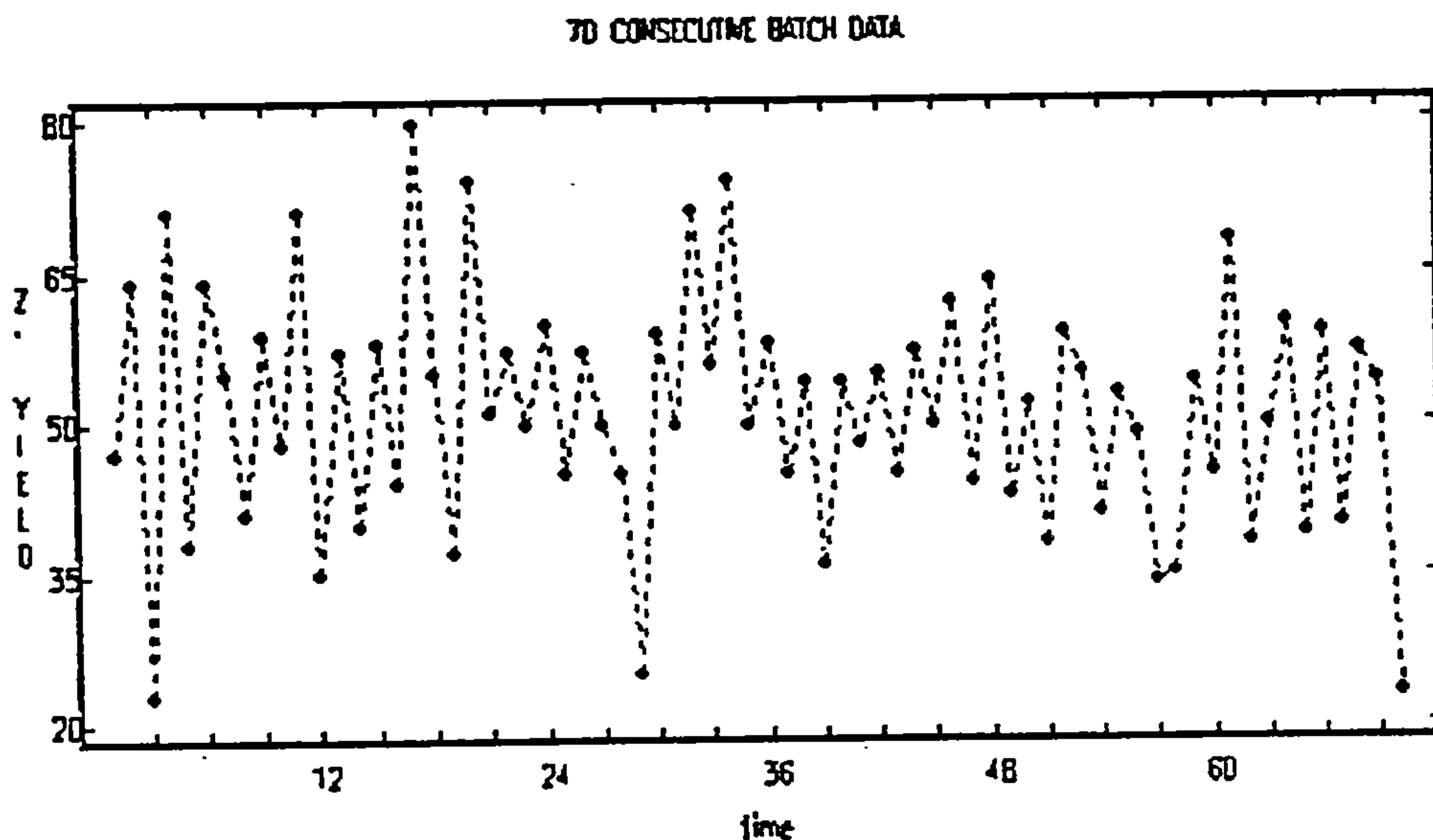
--

#EXAMPLE III: THE BATCH DATA

This example consists of 70 consecutive observations from a chemical reactor, taken from Box, Jenkins and Reinsel (Series F).

```
--
input z.file 'c:\b325\bjf_chmy.dat'

Z      . A  70 BY  1 VARIABLE. IS STORED IN THE WORKSPACE
--
graph z
```



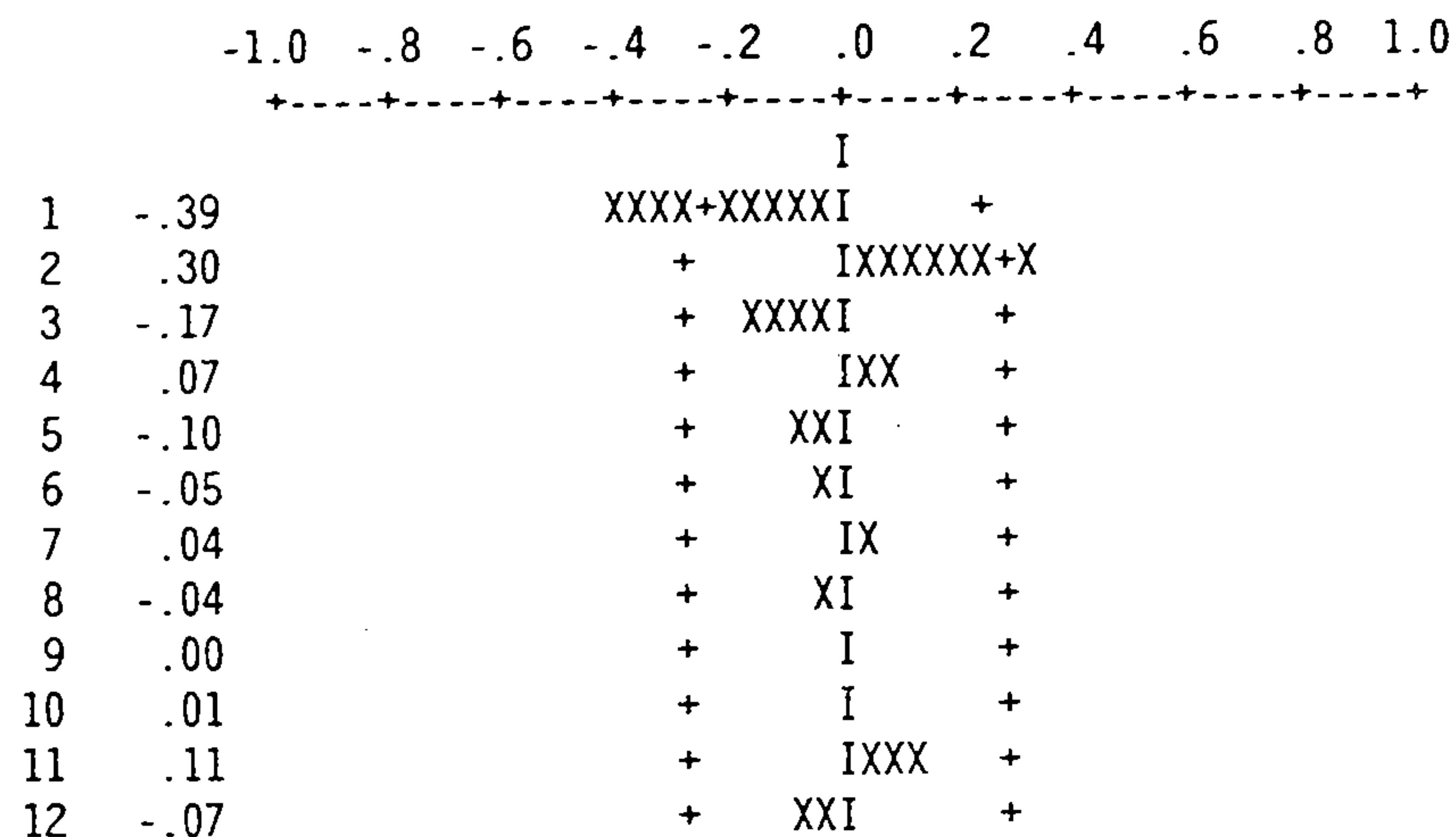
#The following command, uiden, will give both SACF and SPACF.

```
uiden z.maxlag 12
```

```
TIME PERIOD ANALYZED . . . . . 1 TO 70
NAME OF THE SERIES . . . . . Z
EFFECTIVE NUMBER OF OBSERVATIONS . . . 70
STANDARD DEVIATION OF THE SERIES . . . 11.8236
MEAN OF THE (DIFFERENCED) SERIES . . . 51.1286
STANDARD DEVIATION OF THE MEAN . . . 1.4132
T-VALUE OF MEAN (AGAINST ZERO) . . . 36.1795
```

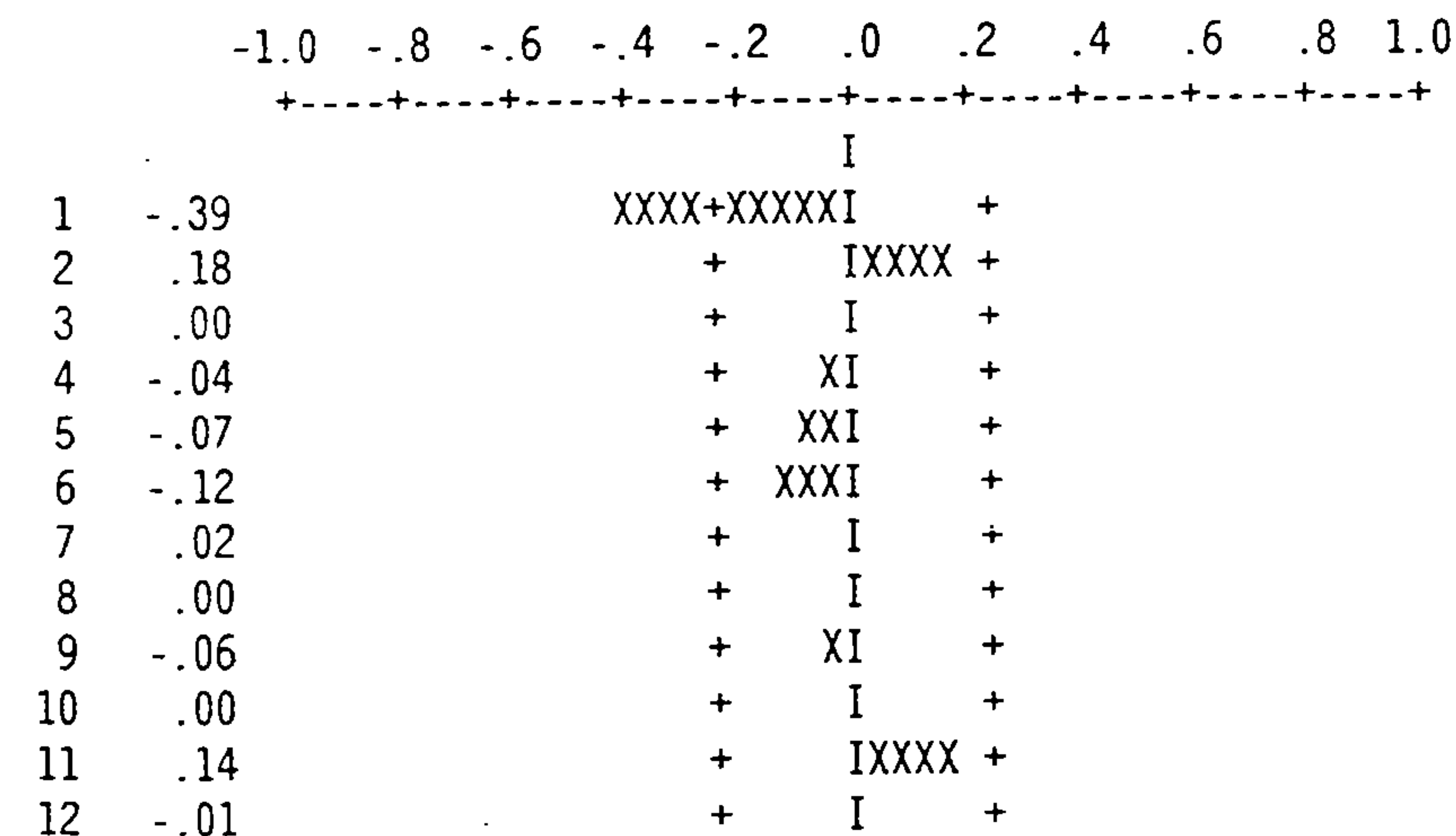
AUTOCORRELATIONS

1- 12	-.39	.30	-.17	.07	-.10	-.05	.04	-.04	-.00	.01	.11	-.07
ST.E.	.12	.14	.15	.15	.15	.15	.15	.15	.15	.15	.15	.15
Q	11.1	18.0	20.0	20.4	21.1	21.3	21.4	21.6	21.6	21.6	22.6	23.0



PARTIAL AUTOCORRELATIONS

1- 12	-.39	.18	.00	-.04	-.07	-.12	.02	.00	-.06	.00	.14	-.01
ST.E.	.12	.12	.12	.12	.12	.12	.12	.12	.12	.12	.12	.12



--
eacf z

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	-.39	.30	-.17	.07	-.10	-.05	.04	-.04	-.00	.01	.11	-.07	.15
(P= 1)	.36	.11	-.06	-.09	-.10	-.14	-.00	-.04	.04	-.05	.13	.00	.14
(P= 2)	-.04	.21	-.08	.01	-.14	-.13	-.01	-.04	.06	-.06	.12	-.12	.13
(P= 3)	.16	.12	-.07	.04	-.23	-.02	-.02	-.06	.02	-.06	.08	-.04	-.06
(P= 4)	-.45	-.28	-.02	.13	-.24	-.00	-.03	-.05	.04	-.08	.05	-.07	-.06
(P= 5)	-.36	-.10	.12	.26	-.25	.02	.05	-.04	.08	-.13	.03	-.04	-.03
(P= 6)	.26	.10	-.31	-.05	.14	.14	.04	.01	-.03	.02	-.03	-.01	-.04

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->) 0 1 2 3 4 5 6 7 8 9 10 11 12

```

(P= 0) X X 0 0 0 0 0 0 0 0 0 0 0
(P= 1) X 0 0 0 0 0 0 0 0 0 0 0 0
(P= 2) 0 0 0 0 0 0 0 0 0 0 0 0 0
(P= 3) 0 0 0 0 0 0 0 0 0 0 0 0 0
(P= 4) X X 0 0 0 0 0 0 0 0 0 0 0
(P= 5) X 0 0 0 0 0 0 0 0 0 0 0 0
(P= 6) X 0 X 0 0 0 0 0 0 0 0 0 0

```

#The following command specifies an AR(2) model for this data set.

```
tsmodel m1.model (1,2)z=c+noise
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
Z	RANDOM	ORIGINAL	NONE

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAIT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	.0000		
2	Z	AR	1	1	NONE	.1000		
3	Z	AR	1	2	NONE	.1000		

#The following command estimates the parameters of the specified model and stores residuals under the variable rz.

```
estim m1.hold resi(rz)
```

#The following section give some details in the estimation process. It can be skipped and we will go directly to the summary.

THE FOLLOWING ANALYSIS IS BASED ON TIME SPAN 1 THRU 70
>> HEAVY COMPUTATION FOLLOWS. PLEASE WAIT !!! <<
ITERATION 1, USING STANDARD ERROR = 15.93720072

ITER.	OBJ.	PARAMETER ESTIMATES		
1	.7820E+04	53.0	-.250	.212
2	.7758E+04	58.2	-.334	.195
3	.7758E+04	58.7	-.340	.190

ITERATION TERMINATED DUE TO:
 RELATIVE CHANGE IN (OBJECTIVE FUNCTION)**0.5 LESS THAN .1000D-03
 TOTAL NUMBER OF ITERATIONS 3
 RELATIVE CHANGE IN (OBJECTIVE FUNCTION)**0.5 1966D-04
 MAXIMUM RELATIVE CHANGE IN THE ESTIMATES 2439D-01
 REDUCED CORRELATION MATRIX OF PARAMETER ESTIMATES

	1	2	3
1	1.00		
2	-.84	1.00	
3	-.84	.42	1.00

THE RECIPROCAL CONDITION VALUE FOR THE CROSS PRODUCT MATRIX OF
 THE PARAMETER PARTIAL DERIVATIVES IS .319203D-02

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
Z	RANDOM	ORIGINAL	NONE					
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	58.7190	10.8515	5.41
2	Z	AR	1	1	NONE	-.3397	.1242	-2.74
3	Z	AR	1	2	NONE	.1905	.1241	1.54

TOTAL SUM OF SQUARES 978584E+04
 TOTAL NUMBER OF OBSERVATIONS 70
 RESIDUAL SUM OF SQUARES. 775756E+04
 R-SQUARE 184
 EFFECTIVE NUMBER OF OBSERVATIONS 68
 RESIDUAL VARIANCE ESTIMATE 114082E+03
 RESIDUAL STANDARD ERROR. 106809E+02

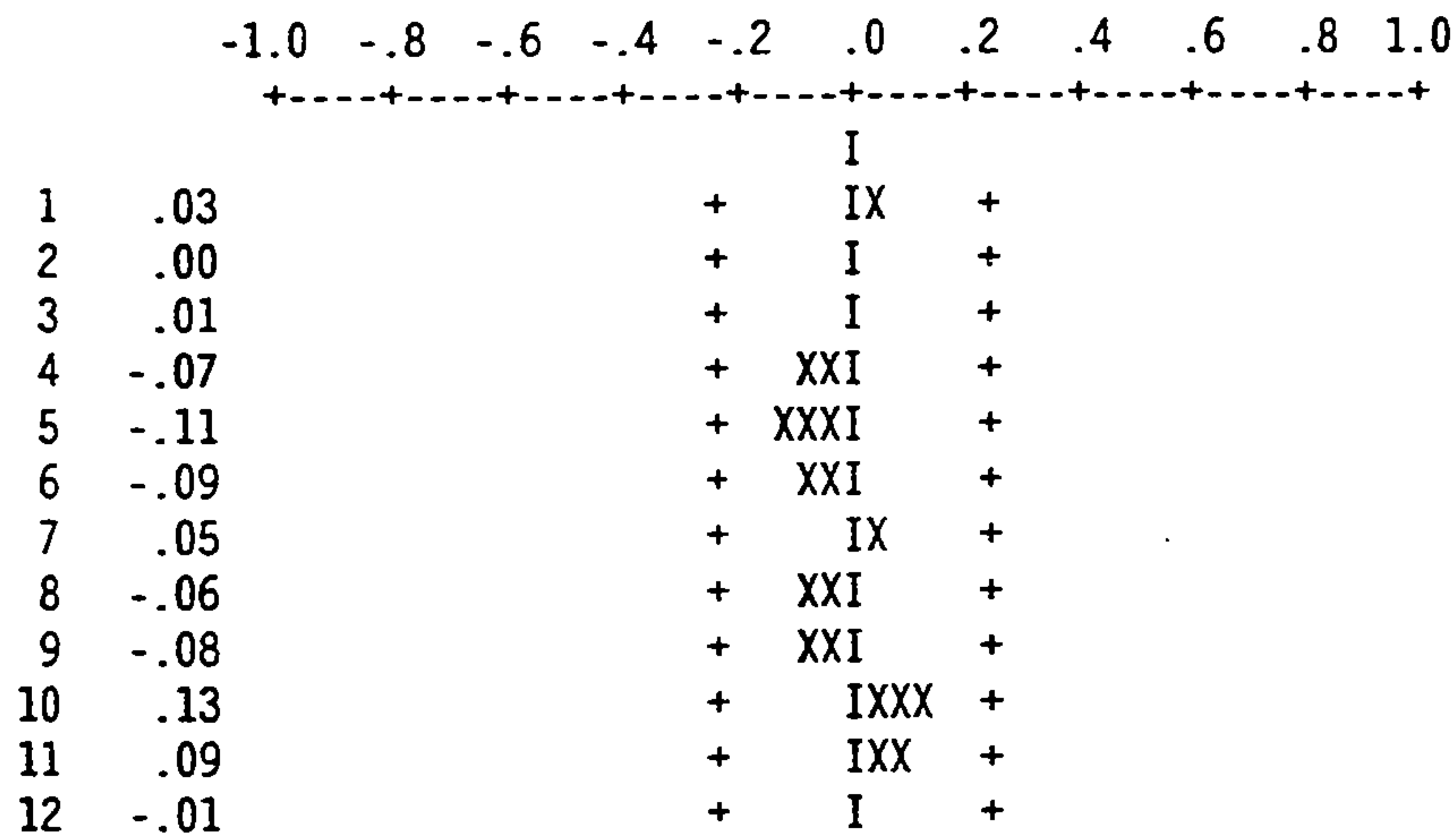
--
 #An useful diagnostic tool is to check the SACF of the residuals to see
 if they exhibit white noise behavior.

acf rz.maxlag 12

TIME PERIOD ANALYZED	3	TO	70
NAME OF THE SERIES			RZ
EFFECTIVE NUMBER OF OBSERVATIONS			68
STANDARD DEVIATION OF THE SERIES			10.6809
MEAN OF THE (DIFFERENCED) SERIES			.0001
STANDARD DEVIATION OF THE MEAN			1.2952
T-VALUE OF MEAN (AGAINST ZERO)			.0000

AUTOCORRELATIONS

1- 12	.03	-.00	.01	-.07	-.11	-.09	.05	-.06	-.08	.13	.09	-.01
ST.E.	.12	.12	.12	.12	.12	.12	.12	.12	.13	.13	.13	.13
Q	.1	.1	.1	.4	1.3	1.8	2.1	2.4	2.9	4.3	5.0	5.0



--
#Although the fit seems adequate, the lagged 2 coefficient in the AR model is not statistically significant at the usual 5% level. So we may eliminate this term and try an AR(1) model.

tsm m2.model (1)z=c+noise

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M2

VARIABLE		TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING				
Z		RANDOM	ORIGINAL	NONE				
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	58.7190		
2	Z	AR	1	1	NONE	.1000		

--
 estim m2.hold resi(rz1)

#Intermediate details are skipped.

VARIABLE		TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
Z		RANDOM	ORIGINAL	NONE					
PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRRAINT	VALUE	STD ERROR	T VALUE
1	C		CNST	1	0	NONE	73.0852	6.0513	12.08
2		Z	AR	1	1	NONE	-.4249	.1146	-3.71

```
--
acf rz1.maxlag 12
```

```

TIME PERIOD ANALYZED . . . . . 2 TO 70
NAME OF THE SERIES . . . . . RZ1
EFFECTIVE NUMBER OF OBSERVATIONS . . . 69
STANDARD DEVIATION OF THE SERIES . . . 10.8659
MEAN OF THE (DIFFERENCED) SERIES . . . .0000
STANDARD DEVIATION OF THE MEAN . . . .1.3081
T-VALUE OF MEAN (AGAINST ZERO) . . . .0000

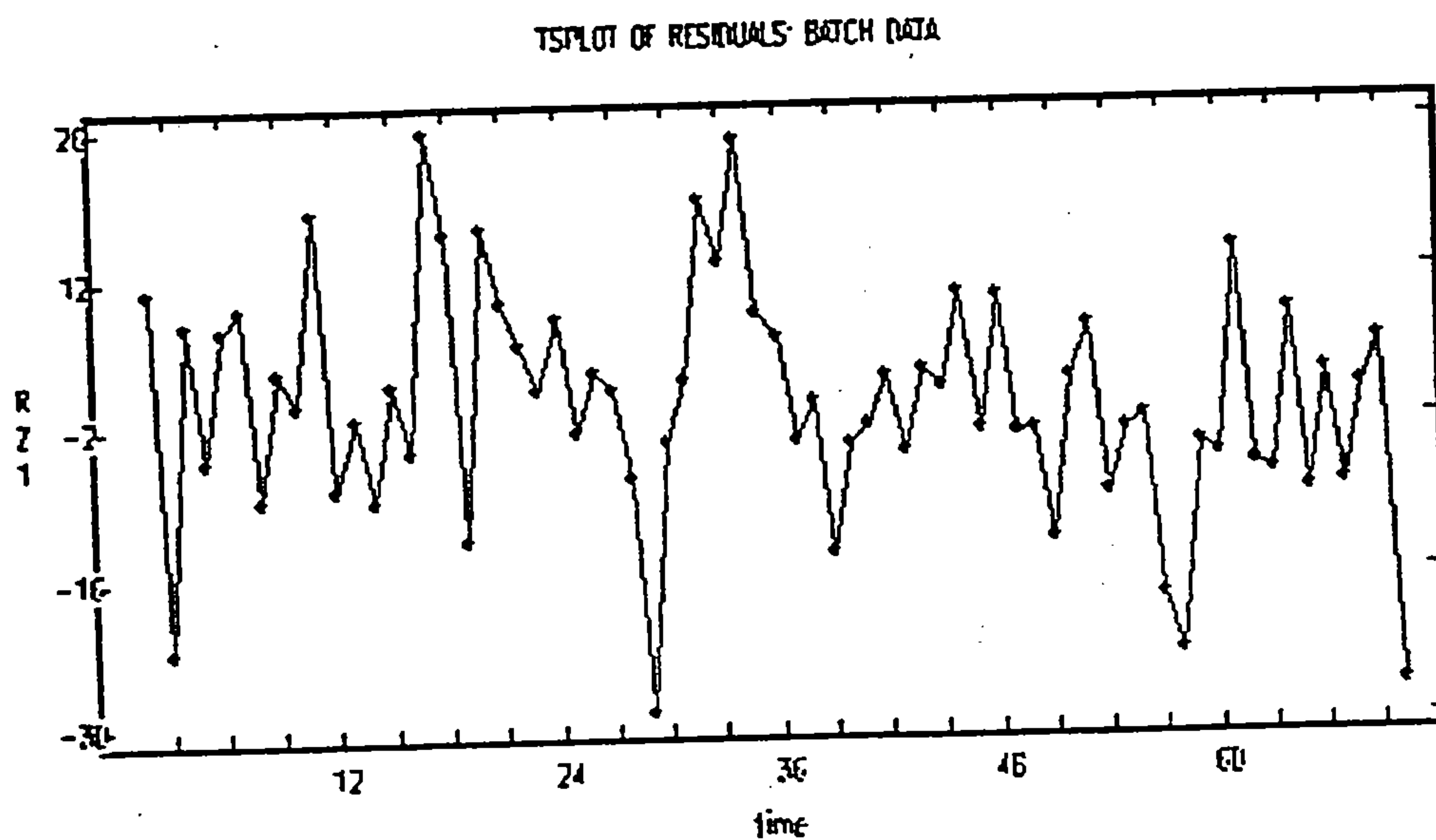
```

1- 12	.08	.14	-.04	-.05	-.12	-.11	.01	-.06	-.03	.10	.12	.03
ST.E.	.12	.12	.12	.12	.12	.13	.13	.13	.13	.13	.13	.13
Q	.4	1.9	2.0	2.2	3.3	4.2	4.2	4.5	4.5	5.4	6.6	6.6



--
The fit is about the same and the SACF of the residuals r21 again indicates white noise behavior. To look for other possible inadequacies, one useful way is to plot the residual series.

graph r21



#Compared with the original series, we see that the persistent seesawing behavior now disappears and the residuals look very much like independent noise.

#Based on the fitted AR(1) model, the following command produces forecasts for 24 periods from the origin $t=70$.

--

uforecast m2

24 FORECASTS. BEGINNING AT 70

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
71	63.3129	10.8659	
72	46.1847	11.8060	
73	53.4622	11.9679	
74	50.3701	11.9969	
75	51.6839	12.0021	
76	51.1257	12.0030	
77	51.3628	12.0032	
78	51.2621	12.0032	
79	51.3049	12.0032	
80	51.2867	12.0032	
81	51.2944	12.0032	
82	51.2911	12.0032	
83	51.2925	12.0032	
84	51.2919	12.0032	
85	51.2922	12.0032	
86	51.2921	12.0032	
87	51.2921	12.0032	
88	51.2921	12.0032	
89	51.2921	12.0032	
90	51.2921	12.0032	
91	51.2921	12.0032	
92	51.2921	12.0032	
93	51.2921	12.0032	
94	51.2921	12.0032	

--

#EXAMPLE IV: ANALYSIS OF A GENERATED (0,1,1) SERIES

We here consider 150 observations generated from the following nonstationary model,

$$Z_t - Z_{t-1} = a_t - \theta a_{t-1}$$

with $\theta = .4$.

--

tsmodel m1.model x(1)=(1-th*b)noise.simu

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE		TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING				
				1				
X		RANDOM	ORIGINAL	(1-B)				
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 TH	X	MA	1	1	NONE	.1000		

--

th=.4

--

simu model m1.noise n(0.,1.).nobs 200

THE UNIVARIATE TIME SERIES X IS SIMULATED USING MODEL M1

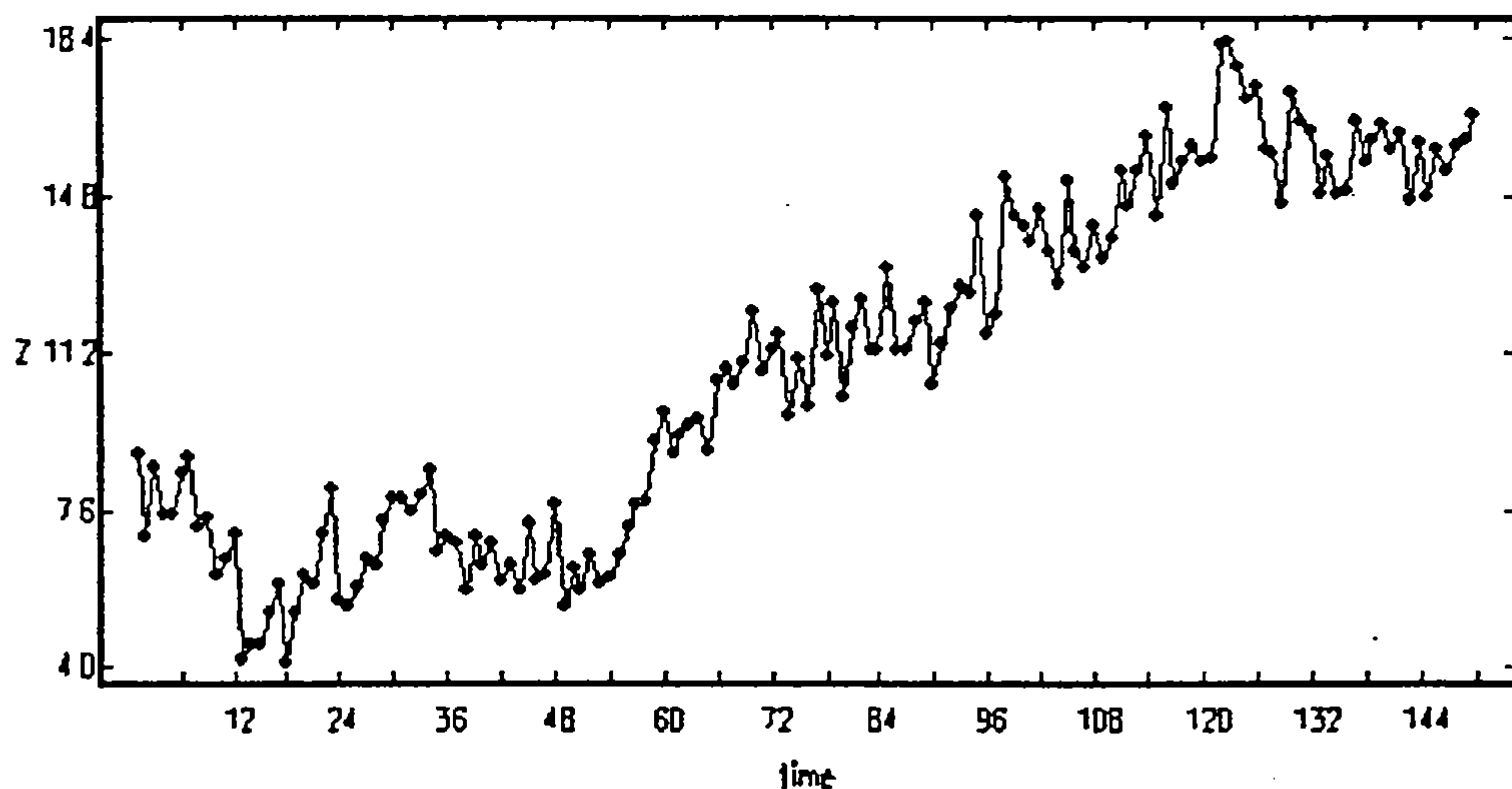
--

select x.new z.span(51,200)

VARIABLE X IS EDITED, THE RESULT IS STORED IN VARIABLE Z
VARIABLE Z IS A 150 BY 1 MATRIX

#It is a common practice in simulation to throw away a number of data points at the beginning to achieve *stability*.

graph z



#It is clear that the series exhibits a drifting behavior. The observations do not cling to any stable mean level.

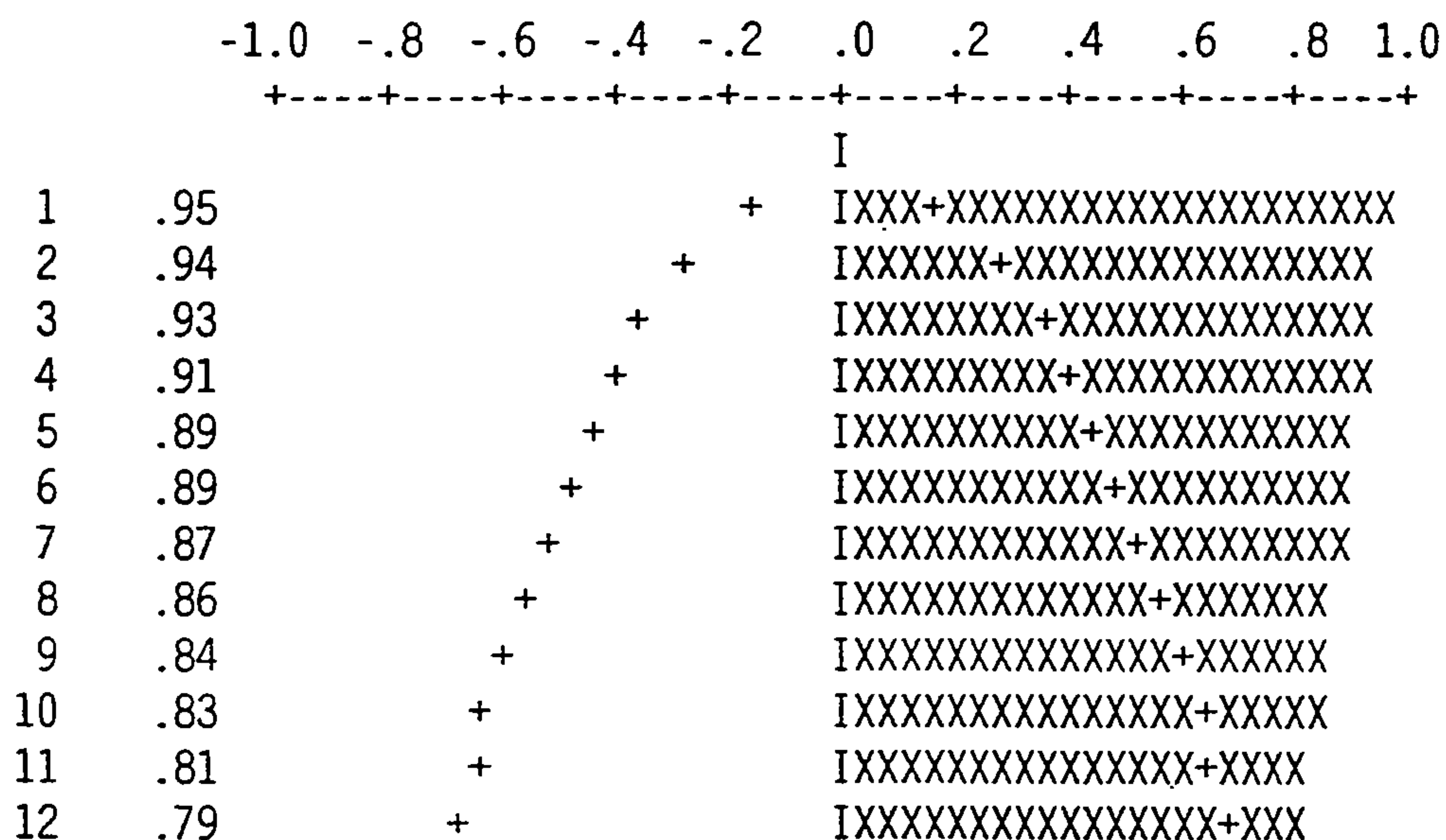
--

uiden z.maxlag 12

TIME PERIOD ANALYZED	1	TO	150
NAME OF THE SERIES			Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .			150
STANDARD DEVIATION OF THE SERIES . . .			3.9726
MEAN OF THE (DIFFERENCED) SERIES . . .			10.9601
STANDARD DEVIATION OF THE MEAN . . .			.3244
T-VALUE OF MEAN (AGAINST ZERO) . . .			33.7901

AUTOCORRELATIONS

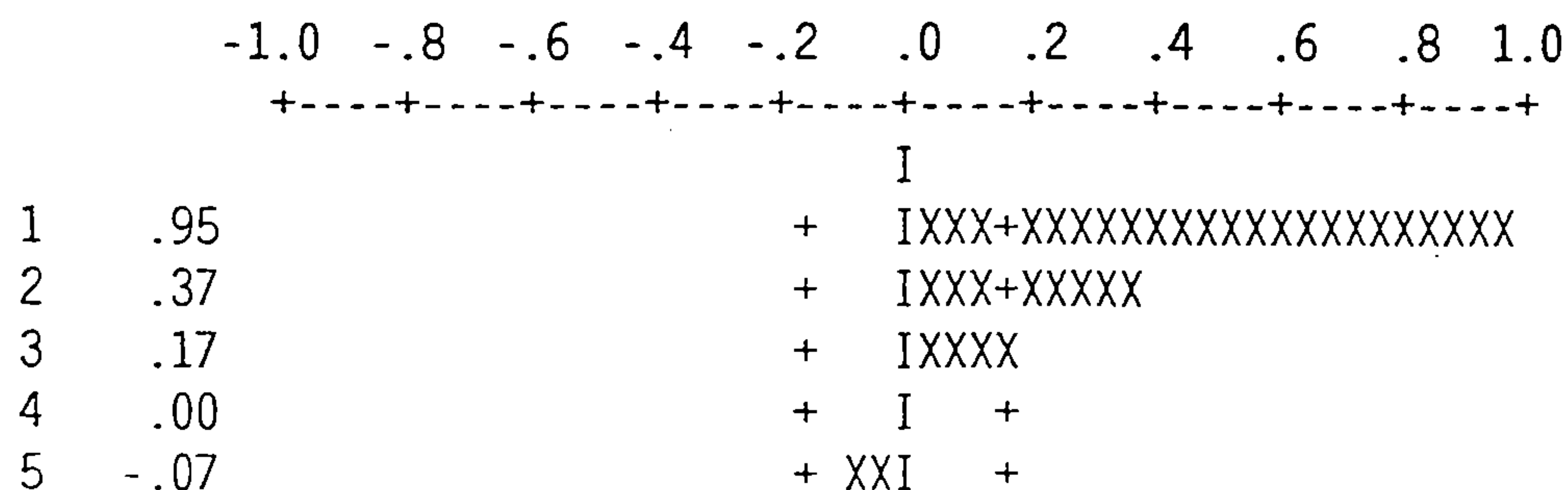
1- 12	.95	.94	.93	.91	.89	.89	.87	.86	.84	.83	.81	.79
ST.E.	.08	.14	.17	.20	.23	.25	.27	.29	.31	.32	.34	.35
Q	138	274	408	537	662	787	908	1026	1140	1251	1359	1461



#We see that the sample autocorrelations stay persistently high and close to one. This type of behavior suggests that the series is nonstationary, calling for differencing the data

PARTIAL AUTOCORRELATIONS

1- 12	.95	.37	.17	.00	-.07	.12	.01	-.04	-.08	-.02	-.01	-.09
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08



6	.12	+	IXXX+
7	.01	+	I +
8	-.04	+	XI +
9	-.08	+	XXI +
10	-.02	+	I +
11	-.01	+	I +
12	-.09	+	XXI +

--

diff z.new dz

1

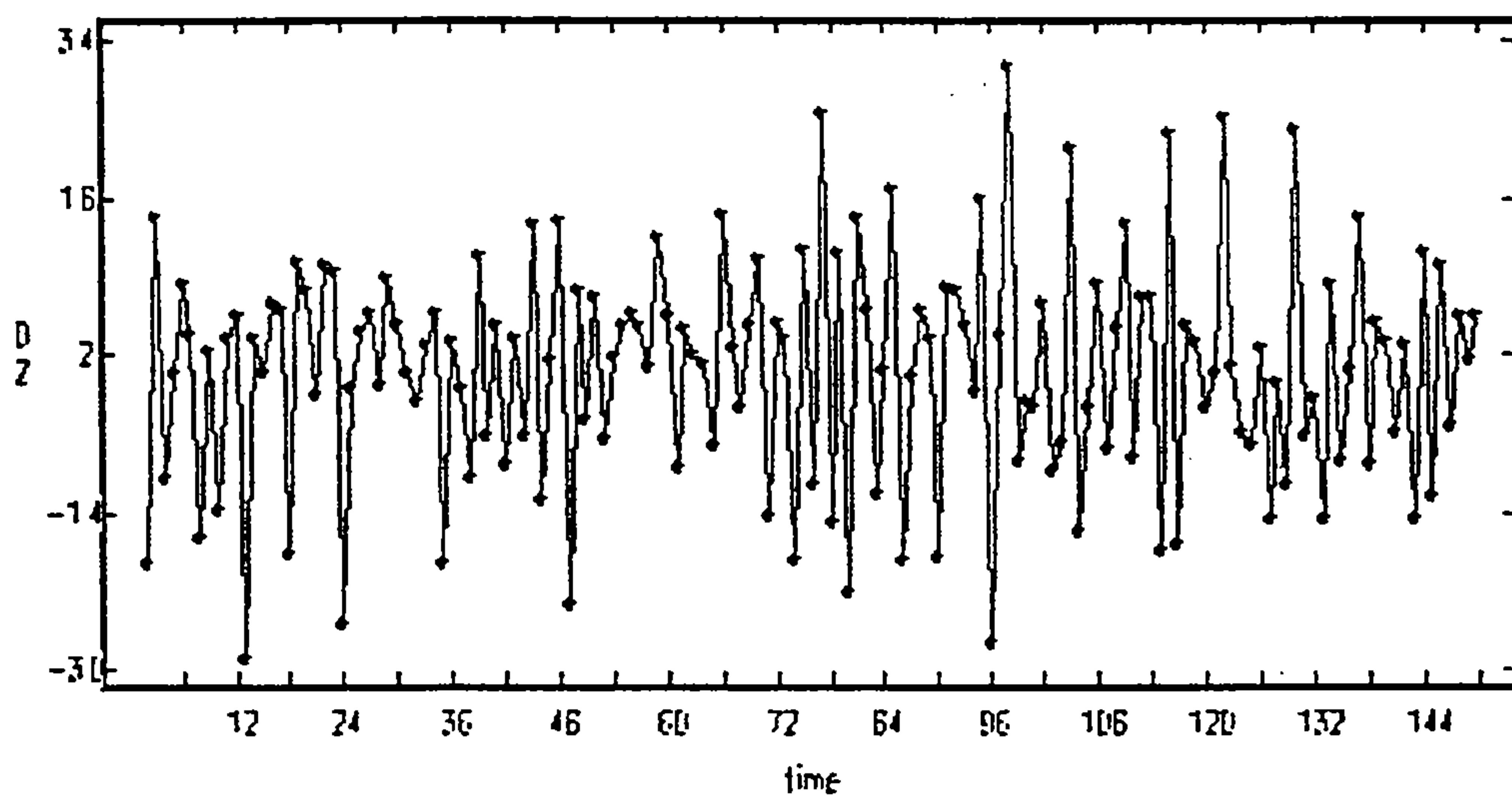
DIFFERENCE ORDERS ARE (1-B)

SERIES Z IS DIFFERENCED, THE RESULT IS STORED IN VARIABLE DZ

SERIES DZ HAS 150 ENTRIES

--

graph dz

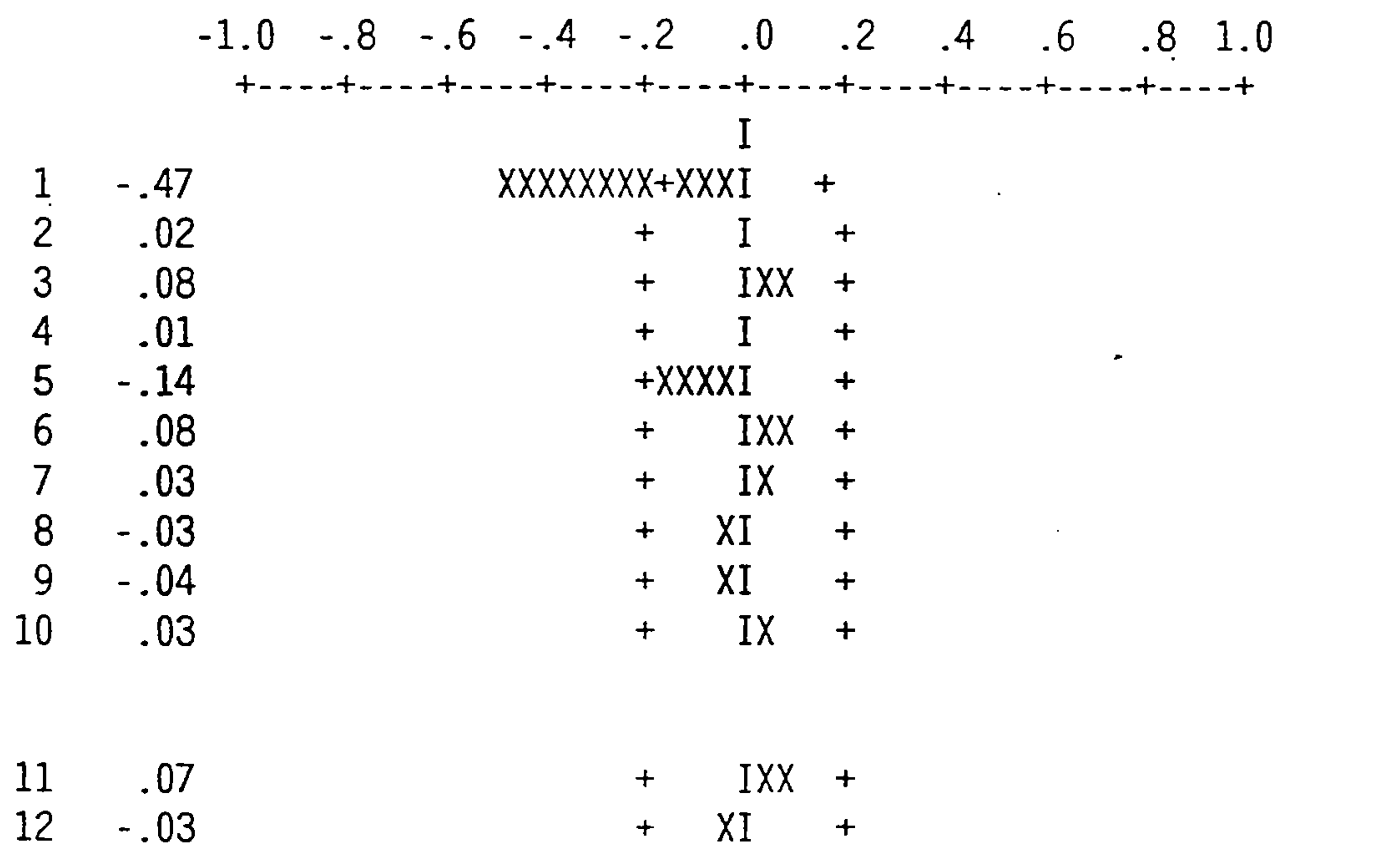


```
uiden dz.maxlag 12
```

TIME PERIOD ANALYZED	2	TO	150
NAME OF THE SERIES			DZ
EFFECTIVE NUMBER OF OBSERVATIONS . . .			149
STANDARD DEVIATION OF THE SERIES . . .			1.1519
MEAN OF THE (DIFFERENCED) SERIES . . .			.0524
STANDARD DEVIATION OF THE MEAN			.0944
T-VALUE OF MEAN (AGAINST ZERO)			.5549

AUTOCORRELATIONS

1- 12	-.47	.02	.08	.01	-.14	.08	.03	-.03	-.04	.03	.07	-.03
ST.E.	.08	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10
Q	33.2	33.2	34.3	34.3	37.4	38.5	38.6	38.7	39.0	39.1	39.9	40.0



#The first difference series is obviously stationary and its SACF indicates that a MA(1) model would be appropriate.

PARTIAL AUTOCORRELATIONS

[illegible]

-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
+---+---+---+---+---+---+---+---+---+---+---+

		I	
1	-.47	XXXXXXXXXX+XXXI	+
2	-.26	XXX+XXXI	+
3	-.04	+ XI	+
4	.06	+ IXX	+
5	-.11	+XXXI	+
6	-.06	+ XXI	+
7	.03	+ IX	+
8	.05	+ IX	+
9	-.03	+ XI	+
10	-.06	+ XI	+
11	.07	+ IXX	+
12	.09	+ IXX	+

#The following command on the original Z series will produce the same SACF and SPACF for the differenced series.

```

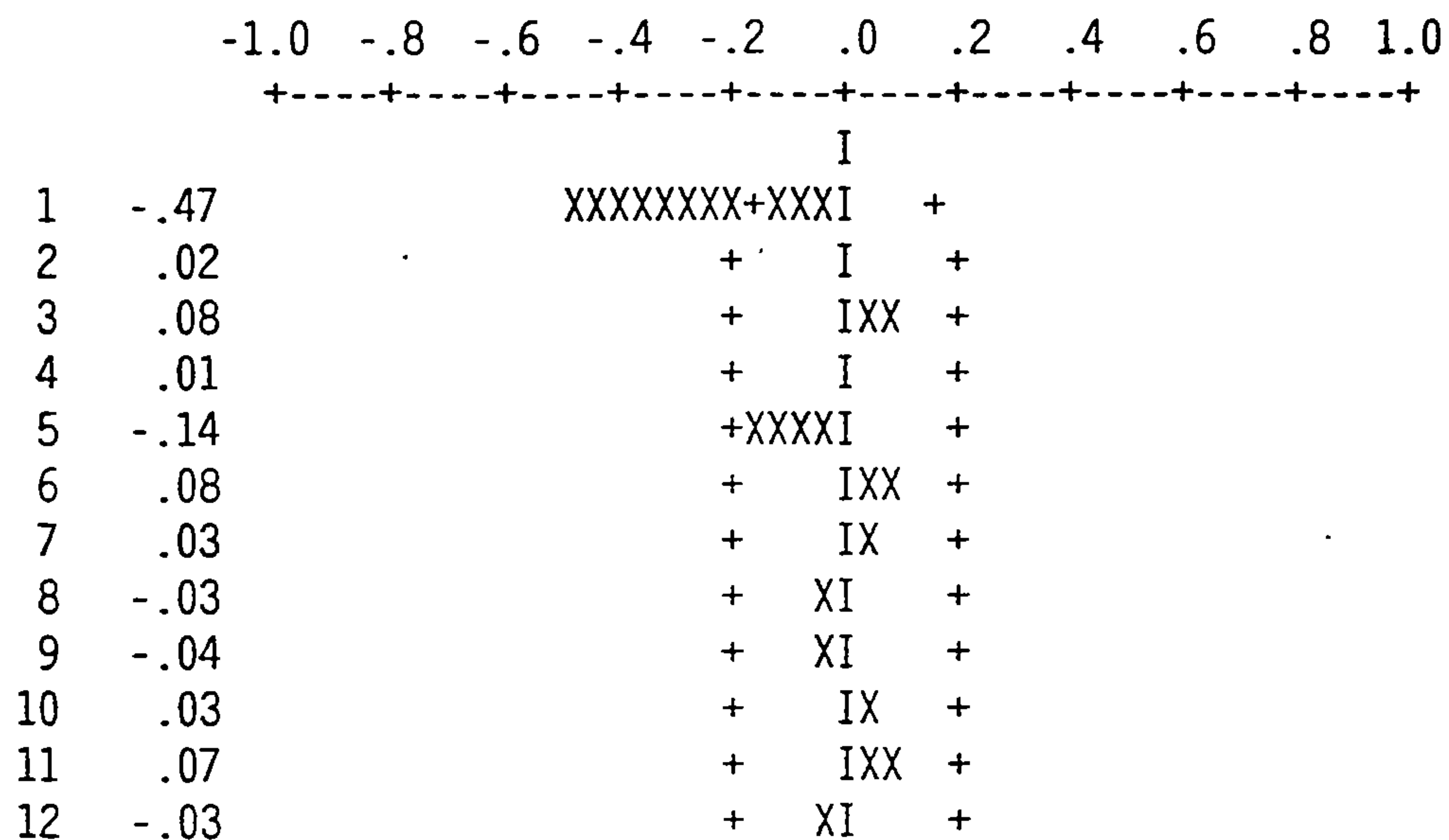
        uiden z.dfor 1.maxlag 12

```

	1	
DIFFERENCE ORDERS.	(1-B)	
TIME PERIOD ANALYZED	1 TO	150
NAME OF THE SERIES		Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .		149
STANDARD DEVIATION OF THE SERIES . . .		1.1519
MEAN OF THE (DIFFERENCED) SERIES . . .		.0524
STANDARD DEVIATION OF THE MEAN		.0944
T-VALUE OF MEAN (AGAINST ZERO)		.5549

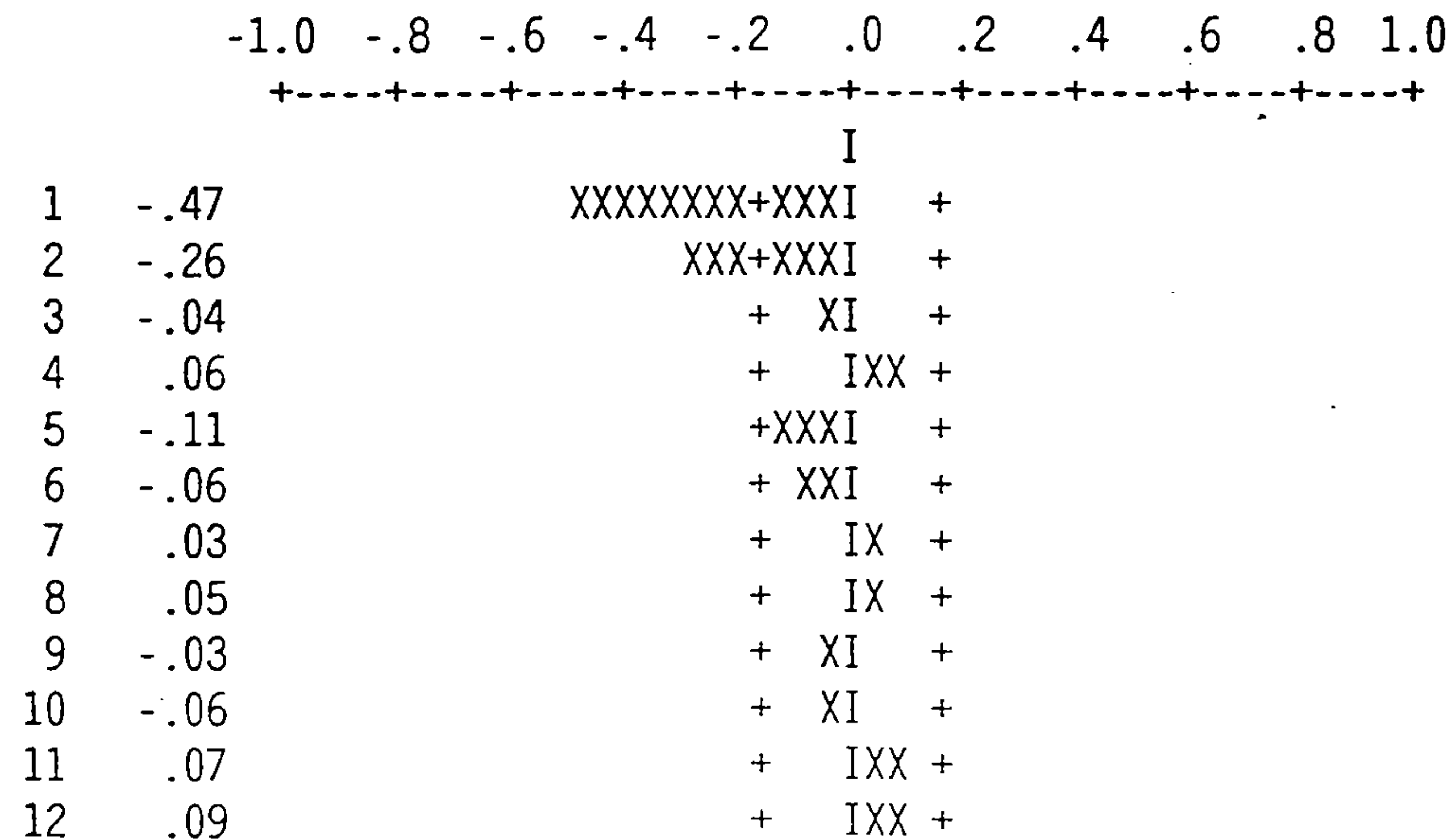
AUTOCORRELATIONS

1- 12	-.47	.02	.08	.01	-.14	.08	.03	-.03	-.04	.03	.07	-.03
ST.E.	.08	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10
Q	33.2	33.2	34.3	34.3	37.4	38.5	38.6	38.7	39.0	39.1	39.9	40.0



PARTIAL AUTOCORRELATIONS

1- 12	-.47	-.26	-.04	.06	-.11	-.06	.03	.05	-.03	-.06	.07	.09
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08



eacf z

```

TIME PERIOD ANALYZED . . . . . 1 TO 150
NAME OF THE SERIES . . . . . Z
EFFECTIVE NUMBER OF OBSERVATIONS . . . 150
STANDARD DEVIATION OF THE SERIES . . . 3.9726
MEAN OF THE (DIFFERENCED) SERIES . . . 10.9601
STANDARD DEVIATION OF THE MEAN . . . .3244
T-VALUE OF MEAN (AGAINST ZERO) . . . .33.7901

```

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12

(P= 0)	.95	.94	.93	.91	.89	.89	.87	.86	.84	.83	.81	.79	.77
(P= 1)	-.47	.01	.08	.01	-.14	.08	.03	-.03	-.04	.03	.07	-.03	-.05
(P= 2)	-.44	.07	.07	.00	-.16	.08	.09	-.02	-.08	.01	.06	-.02	-.03
(P= 3)	-.14	.22	-.13	.04	-.09	.07	-.02	.01	.01	.00	.01	.03	-.04
(P= 4)	.36	.01	-.22	-.23	-.04	.01	-.02	.00	.00	.01	-.01	.01	-.05
(P= 5)	.46	.09	-.36	-.24	-.12	.00	-.02	.01	.03	-.01	.00	.01	-.05
(P= 6)	-.37	.34	.16	-.08	-.02	.01	-.01	.00	.00	-.00	-.00	.01	-.05

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12

(P= 0)	X	X	X	X	X	X	X	X	X	X	X	X	X
(P= 1)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 2)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 3)	0	0	0	0	0	0	0	0	0	0	0	0	0
(P= 4)	X	0	X	X	0	0	0	0	0	0	0	0	0
(P= 5)	X	0	X	X	0	0	0	0	0	0	0	0	0
(P= 6)	X	X	0	0	0	0	0	0	0	0	0	0	0

--
#The usefulness of the EACF for this example will be discussed later. We now turn to model estimation.

```
tsmodel mx.model z(1)=(1)noise
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MX

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
				1				
Z	RANDOM	ORIGINAL	(1-B	)				
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRRAINT	VALUE	STD ERROR	T VALUE
1	Z	MA	1	1	NONE	.1000		
--								

```
estim mx.method exact.hold resi(r1)
```

#Some details of the estimating process are omitted.

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MX

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
				1				
Z	RANDOM	ORIGINAL	(1-B	)				
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRRAINT	VALUE	STD ERROR	T VALUE
1	Z	MA	1	1	NONE	.5376	.0690	7.79

```
TOTAL SUM OF SQUARES . . . . . .236721E+04
TOTAL NUMBER OF OBSERVATIONS . . . . .150
RESIDUAL SUM OF SQUARES. . . . . .146172E+03
R-SQUARE . . . . . .937
EFFECTIVE NUMBER OF OBSERVATIONS . . .148
RESIDUAL VARIANCE ESTIMATE . . . . .987651E+00
RESIDUAL STANDARD ERROR. . . . . .993806E+00
```

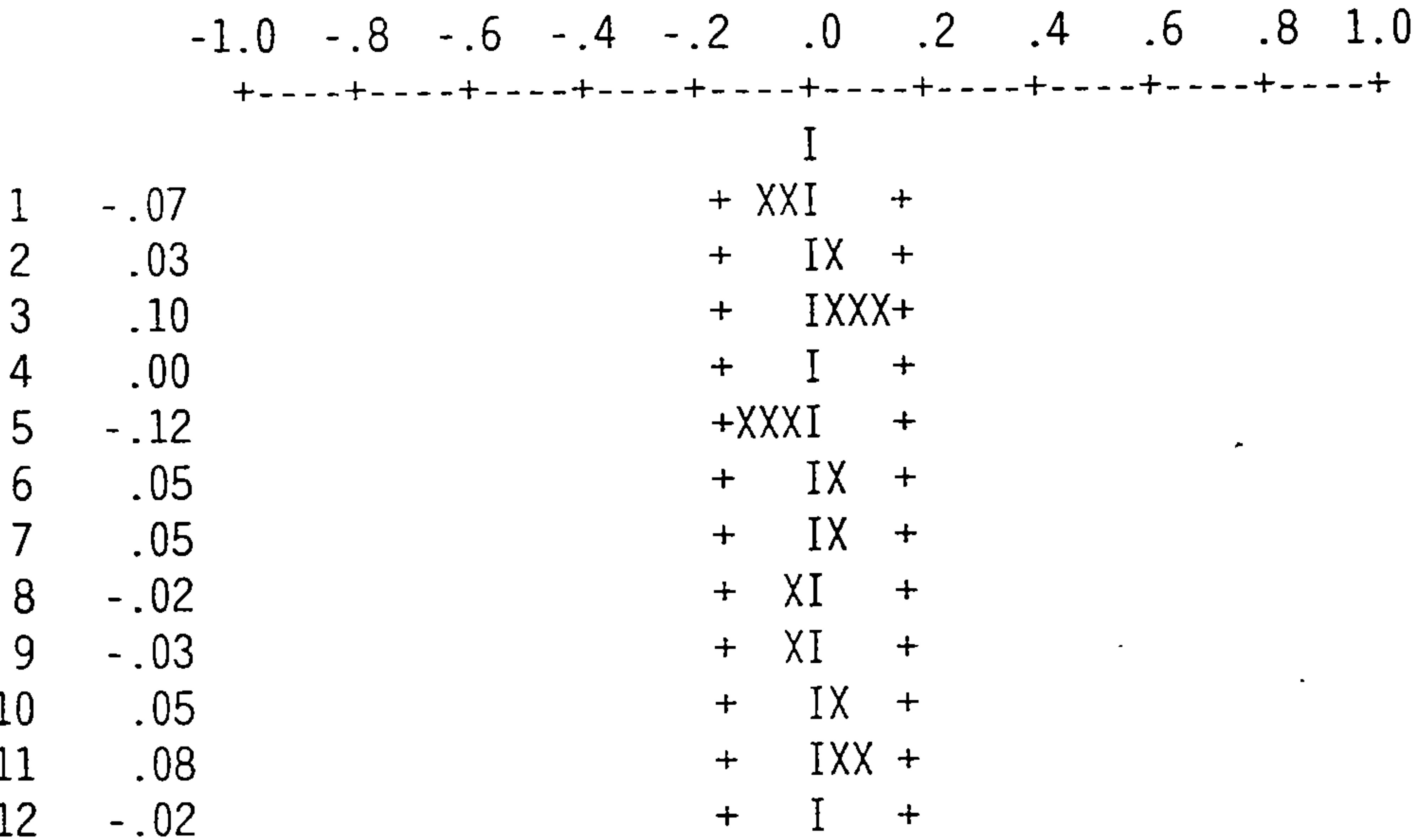
#Useful diagnostic checks of the adequacy of the model fit include examination of the SACF of the residuals and a time series plot of the residuals.

acf r1.maxlag 12

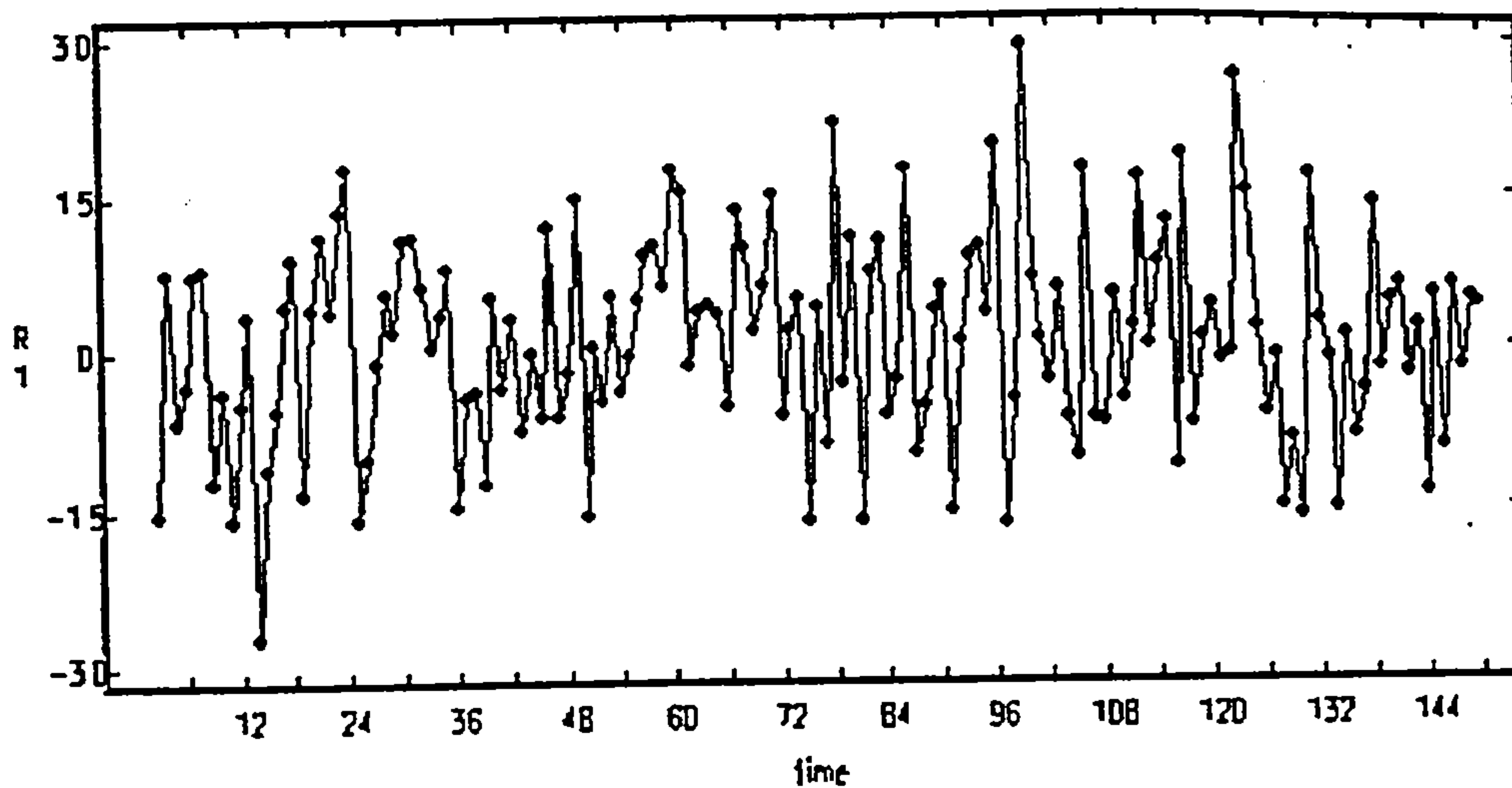
TIME PERIOD ANALYZED 3 TO 150
NAME OF THE SERIES R1
EFFECTIVE NUMBER OF OBSERVATIONS . . . 148
STANDARD DEVIATION OF THE SERIES 9874
MEAN OF THE (DIFFERENCED) SERIES 1076
STANDARD DEVIATION OF THE MEAN 0812
T-VALUE OF MEAN (AGAINST ZERO) 1.3253

AUTOCORRELATIONS

1- 12	-.07	.03	.10	.00	-.12	.05	.05	-.02	-.03	.05	.08	-.02
ST.E.	.08	.08	.08	.08	.08	.08	.08	.09	.09	.09	.09	.09
Q	.7	.8	2.3	2.3	4.6	5.0	5.5	5.6	5.7	6.1	7.0	7.1



graph r1



#We now illustrate forecasts for this model.

--
uforecast mx

NOTE: THE EXACT METHOD FOR COMPUTING RESIDUALS IS USED

24 FORECASTS, BEGINNING AT 150

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
151	16.2642	.9938	
152	16.2642	1.0949	
153	16.2642	1.1875	
154	16.2642	1.2733	
155	16.2642	1.3537	
156	16.2642	1.4296	
157	16.2642	1.5016	
158	16.2642	1.5704	
159	16.2642	1.6363	
160	16.2642	1.6996	
161	16.2642	1.7606	
162	16.2642	1.8196	
163	16.2642	1.8768	
164	16.2642	1.9322	
165	16.2642	1.9861	
166	16.2642	2.0386	

167	16.2642	2.0898
168	16.2642	2.1397
169	16.2642	2.1885
170	16.2642	2.2362
171	16.2642	2.2830
172	16.2642	2.3288
173	16.2642	2.3737
174	16.2642	2.4178

--
#We see that forecasts for all future periods are the same, but their standard errors keep increasing with time.

#In what follows we show an alternative way to build a model for this series. The EACF on pp. C-42 indicates that an ARMA(1,1) model might be appropriate.

$$Z_t - \phi Z_{t-1} = a_t - \theta a_{t-1}.$$

```
tsmodel mxx.model (1)z=(1)noise
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MXX

VARIABLE	TYPE OF	ORIGINAL	DIFFERENCING					
	VARIABLE	OR CENTERED						
Z	RANDOM	ORIGINAL	NONE					

PARAMETER	VARIABLE	NUM./	FACTOR	ORDER	CONS-	VALUE	STD	T
LABEL	NAME	DENOM.			TRAI NT		ERROR	VALUE
1	Z	MA	1	1	NONE	.1000		
2	Z	AR	1	1	NONE	.1000		

--
 estim mxx.hold resi(r2).method exact

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MXX

VARIABLE	TYPE OF	ORIGINAL	DIFFERENCING
	VARIABLE	OR CENTERED	
Z	RANDOM	ORIGINAL	NONE

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	Z	MA	1	1	NONE	.5582	.0676	8.26
2	Z	AR	1	1	NONE	1.0043	.0031	321.79

TOTAL SUM OF SQUARES 236721E+04
 TOTAL NUMBER OF OBSERVATIONS 150
 RESIDUAL SUM OF SQUARES. 144416E+03
 R-SQUARE 938
 EFFECTIVE NUMBER OF OBSERVATIONS . . .148
 RESIDUAL VARIANCE ESTIMATE 975781E+00
 RESIDUAL STANDARD ERROR. 987816E+00

--

#The fit is almost identical. In particular, the estimated ϕ is 1.0043 which is very close to $\phi = 1$ corresponding to differencing the data.

acf r2.maxlag 12

TIME PERIOD ANALYZED 3 TO 150
 NAME OF THE SERIES R2
 EFFECTIVE NUMBER OF OBSERVATIONS . . .148
 STANDARD DEVIATION OF THE SERIES 9875
 MEAN OF THE (DIFFERENCED) SERIES 0089
 STANDARD DEVIATION OF THE MEAN 0812
 T-VALUE OF MEAN (AGAINST ZERO) 1098

AUTOCORRELATIONS

1- 12	-.05	.04	.10	.00	-.12	.05	.05	-.02	-.03	.05	.07	-.02
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.09	.09	.09
Q	.3	.5	2.2	2.2	4.4	4.8	5.2	5.3	5.4	5.8	6.6	6.7

-1.0	-.8	-.6	-.4	-.2	.0	.2	.4	.6	.8	1.0
+-----+-----+-----+-----+-----+-----+-----+-----+-----+-----+										

						I						
1	-.05					+ XI	+					
2	.04					+ IX	+					
3	.10					+ IXXX						
4	.00					+ I	+					

5	-.12	+XXXI	+
6	.05	+ IX	+
7	.05	+ IX	+
8	-.02	+ XI	+
9	-.03	+ XI	+
10	.05	+ IX	+
11	.07	+ IXX	+
12	-.02	+ I	+
--			

stop

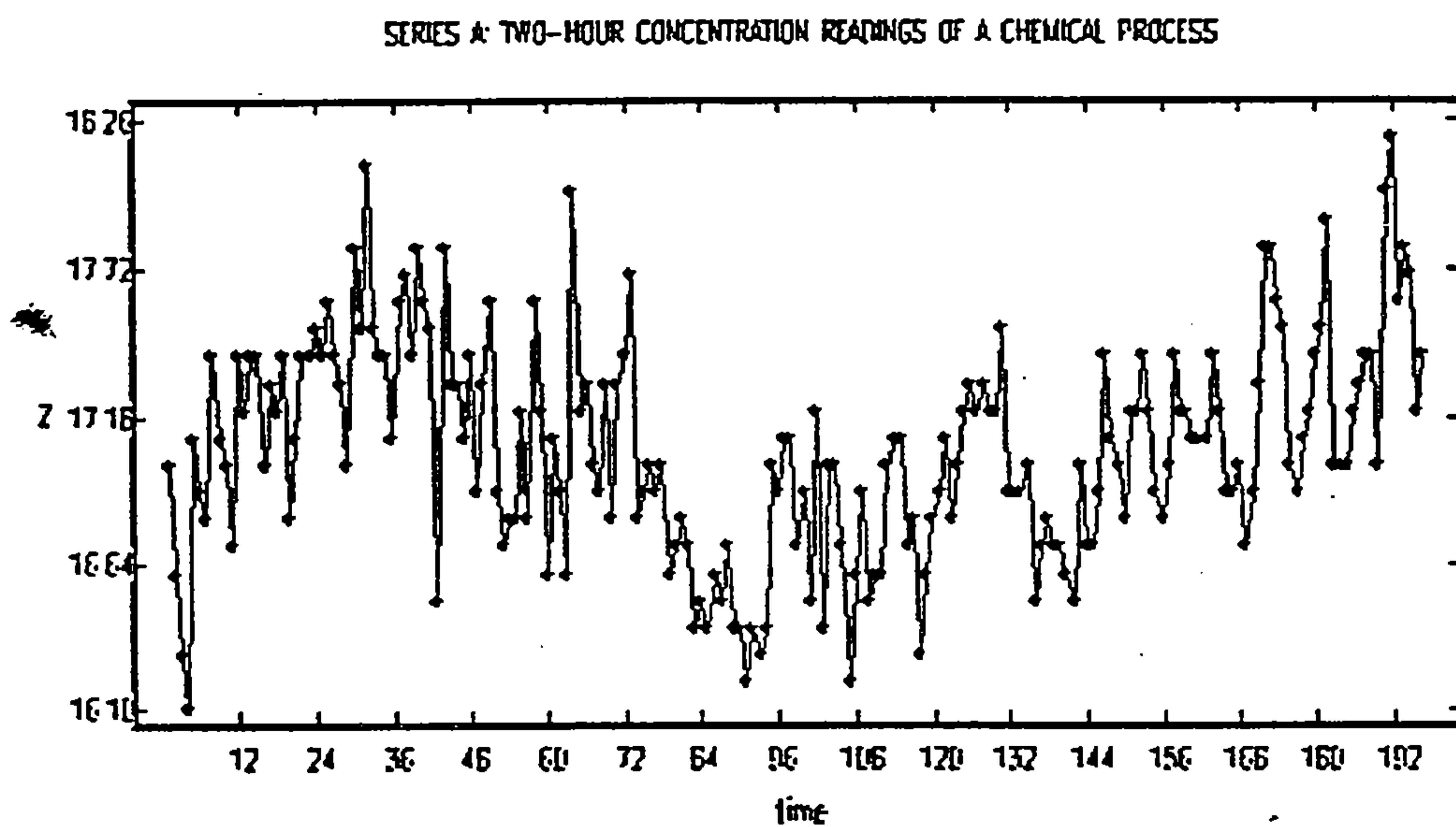
#EXAMPLE V: SERIES A OF BOX, JENKINS AND REINSEL

input z.file'c:\b325\bja_chmp.dat'

Z . A 197 BY 1 VARIABLE, IS STORED IN THE WORKSPACE

--

graph z

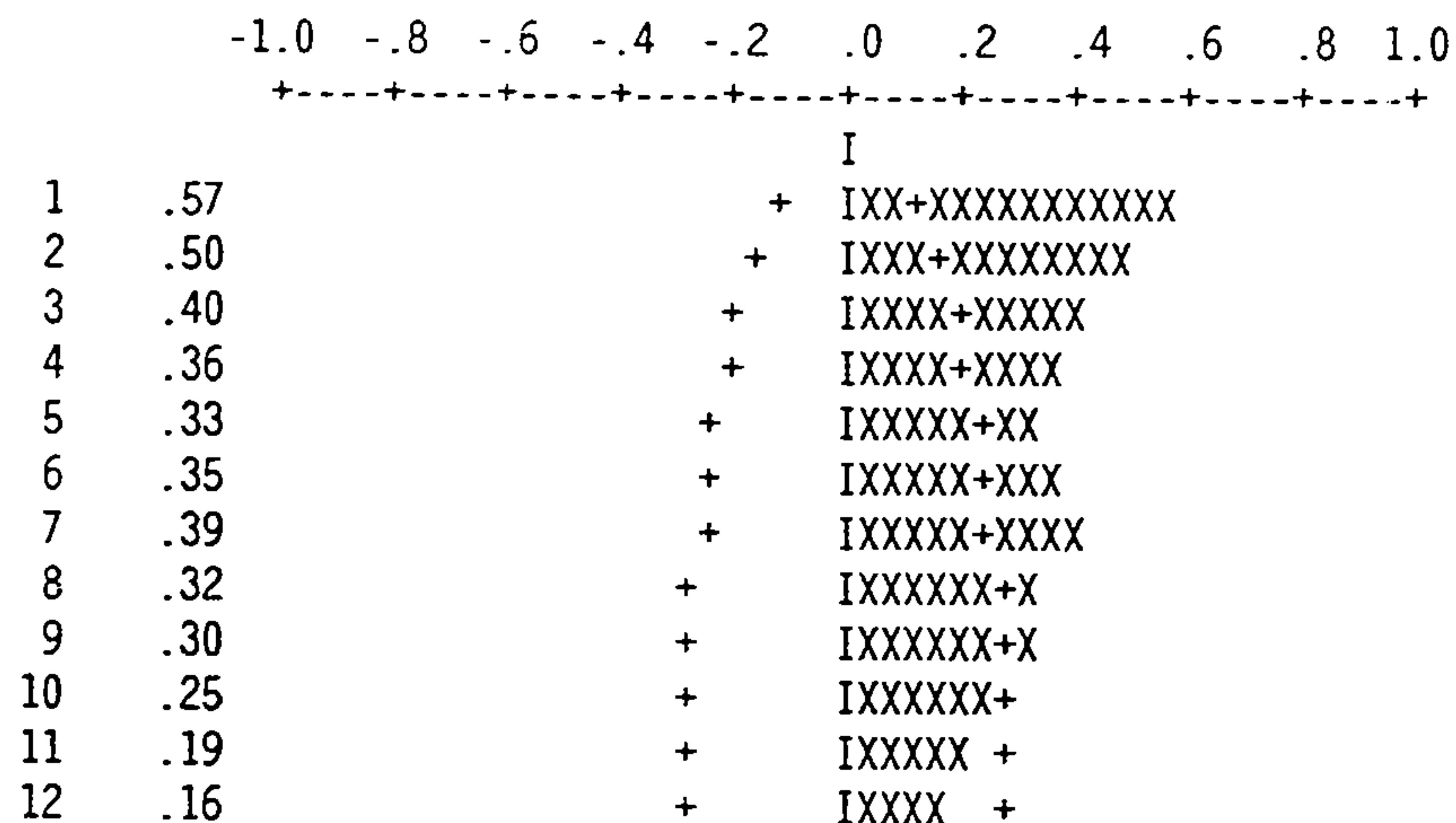


iden z.maxlag 12

TIME PERIOD ANALYZED	1 TO 197
NAME OF THE SERIES	Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .	197
STANDARD DEVIATION OF THE SERIES . . .	.3982
MEAN OF THE (DIFFERENCED) SERIES . . .	17.0624
STANDARD DEVIATION OF THE MEAN	.0284
T-VALUE OF MEAN (AGAINST ZERO)	601.3644

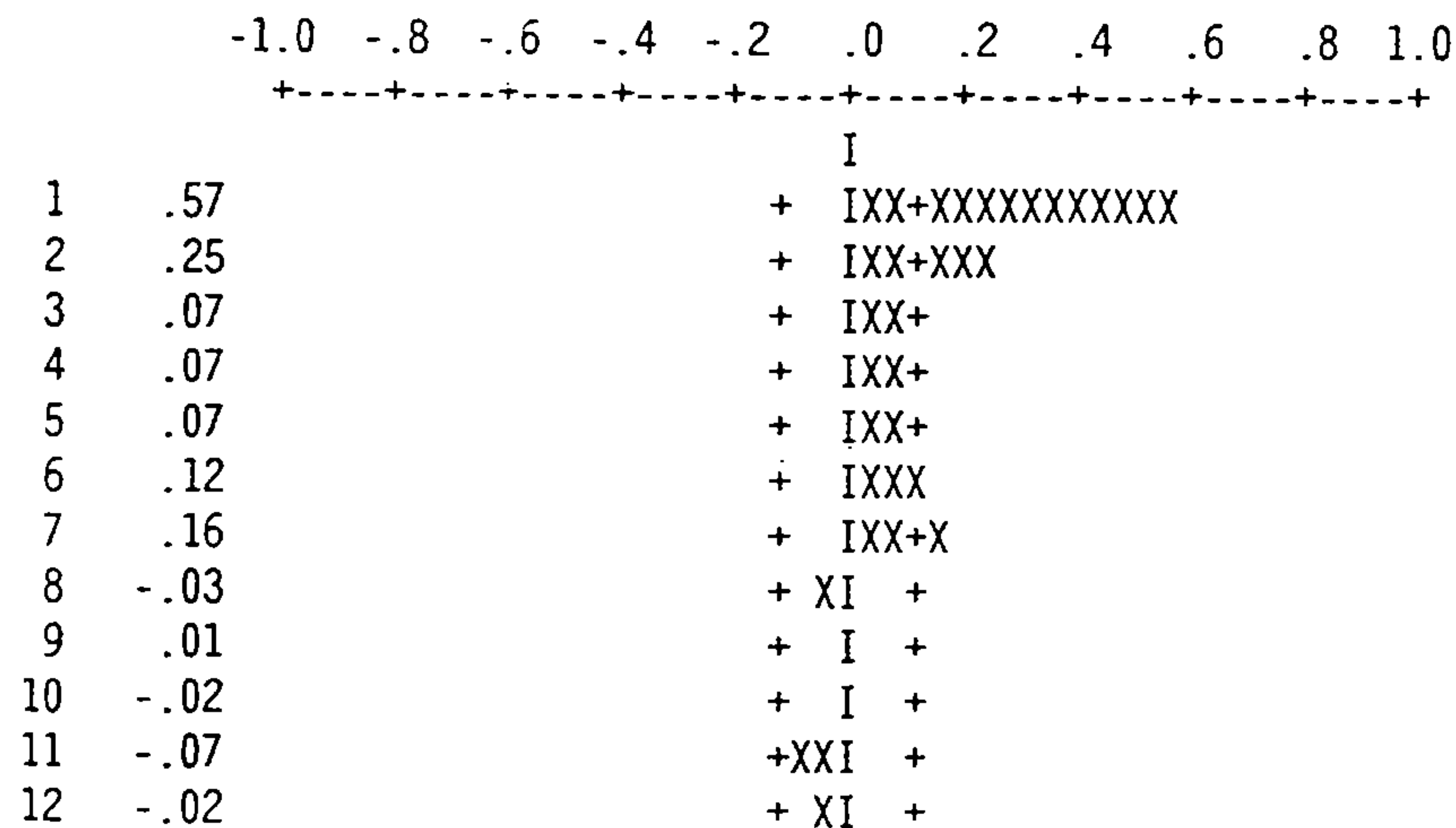
AUTOCORRELATIONS

1- 12	.57	.50	.40	.36	.33	.35	.39	.32	.30	.25	.19	.16
ST.E.	.07	.09	.10	.11	.12	.12	.13	.13	.14	.14	.14	.14
Q	65.0	114	146	172	194	219	251	272	291	305	312	318



PARTIAL AUTOCORRELATIONS

1- 12	.57	.25	.07	.07	.07	.12	.16	-.03	.01	-.02	-.07	-.02
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07



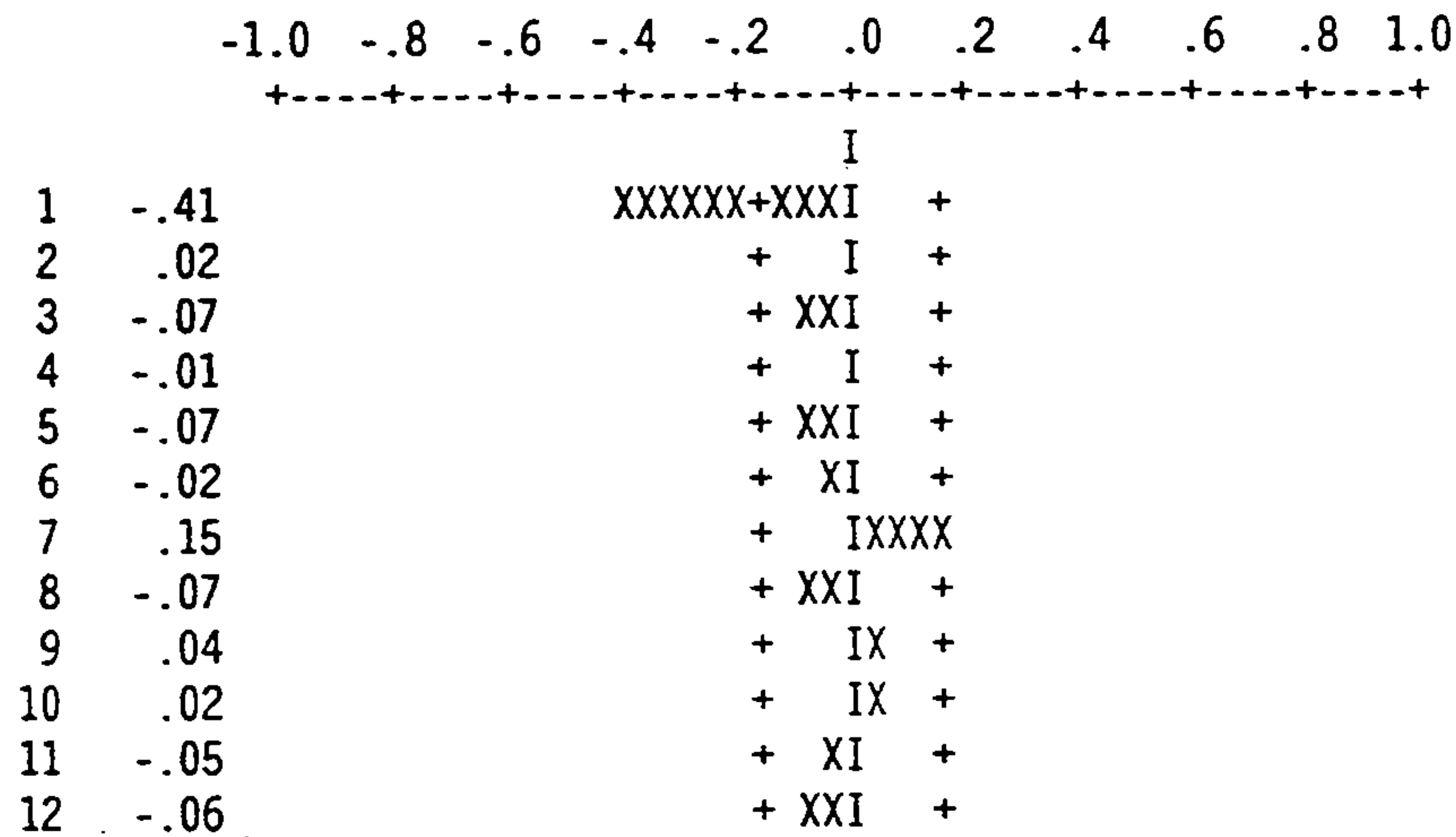
iden z.dfor 1.maxlag 12

DIFFERENCE ORDERS. (1-B)
 TIME PERIOD ANALYZED 1 TO 197

NAME OF THE SERIES	Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .	196
STANDARD DEVIATION OF THE SERIES . . .	.3694
MEAN OF THE (DIFFERENCED) SERIES . . .	.0020
STANDARD DEVIATION OF THE MEAN	.0264
T-VALUE OF MEAN (AGAINST ZERO)	.0774

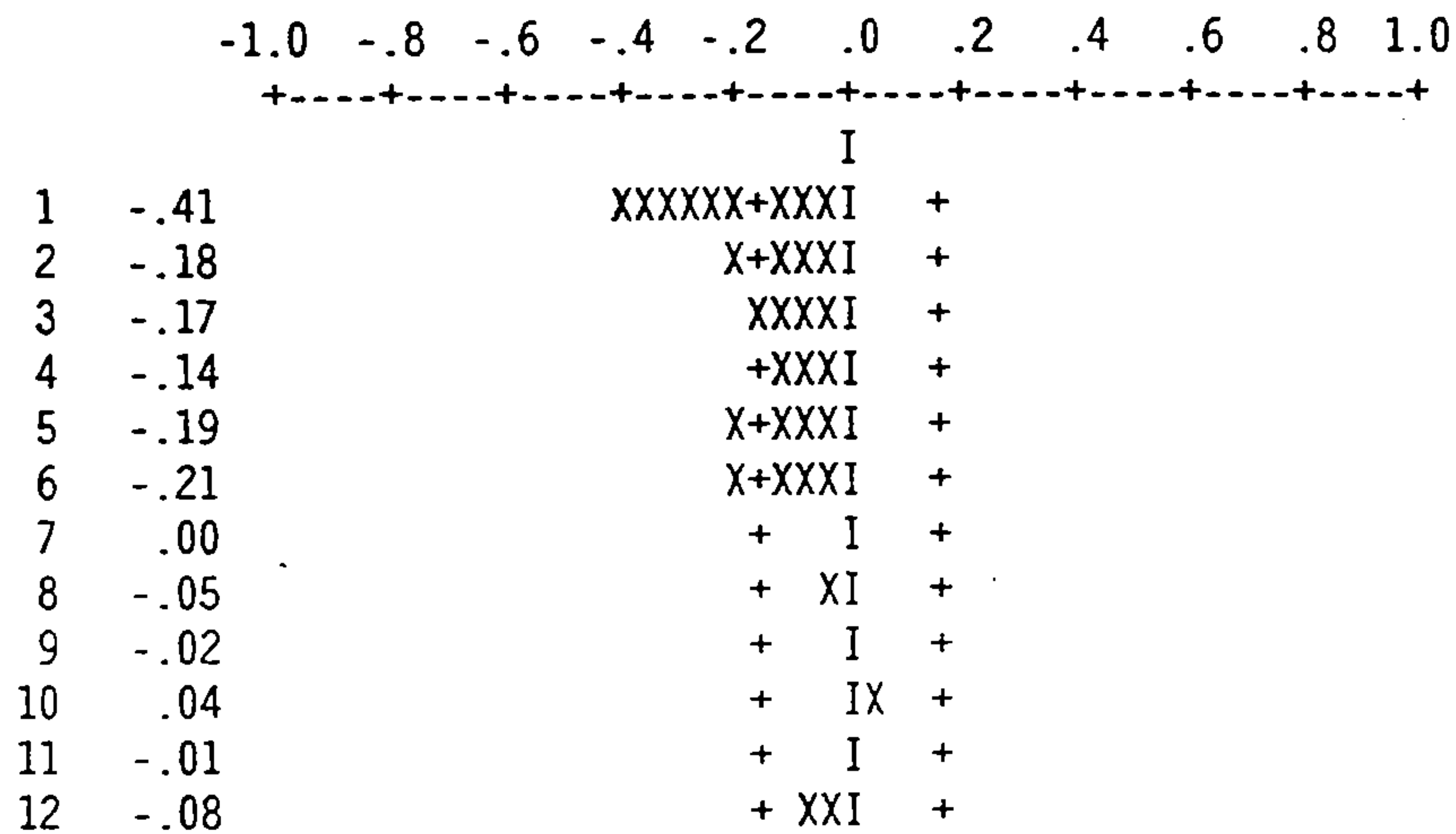
AUTOCORRELATIONS

1- 12	-.41	.02	-.07	-.01	-.07	-.02	.15	-.07	.04	.02	-.05	-.06
ST.E.	.07	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.09
Q	33.9	34.0	34.9	34.9	35.9	35.9	40.3	41.2	41.5	41.6	42.1	42.9



PARTIAL AUTOCORRELATIONS

1- 12	-.41	-.18	-.17	-.14	-.19	-.21	-.00	-.05	-.02	.04	-.01	-.08
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07



--
eacf z

TIME PERIOD ANALYZED 1 TO 197
 NAME OF THE SERIES Z
 EFFECTIVE NUMBER OF OBSERVATIONS . . . 197
 STANDARD DEVIATION OF THE SERIES 3982
 MEAN OF THE (DIFFERENCED) SERIES . . . 17.0624
 STANDARD DEVIATION OF THE MEAN 0284
 T-VALUE OF MEAN (AGAINST ZERO) 601.3644

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.57	.50	.40	.36	.33	.35	.39	.32	.30	.25	.19	.16	.19
(P= 1)	-.39	.04	-.06	-.01	-.07	-.01	.16	-.07	.04	.04	-.04	-.06	-.00
(P= 2)	-.29	-.27	-.04	.01	-.05	-.01	.17	.03	.04	.07	-.02	-.05	-.00
(P= 3)	-.50	-.01	.09	-.01	-.01	-.03	.16	-.03	.11	-.02	-.01	.01	-.06
(P= 4)	-.48	-.02	.08	-.02	-.01	-.04	.14	.03	.09	-.03	-.02	.00	-.08
(P= 5)	-.39	-.41	-.17	.01	-.17	-.02	.10	-.01	.06	.07	-.01	.01	-.06
(P= 6)	-.49	.15	-.18	-.00	-.26	-.06	.09	-.10	.05	.02	-.02	-.03	-.05

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	X	X	X	X	X	0	0	0	0
(P= 1)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 2)	X	X	0	0	0	0	X	0	0	0	0	0	0
(P= 3)	X	0	0	0	0	0	X	0	0	0	0	0	0
(P= 4)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 5)	X	X	0	0	0	0	0	0	0	0	0	0	0
(P= 6)	X	0	0	0	X	0	0	0	0	0	0	0	0

--
#The EACF pattern suggests that an (1,0,1) model might be appropriate.

tsmodel m1.model (1)z=c+(1)noise

estim m1.method exact.hold resi(r1)

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
Z	RANDOM	ORIGINAL	NONE

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAIT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	1.4964	.7272	2.06
2	Z	MA	1	1	NONE	.5906	.0828	7.13
3	Z	AR	1	1	NONE	.9125	.0426	21.41

TOTAL SUM OF SQUARES 312420E+02
 TOTAL NUMBER OF OBSERVATIONS 197
 RESIDUAL SUM OF SQUARES. 190225E+02
 R-SQUARE 385
 EFFECTIVE NUMBER OF OBSERVATIONS . . .195
 RESIDUAL VARIANCE ESTIMATE 975511E-01
 RESIDUAL STANDARD ERROR. 312332E+00

--

acf r1.maxlag 12

TIME PERIOD ANALYZED 3 TO 197
 NAME OF THE SERIES R1
 EFFECTIVE NUMBER OF OBSERVATIONS . . .195
 STANDARD DEVIATION OF THE SERIES . . .3123
 MEAN OF THE (DIFFERENCED) SERIES . . .-.0015
 STANDARD DEVIATION OF THE MEAN 0224
 T-VALUE OF MEAN (AGAINST ZERO)-.0661

AUTOCORRELATIONS

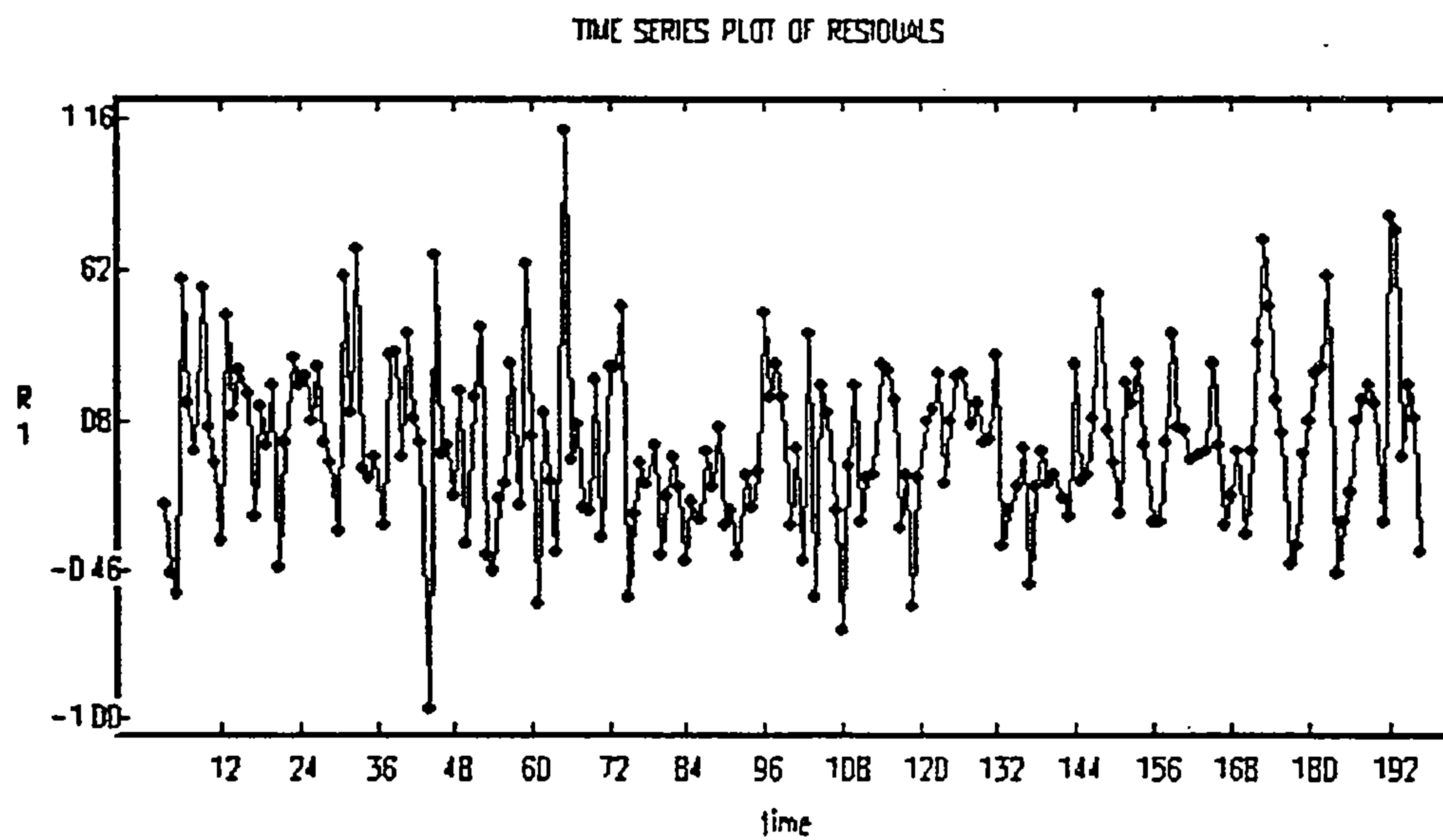
1- 12	.05	.00	-.09	-.08	-.09	.03	.17	.04	.06	.02	-.07	-.09
ST.E.	.07	.07	.07	.07	.07	.07	.07	.08	.08	.08	.08	.08
Q	.5	.5	2.0	3.4	5.0	5.1	10.9	11.3	12.1	12.3	13.3	15.0

-1.0 - .8 - .6 - .4 - .2 .0 .2 .4 .6 .8 1.0
 +-----+-----+-----+-----+-----+-----+

		I	
1	.05	+ IX +	
2	.00	+ I +	
3	-.09	+ XXI +	
4	-.08	+ XXI +	
5	-.09	+ XXI +	
6	.03	+ IX +	
7	.17	+ IXXXX	
8	.04	+ IX +	
9	.06	+ IXX +	
10	.02	+ IX +	
11	-.07	+ XXI +	
12	-.09	+ XXI +	

--

graph r1



outlier ml

INITIAL RESIDUAL STANDARD ERROR = .31151

TIME	ESTIMATE	T-VALUE	TYPE
64	1.12	3.72	I0
43	-1.00	-3.68	A0

ADJUSTED RESIDUAL STANDARD ERROR = .29089

--

uforest ml.origin 180

NOTE: THE EXACT METHOD FOR COMPUTING RESIDUALS IS USED

24 FORECASTS, BEGINNING AT 180

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
181	17.2239	.3123	17.5000
182	17.2133	.3281	17.9000
183	17.2037	.3407	17.0000
184	17.1949	.3508	17.0000
185	17.1869	.3591	17.0000
186	17.1796	.3658	17.2000
187	17.1729	.3713	17.3000
188	17.1668	.3758	17.4000
189	17.1613	.3795	17.4000
190	17.1562	.3826	17.0000
191	17.1516	.3851	18.0000
192	17.1474	.3872	18.2000
193	17.1435	.3889	17.6000
194	17.1400	.3904	17.8000
195	17.1368	.3916	17.7000
196	17.1339	.3926	17.2000
197	17.1312	.3934	17.4000
198	17.1287	.3941	
199	17.1265	.3946	
200	17.1245	.3951	
201	17.1226	.3955	
202	17.1209	.3958	
203	17.1194	.3961	
204	17.1180	.3963	

--

tsmodel m2.model z(1)=(1)noise

estim m2.method exact.hold resi(r2)

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M2

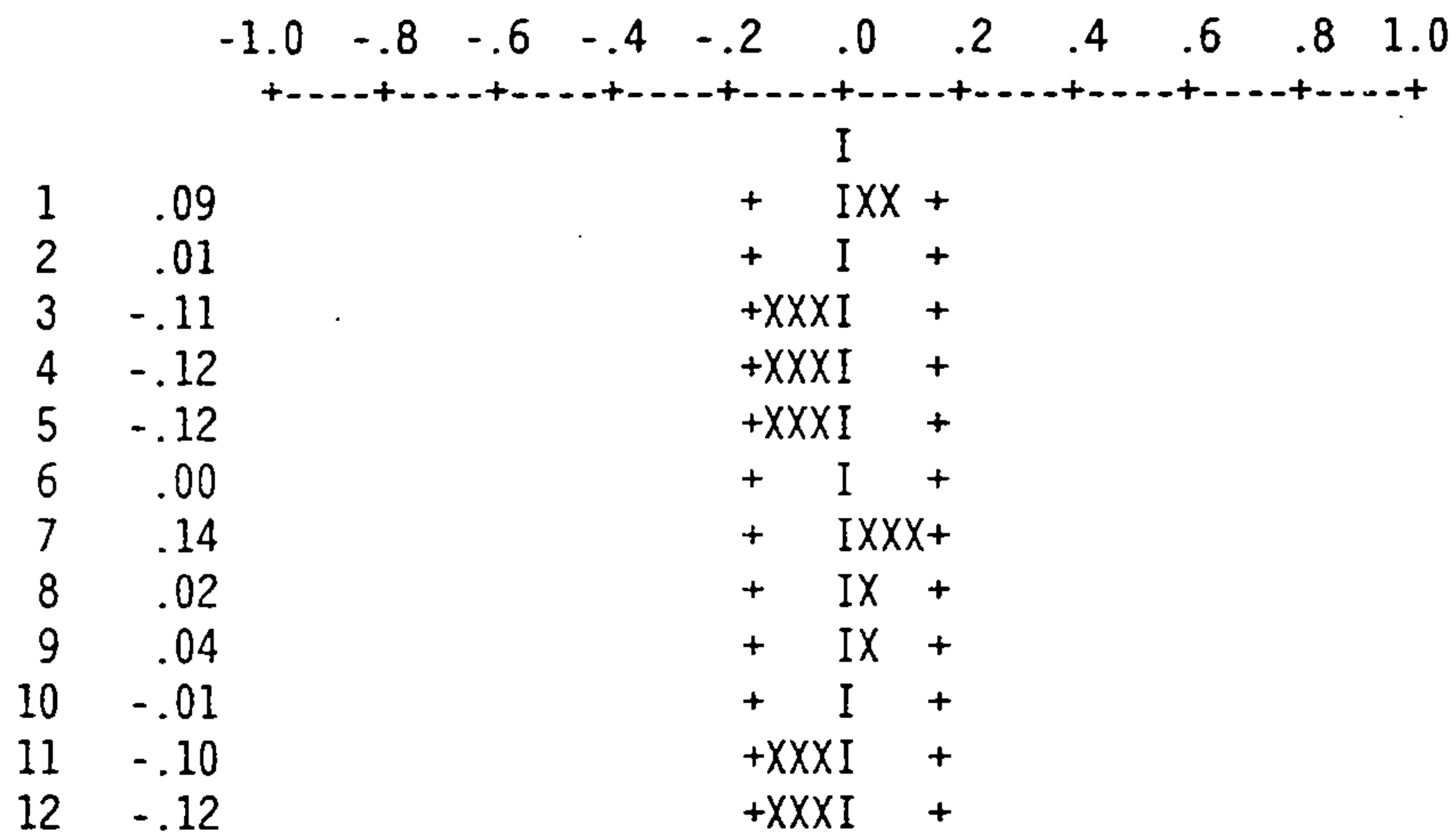
VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
			1					
Z	RANDOM	ORIGINAL	(1-B)					
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	Z	MA	1	1	NONE	.6992	.0512	13.65
TOTAL SUM OF SQUARES				.312420E+02				
TOTAL NUMBER OF OBSERVATIONS				197				
RESIDUAL SUM OF SQUARES.				.196427E+02				
R-SQUARE				.365				
EFFECTIVE NUMBER OF OBSERVATIONS . .				195				
RESIDUAL VARIANCE ESTIMATE				.100732E+00				
RESIDUAL STANDARD ERROR.				.317383E+00				

--
acf r2.maxlag 12

TIME PERIOD ANALYZED 3 TO 197
NAME OF THE SERIES R2
EFFECTIVE NUMBER OF OBSERVATIONS . . . 195
STANDARD DEVIATION OF THE SERIES 3172
MEAN OF THE (DIFFERENCED) SERIES 0126
STANDARD DEVIATION OF THE MEAN 0227
T-VALUE OF MEAN (AGAINST ZERO) 5531

AUTOCORRELATIONS

1- 12	.09	.01	-.11	-.12	-.12	.00	.14	.02	.04	-.01	-.10	-.12
ST.E.	.07	.07	.07	.07	.07	.07	.07	.08	.08	.08	.08	.08
Q	1.6	1.6	3.9	6.7	9.6	9.6	13.6	13.7	14.0	14.0	16.2	19.5



--

uforecast m2.origin 180

NOTE: THE EXACT METHOD FOR COMPUTING RESIDUALS IS USED

24 FORECASTS, BEGINNING AT 180

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
181	17.2479	.3174	17.5000
182	17.2479	.3314	17.9000
183	17.2479	.3449	17.0000
184	17.2479	.3579	17.0000
185	17.2479	.3704	17.0000
186	17.2479	.3825	17.2000
187	17.2479	.3942	17.3000
188	17.2479	.4056	17.4000
189	17.2479	.4167	17.4000
190	17.2479	.4275	17.0000
191	17.2479	.4381	18.0000
192	17.2479	.4483	18.2000
193	17.2479	.4584	17.6000
194	17.2479	.4682	17.8000
195	17.2479	.4779	17.7000
196	17.2479	.4873	17.2000
197	17.2479	.4966	17.4000
198	17.2479	.5057	
199	17.2479	.5146	
200	17.2479	.5234	
201	17.2479	.5320	
202	17.2479	.5405	
203	17.2479	.5489	
204	17.2479	.5571	

--

#In the above, we have fitted two models to the data:

$$(1) Z_t - .91Z_{t-1} = 1.496 + a_t - .59a_{t-1}$$

$$(2) Z_t - Z_{t-1} = a_t - .70a_{t-1}$$

The first one is a stationary ARIMA(1,0,1) model, but the second is a nonstationary ARIMA(0,1,1) model. The fits are very close. However, we see from the forecasts made at the origin 180 that for the stationary model the forecasted values gradually converge toward 17.06, the mean of the series. On the other hand, for the nonstationary model they stay constant at 17.2479. It has been shown in the literature that for long horizon forecasts, the stationary model does a better job for this series.

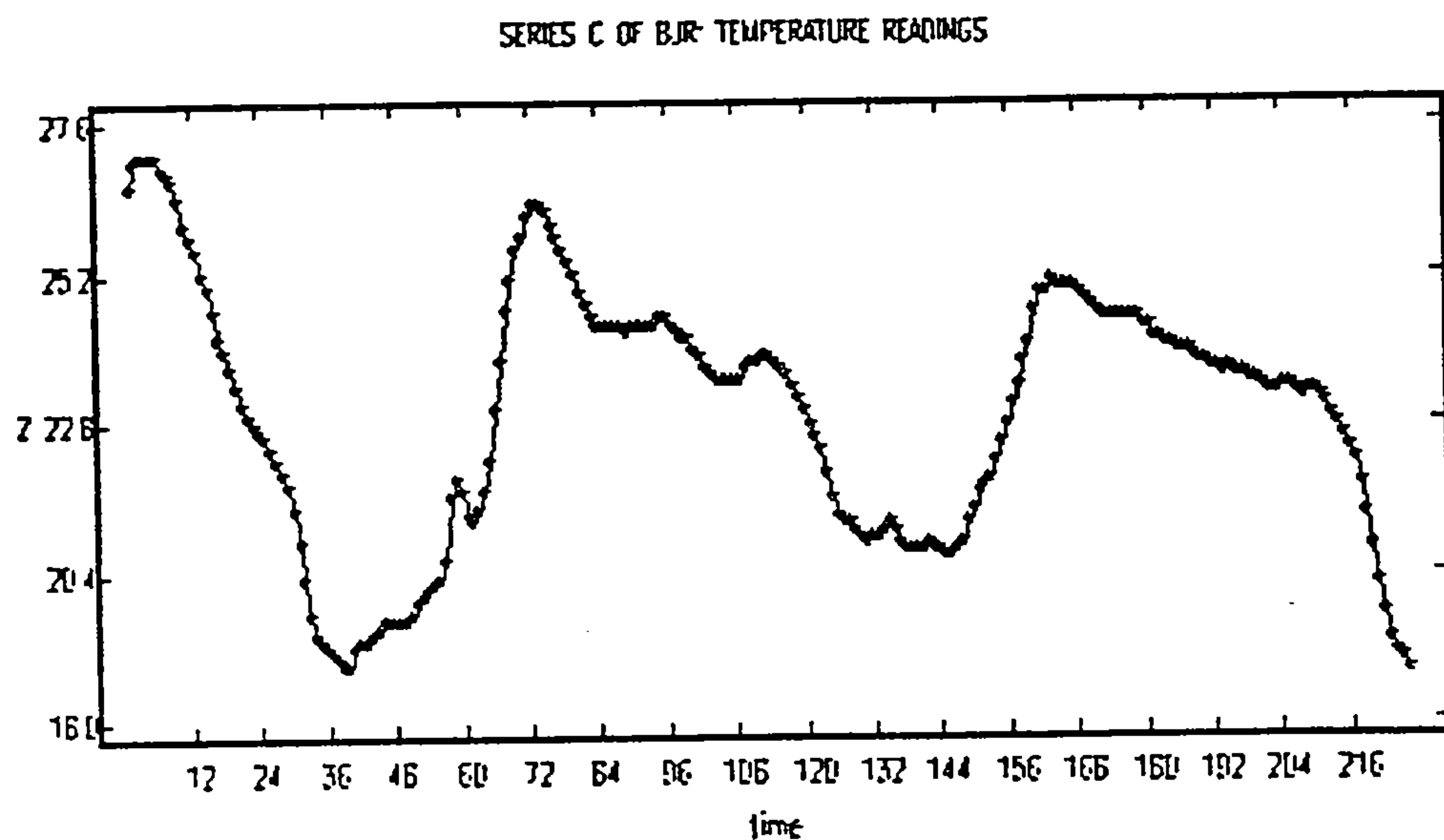
#EXAMPLE VI: SERIES C OF BOX, JENKINS AND REINSEL

input z.file'c:\b325\bjc_temp.dat'

Z , A 226 BY 1 VARIABLE, IS STORED IN THE WORKSPACE

--

graph z



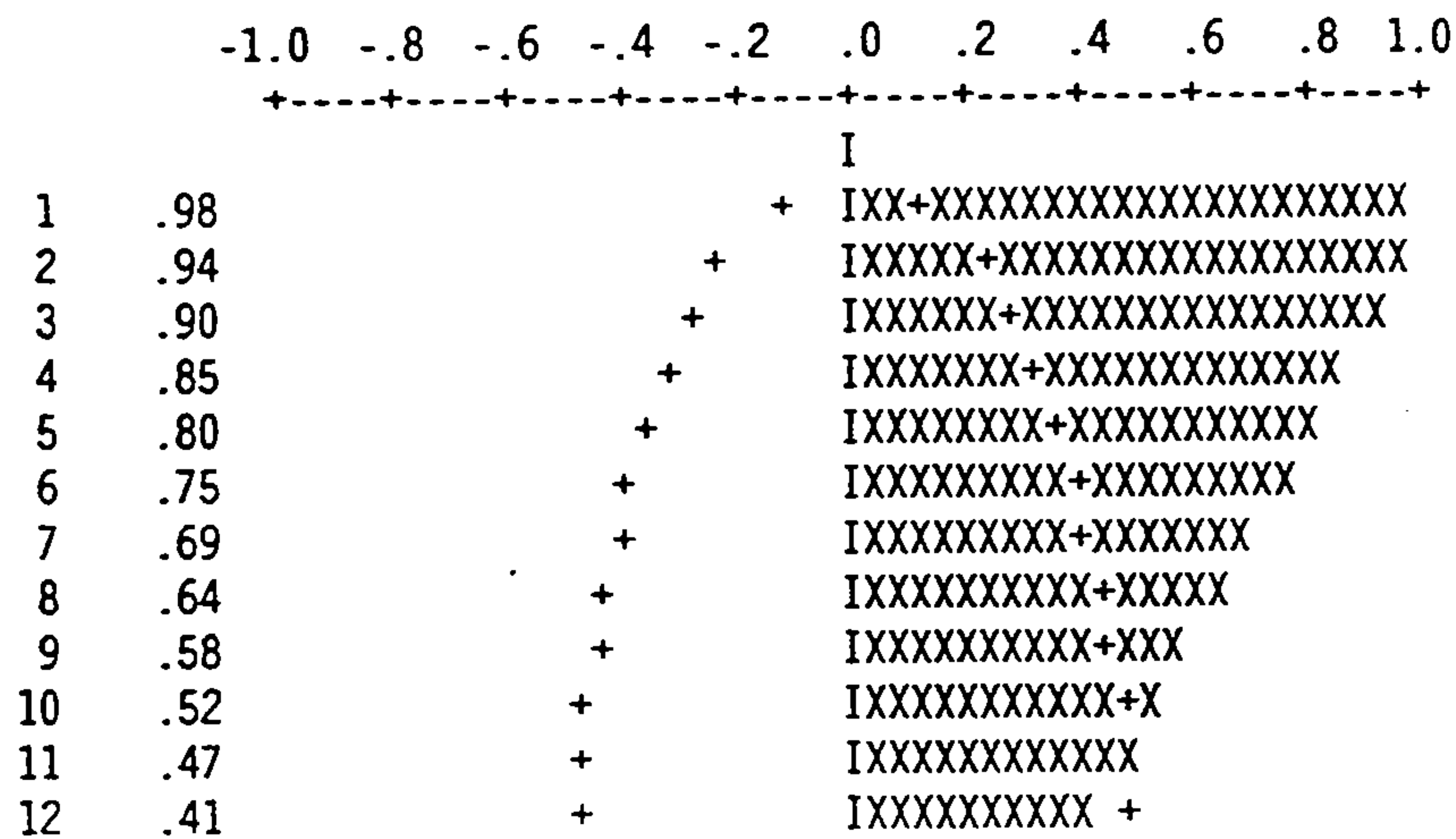
--

iden z.maxlag 12

TIME PERIOD ANALYZED	1 TO 226
NAME OF THE SERIES	Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .	226
STANDARD DEVIATION OF THE SERIES . . .	2.0549
MEAN OF THE (DIFFERENCED) SERIES . . .	22.9739
STANDARD DEVIATION OF THE MEAN	.1367
T-VALUE OF MEAN (AGAINST ZERO)	168.0709

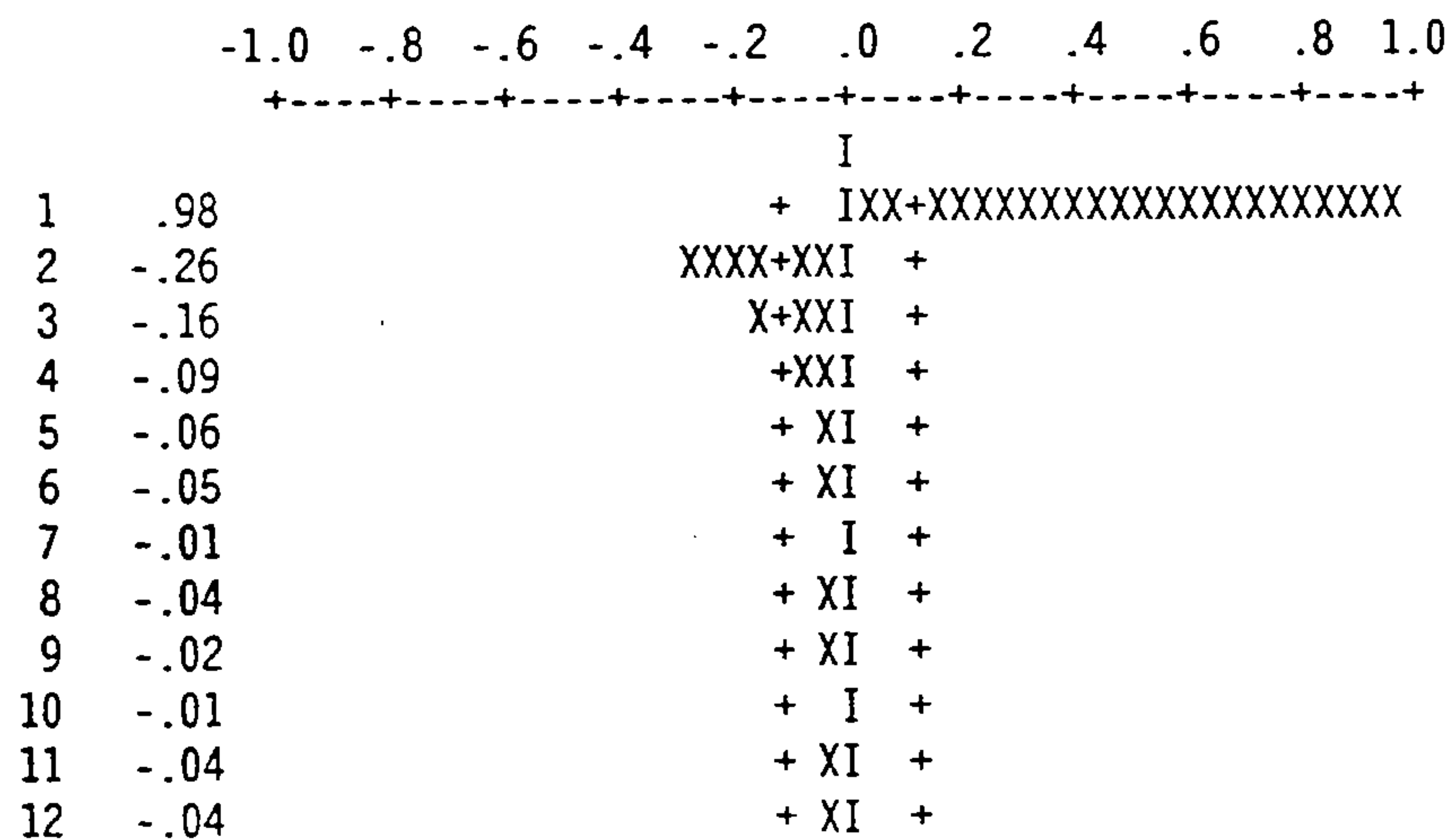
AUTOCORRELATIONS

1- 12	.98	.94	.90	.85	.80	.75	.69	.64	.58	.52	.47	.41
ST.E.	.07	.11	.14	.17	.19	.20	.21	.22	.23	.24	.24	.25
Q	219	424	612	781	931	1062	1175	1270	1350	1415	1468	1509



PARTIAL AUTOCORRELATIONS

1- 12	.98	-.26	-.16	-.09	-.06	-.05	-.01	-.04	-.02	-.01	-.04	-.04
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07



--
iden z.dfor 1.maxlag 12

DIFFERENCE ORDERS.	(1-B)
TIME PERIOD ANALYZED	1 TO 226
NAME OF THE SERIES	Z
EFFECTIVE NUMBER OF OBSERVATIONS	225
STANDARD DEVIATION OF THE SERIES	.2306
MEAN OF THE (DIFFERENCED) SERIES	-.0347
STANDARD DEVIATION OF THE MEAN	.0154
T-VALUE OF MEAN (AGAINST ZERO)	-2.2545

1- 12	.81	.65	.53	.44	.38	.32	.26	.19	.14	.14	.10	.09
ST.E.	.07	.10	.12	.13	.13	.14	.14	.14	.15	.15	.15	.15
Q	148	245	309	354	388	411	427	436	440	445	447	449

[illegible]

```

TIME PERIOD ANALYZED . . . . . 1 TO 226
NAME OF THE SERIES . . . . . Z
EFFECTIVE NUMBER OF OBSERVATIONS . . . 226
STANDARD DEVIATION OF THE SERIES . . . 2.0549

```

MEAN OF THE (DIFFERENCED) SERIES . . . 22.9739
 STANDARD DEVIATION OF THE MEAN 1367
 T-VALUE OF MEAN (AGAINST ZERO) 168.0709

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.98	.94	.90	.85	.80	.75	.69	.64	.58	.52	.47	.41	.36
(P= 1)	.81	.66	.55	.48	.43	.38	.34	.28	.25	.25	.22	.22	.20
(P= 2)	-.04	-.03	-.12	-.06	.02	-.01	.07	-.04	-.12	.13	-.12	.08	-.08
(P= 3)	-.50	.01	-.07	-.11	-.01	-.00	.03	-.03	-.10	.01	-.05	-.03	-.06
(P= 4)	-.25	-.27	-.05	-.11	-.01	.03	.00	-.02	-.09	-.01	-.04	.02	-.06
(P= 5)	-.48	.28	-.29	-.07	.04	-.05	-.00	-.01	-.08	.07	.00	-.01	-.00
(P= 6)	-.08	-.32	.14	.04	-.03	-.04	-.01	-.03	-.08	.07	.00	-.02	.00

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	X	X	X	X	X	X	X	X	X
(P= 1)	X	X	X	X	X	X	X	0	0	0	0	0	0
(P= 2)	0	0	0	0	0	0	0	0	0	0	0	0	0
(P= 3)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 4)	X	X	0	0	0	0	0	0	0	0	0	0	0
(P= 5)	X	X	X	0	0	0	0	0	0	0	0	0	0
(P= 6)	0	X	0	0	0	0	0	0	0	0	0	0	0

AR(1) diff

$$(1 - \phi B)(1 - B) Z_t = a_t$$

$$(1 - (1 + \phi)B + \phi B^2) Z_t = a_t$$

AR(2)

#From iden of Z_t and that of $(1-B)Z_t$, a (1,1,0) model is suggested. On the other hand the EACF pattern a (2,0,0) model seems appropriate. We will try an (2,0,0) model first.

tsm mz.model (1,2)z=c+noise.

estim mz.hold resi(r1)

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MZ

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
----------	------------------	----------------------	--------------

Z	RANDOM	ORIGINAL	NONE
---	--------	----------	------

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONSTRAINT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	.2766	.0999	2.77
2	Z	AR	1	1	NONE	1.8027	.0376	47.92
3	Z	AR	1	2	NONE	-.8151	.0377	-21.63

TOTAL SUM OF SQUARES 954336E+03
 TOTAL NUMBER OF OBSERVATIONS 226
 RESIDUAL SUM OF SQUARES. 385337E+01
 R-SQUARE 996
 EFFECTIVE NUMBER OF OBSERVATIONS 224
 RESIDUAL VARIANCE ESTIMATE 172026E-01
 RESIDUAL STANDARD ERROR. 131159E+00

--

acf r1.maxlag 12

TIME PERIOD ANALYZED 3 TO 226
 NAME OF THE SERIES R1
 EFFECTIVE NUMBER OF OBSERVATIONS 224
 STANDARD DEVIATION OF THE SERIES 1312
 MEAN OF THE (DIFFERENCED) SERIES 0000
 STANDARD DEVIATION OF THE MEAN 0088
 T-VALUE OF MEAN (AGAINST ZERO) 0000

AUTOCORRELATIONS

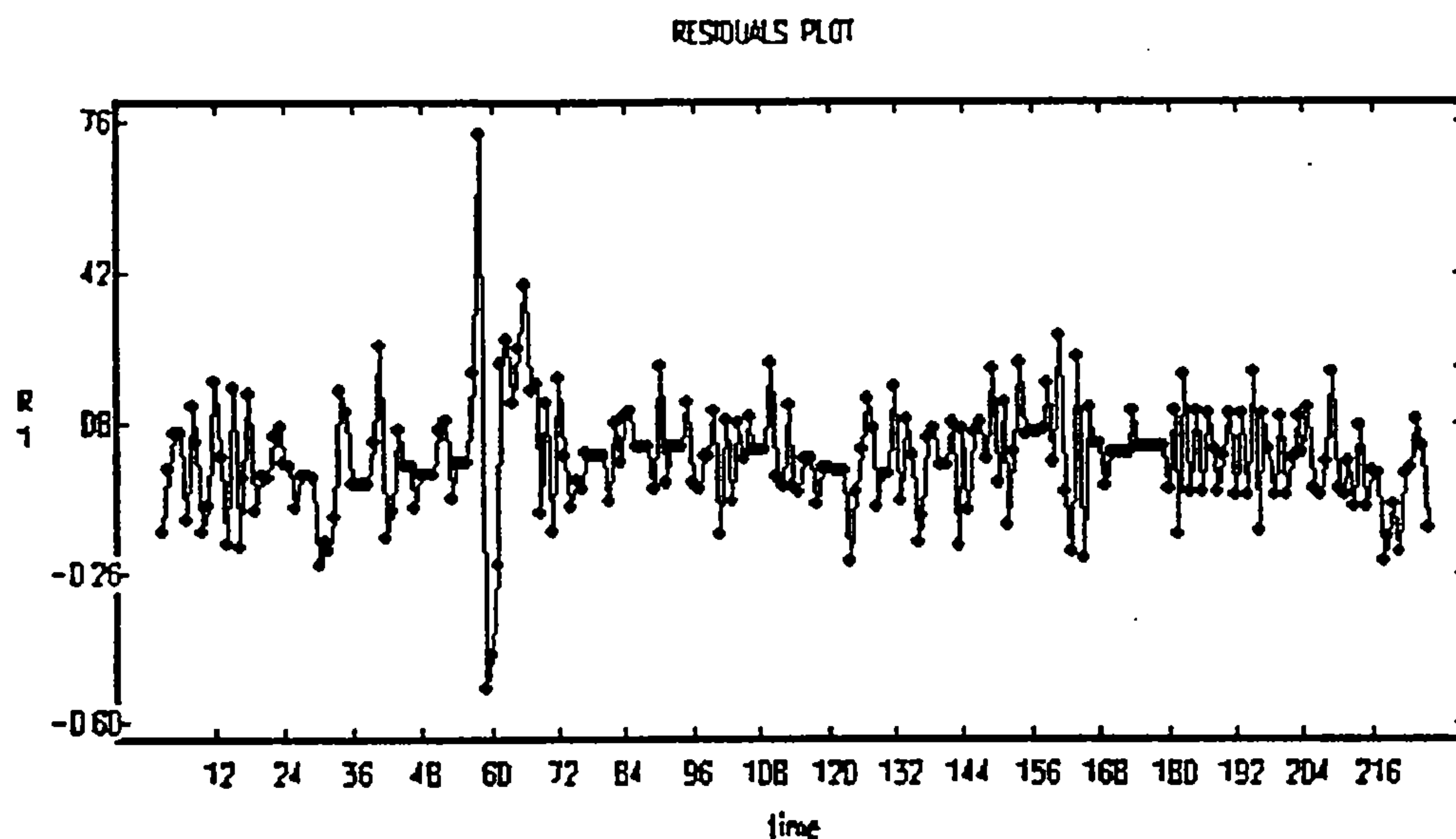
1- 12	-.01	-.00	-.06	-.01	.06	.02	.08	-.01	-.08	.14	-.09	.09
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07
Q	.1	.1	.9	.9	1.9	2.0	3.6	3.7	5.4	9.9	12.0	14.0

-1.0 -.8 -.6 -.4 -.2 .0 .2 .4 .6 .8 1.0
 +-----+-----+-----+-----+-----+

		I
1	-.01	+ I +
2	.00	+ I +
3	-.06	+XXI +
4	-.01	+ I +
5	.06	+ IXX+
6	.02	+ IX +
7	.08	+ IXX+
8	-.01	+ I +
9	-.08	+XXI +
10	.14	+ IXXX
11	-.09	+XXI +
12	.09	+ IXX+

--

graph r1



outlier mz

INITIAL RESIDUAL STANDARD ERROR = .13116

TIME	ESTIMATE	T-VALUE	TYPE
58	.73	6.03	I0
59	-.52	-4.48	I0
60	-.45	-3.98	I0
66	.39	3.57	I0
163	-.17	-3.39	A0

ADJUSTED RESIDUAL STANDARD ERROR = .10666

--

uforecast mz.origin 214

24 FORECASTS, BEGINNING AT 214

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
215	22.7117	.1312	22.6000
216	22.6341	.2704	22.4000
217	22.5661	.4184	22.2000
218	22.5069	.5672	21.8000

C-64

219	22.4554	.7117	21.3000
220	22.4110	.8491	20.8000
221	22.3728	.9775	20.2000
222	22.3403	1.0959	19.7000
223	22.3127	1.2039	19.3000
224	22.2895	1.3016	19.1000
225	22.2701	1.3893	19.0000
226	22.2542	1.4673	18.8000
227	22.2412	1.5363	
228	22.2307	1.5969	
229	22.2225	1.6499	
230	22.2162	1.6959	
231	22.2116	1.7355	
232	22.2083	1.7696	
233	22.2063	1.7987	
234	22.2052	1.8234	
235	22.2049	1.8443	
236	22.2053	1.8618	
237	22.2062	1.8764	
238	22.2076	1.8886	

--

tsm mz1.model (1)z(1)=c+noise

estim mz1

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MZ1

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING 1 (1-B)
Z	RANDOM	ORIGINAL	

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAI NT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	-.0092	.0090	-1.02
2	Z	AR	1	1	NONE	.8074	.0386	20.89

TOTAL SUM OF SQUARES 954336E+03
TOTAL NUMBER OF OBSERVATIONS 226
RESIDUAL SUM OF SQUARES 399529E+01
R-SQUARE 996
EFFECTIVE NUMBER OF OBSERVATIONS 224
RESIDUAL VARIANCE ESTIMATE 178361E-01
RESIDUAL STANDARD ERROR 133552E+00

--

#We now eliminate the constant term c.

```
tsm mz2.model (1)z(1)=noise
estim mz2.hold resi(r2)
```

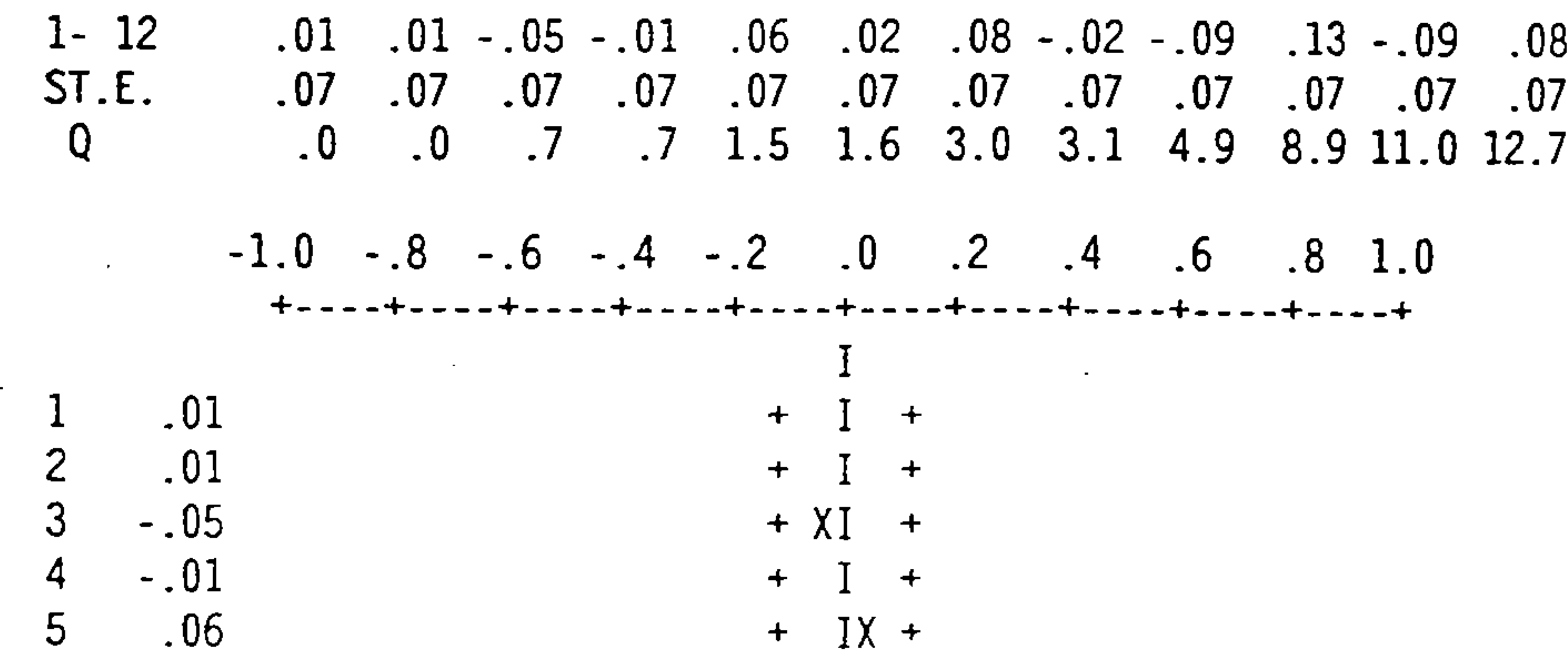
SUMMARY FOR UNIVARIATE TIME SERIES MODEL --- MZ2

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
Z	RANDOM	ORIGINAL	1 (1-B)					
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	Z	AR	1	1	NONE	.8131	.0383	21.22
TOTAL SUM OF SQUARES				.954336E+03				
TOTAL NUMBER OF OBSERVATIONS				226				
RESIDUAL SUM OF SQUARES.				.401391E+01				
R-SQUARE				.996				
EFFECTIVE NUMBER OF OBSERVATIONS . .				224				
RESIDUAL VARIANCE ESTIMATE				.179192E-01				
RESIDUAL STANDARD ERROR.				.133863E+00				

acf r2.maxlag 12

TIME PERIOD ANALYZED	3	TO	226
NAME OF THE SERIES	R2		
EFFECTIVE NUMBER OF OBSERVATIONS . . .	224		
STANDARD DEVIATION OF THE SERIES . . .	.1336		
MEAN OF THE (DIFFERENCED) SERIES . . .	-.0090		
STANDARD DEVIATION OF THE MEAN	.0089		
T-VALUE OF MEAN (AGAINST ZERO)	-1.0107		

AUTOCORRELATIONS



6	.02	+ IX +
7	.08	+ IXX+
8	-.02	+ I +
9	-.09	+XXI +
10	.13	+ IXXX
11	-.09	+XXI +
12	.08	+ IXX+

uforecast mz2.origin 214

24 FORECASTS, BEGINNING AT 214

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
215	22.7187	.1339	22.6000
216	22.6526	.2772	22.4000
217	22.5988	.4319	22.2000
218	22.5551	.5908	21.8000
219	22.5196	.7498	21.3000
220	22.4907	.9064	20.8000
221	22.4672	1.0592	20.2000
222	22.4481	1.2073	19.7000
223	22.4325	1.3504	19.3000
224	22.4199	1.4883	19.1000
225	22.4096	1.6212	19.0000
226	22.4013	1.7490	18.8000
227	22.3945	1.8721	
228	22.3890	1.9907	
229	22.3845	2.1049	
230	22.3808	2.2152	
231	22.3778	2.3216	
232	22.3754	2.4246	
233	22.3735	2.5242	
234	22.3719	2.6208	
235	22.3706	2.7144	
236	22.3695	2.8054	
237	22.3687	2.8939	
238	22.3680	2.9800	

--
#This example has demonstrated the usefulness of EACF in tentative model specification. Instead of judgmentally deciding whether or not to difference a series, EACF suggests fitting an AR(2), i.e. an (2,0,0), model directly. The fitting result is practically the same as that of an (1,1,0) model and short term predictions are also very close. For long horizon forecasts, these two model will, however, lead to different results.

#EXAMPLE VII. MONTHLY RATES OF THREE-MONTH T-BILL 1/81 TO 12/93

input date.z.file'c:\b325\3tb5693.dat'

DATE . A 456 BY 1 VARIABLE. IS STORED IN THE WORKSPACE

Z . A 456 BY 1 VARIABLE. IS STORED IN THE WORKSPACE

--

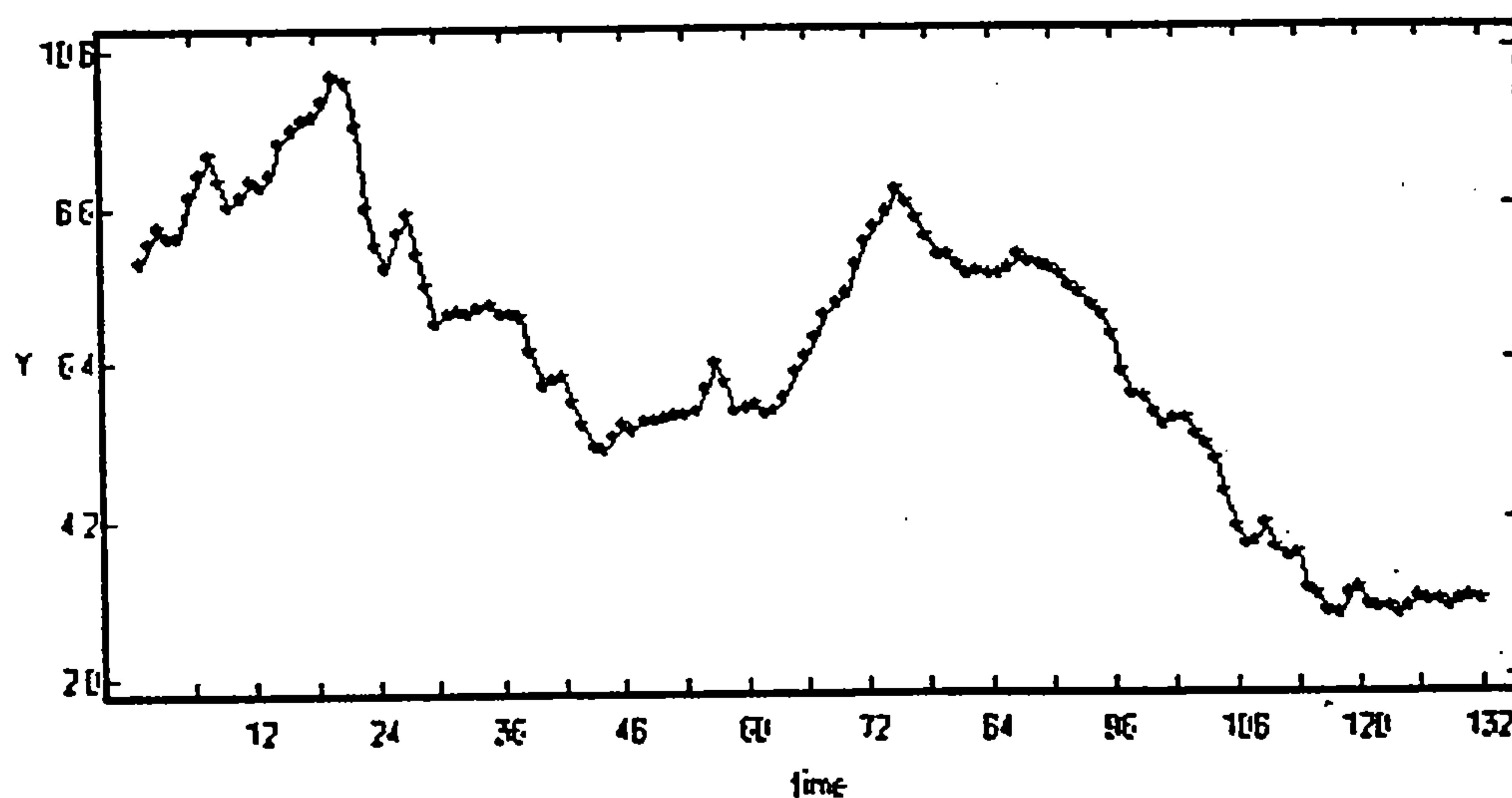
select z.new y.span(325,456)

VARIABLE Z IS EDITED. THE RESULT IS STORED IN VARIABLE Y

VARIABLE Y IS A 132 BY 1 MATRIX

--

graph y



--

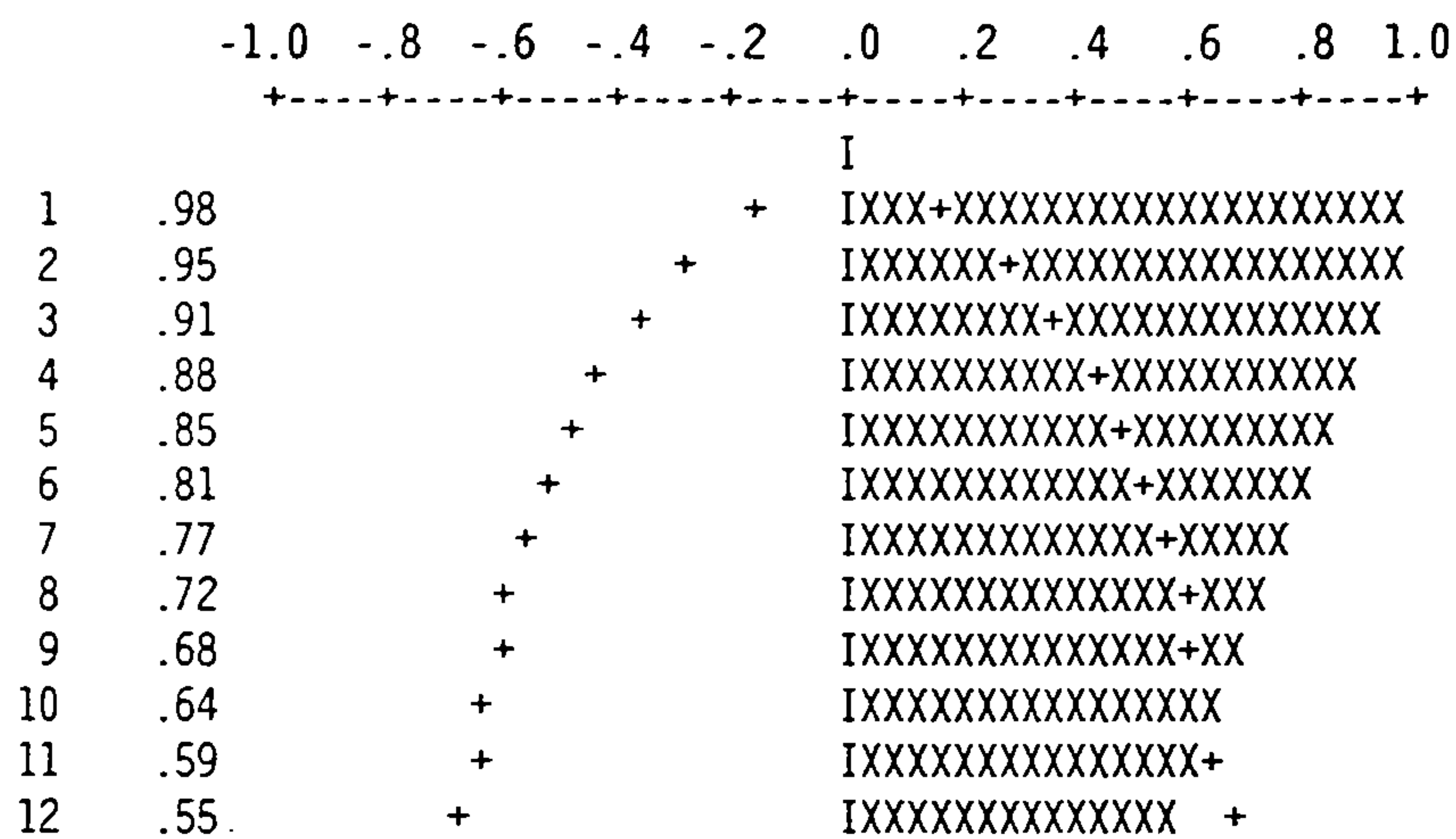
iden y.maxlag 12

TIME PERIOD ANALYZED	1	TO	132
NAME OF THE SERIES	Y		
EFFECTIVE NUMBER OF OBSERVATIONS	132		
STANDARD DEVIATION OF THE SERIES	2.0108		
MEAN OF THE (DIFFERENCED) SERIES	6.4978		
STANDARD DEVIATION OF THE MEAN	.1750		
T-VALUE OF MEAN (AGAINST ZERO)	37.1271		

AUTOCORRELATIONS

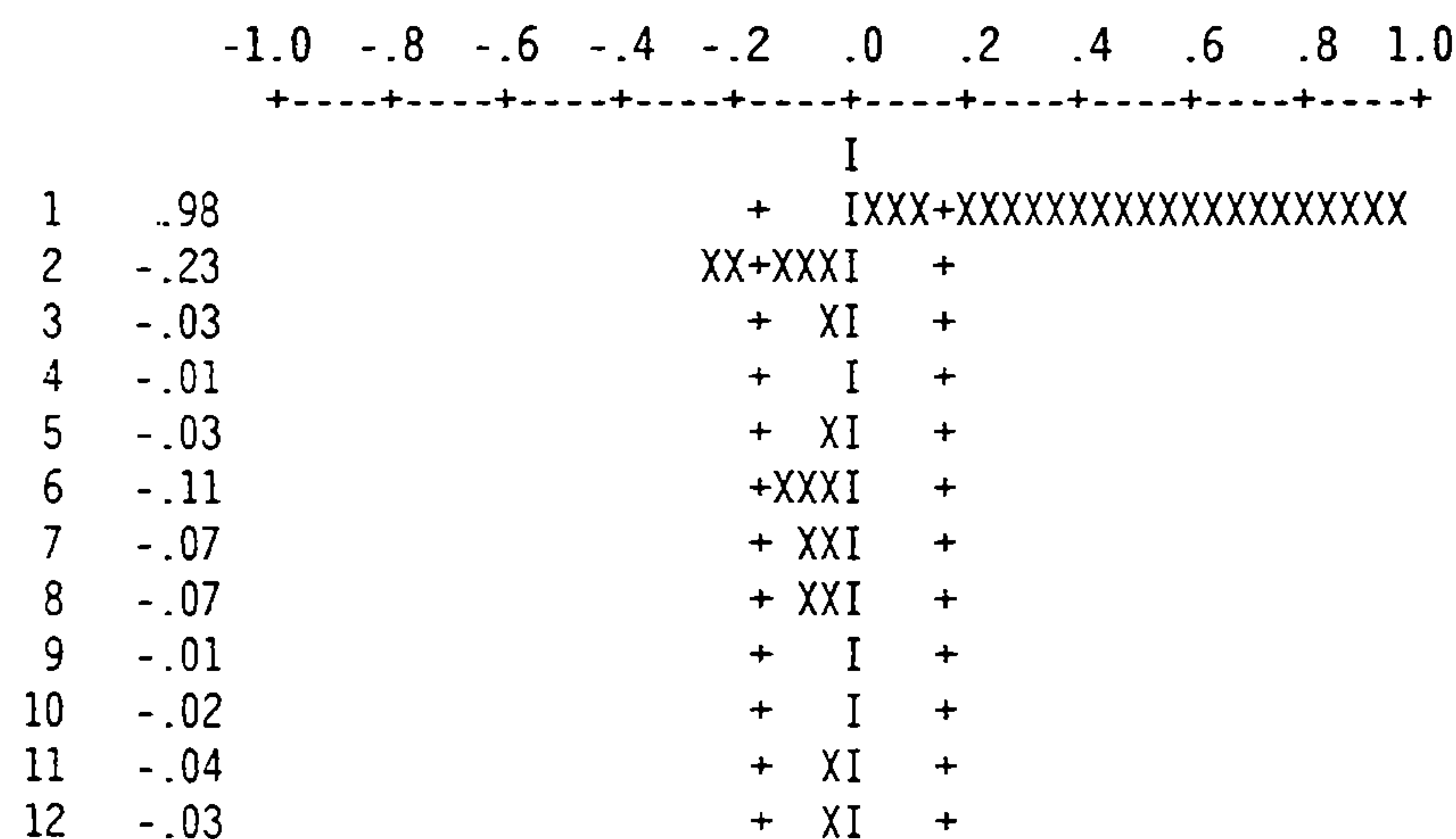
1- 12	.98	.95	.91	.88	.85	.81	.77	.72	.68	.64	.59	.55
ST.E.	.09	.15	.19	.22	.25	.27	.28	.30	.31	.32	.33	.34
Q	129	252	366	474	574	666	750	825	891	950	1001	1045

C-68



PARTIAL AUTOCORRELATIONS

1- 12	.98	-.23	-.03	-.01	-.03	-.11	-.07	-.07	-.01	-.02	-.04	-.03
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09



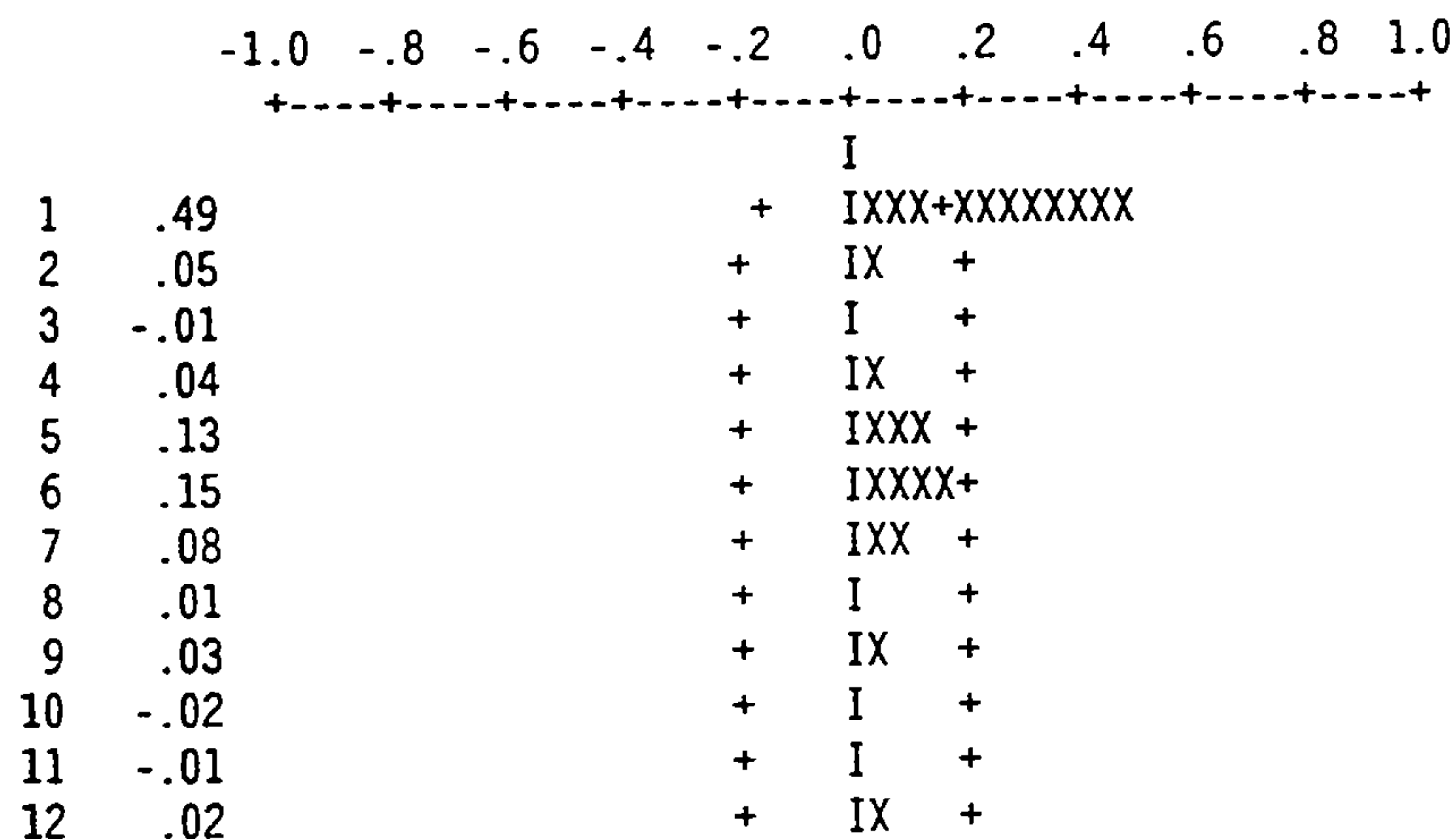
--

iden y.dfor 1.maxlag 12

DIFFERENCE ORDERS.	1	(1-B)
TIME PERIOD ANALYZED	1	TO 132
NAME OF THE SERIES		Y
EFFECTIVE NUMBER OF OBSERVATIONS		131
STANDARD DEVIATION OF THE SERIES		.2641
MEAN OF THE (DIFFERENCED) SERIES		-.0366
STANDARD DEVIATION OF THE MEAN		.0231
T-VALUE OF MEAN (AGAINST ZERO)		-1.5856

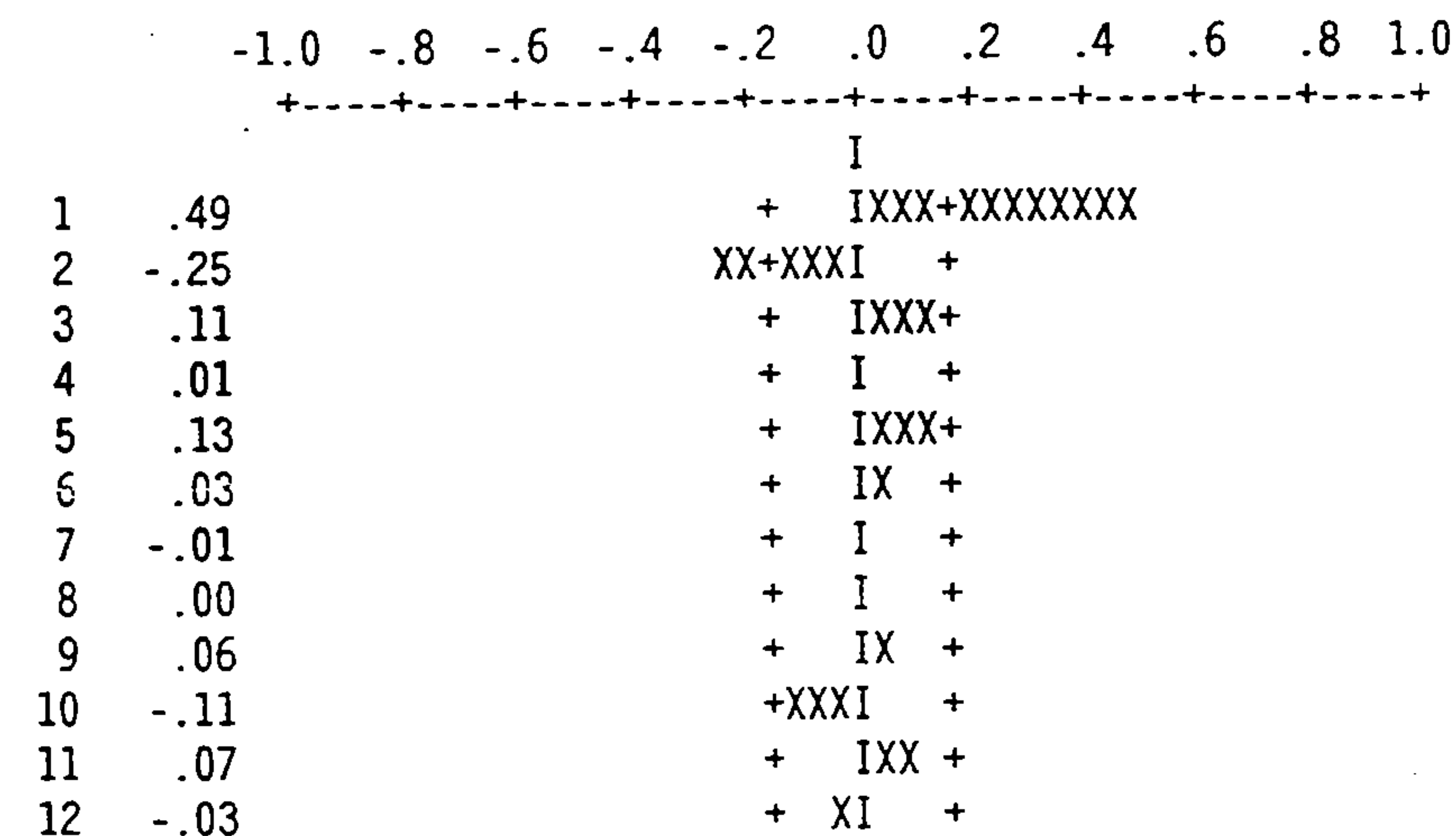
AUTOCORRELATIONS

1- 12	.49	.05	-.01	.04	.13	.15	.08	.01	.03	-.02	-.01	.02
ST.E.	.09	.11	.11	.11	.11	.11	.11	.11	.11	.11	.11	.11
Q	32.4	32.7	32.7	32.9	35.2	38.4	39.2	39.2	39.4	39.4	39.4	39.5



PARTIAL AUTOCORRELATIONS

1- 12	.49	-.25	.11	.01	.13	.03	-.01	-.00	.06	-.11	.07	-.03
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09



eacf y

TIME PERIOD ANALYZED	1 TO 132
NAME OF THE SERIES	Y
EFFECTIVE NUMBER OF OBSERVATIONS	132
STANDARD DEVIATION OF THE SERIES	2.0108
MEAN OF THE (DIFFERENCED) SERIES	6.4978

STANDARD DEVIATION OF THE MEAN 1750
T-VALUE OF MEAN (AGAINST ZERO) 37.1271

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.98	.95	.91	.88	.85	.81	.77	.72	.68	.64	.59	.55	.50
(P= 1)	.50	.07	.01	.06	.16	.19	.13	.07	.10	.05	.07	.10	.07
(P= 2)	.39	-.10	-.05	-.01	.08	.11	.04	-.08	.07	-.06	-.00	.07	-.06
(P= 3)	.42	-.08	-.06	-.01	.05	.08	.04	-.05	-.02	-.06	-.00	.01	-.05
(P= 4)	-.23	-.19	.07	-.20	.00	-.01	.10	-.05	.00	-.07	-.02	-.01	-.02
(P= 5)	-.19	-.48	.29	-.14	-.05	-.00	.07	-.04	-.02	-.01	.01	-.01	-.02
(P= 6)	-.30	-.08	.02	-.13	-.02	-.01	.05	-.07	-.04	-.01	.00	-.02	-.03

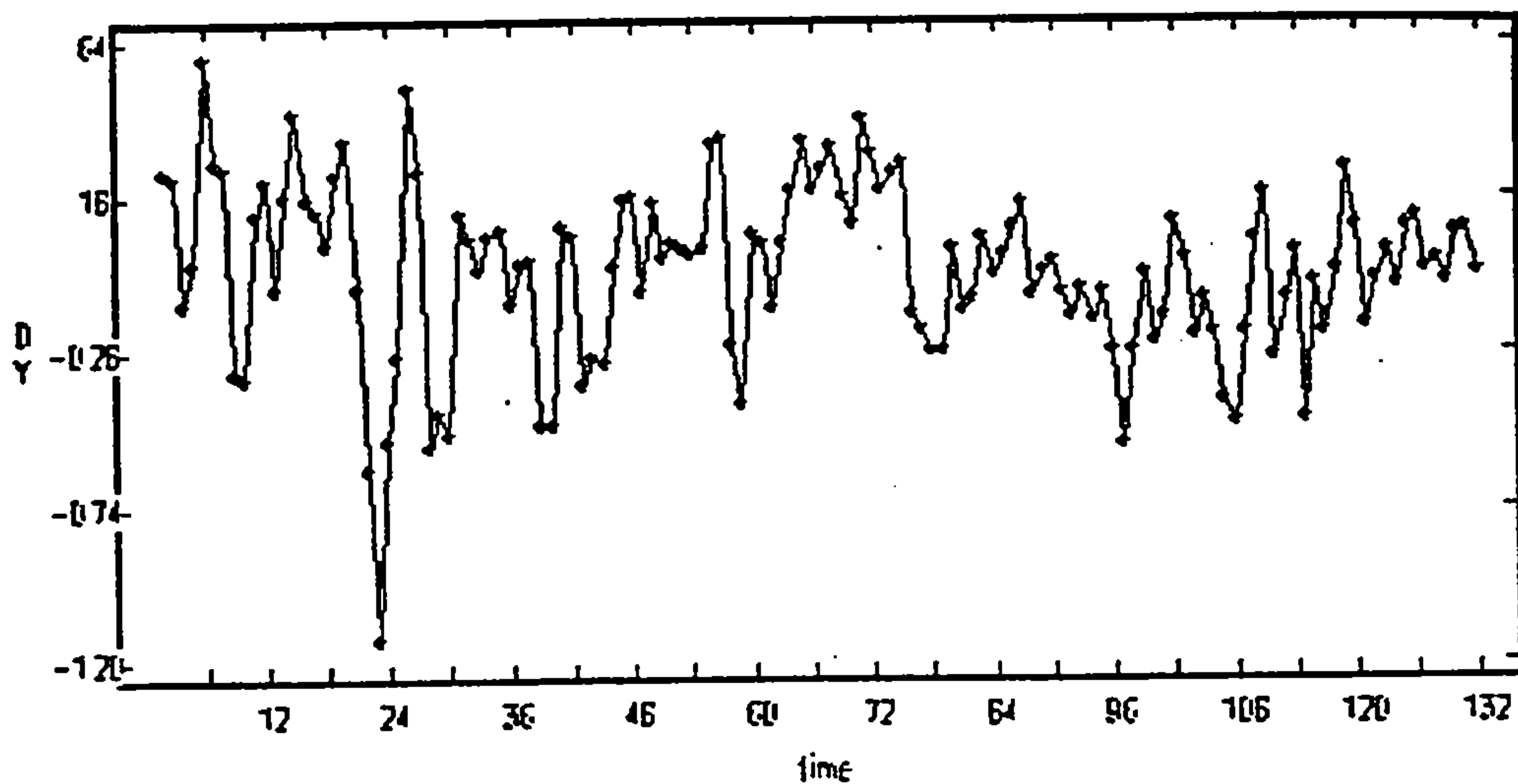
SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	X	X	X	X	X	X	X	X	X
(P= 1)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 2)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 3)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 4)	X	X	0	0	0	0	0	0	0	0	0	0	0
(P= 5)	X	X	X	0	0	0	0	0	0	0	0	0	0
(P= 6)	X	0	0	0	0	0	0	0	0	0	0	0	0

--
diff y.new dy

1
DIFFERENCE ORDERS ARE (1-B)
SERIES Y IS DIFFERENCED. THE RESULT IS STORED IN VARIABLE - DY
SERIES DY HAS 132 ENTRIES

--
graph dy



#From the SACF and SPACF OF y and dy,we now filt an (0,1,1) model to y.

```
tsm my.model y(1)=(1)noise
estim my.method exact.hold resi(ry)
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MY

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
			1					
Y	RANDOM	ORIGINAL	(1-B)					
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	Y	MA	1	1	NONE	-.6113	.0688	-8.88

TOTAL SUM OF SQUARES 533696E+03
TOTAL NUMBER OF OBSERVATIONS 132
RESIDUAL SUM OF SQUARES. 642705E+01
R-SQUARE 988
EFFECTIVE NUMBER OF OBSERVATIONS . . 130
RESIDUAL VARIANCE ESTIMATE 494388E-01
RESIDUAL STANDARD ERROR. 222348E+00

--
acf ry.maxlag 12

TIME PERIOD ANALYZED 3 TO 132
NAME OF THE SERIES RY

AUTOCORRELATIONS

-1.0 -0.8 -0.6 -0.4 -0.2 .0 .2 .4 .6 .8 1.0
+-----+-----+-----+-----+-----+-----+-----+-----+-----+

graph ry



```
--
forecast my.origins 48,72,96,120.nofs 10.hold@
forecasts(y48,y72,y96,y120).std_errs(sy1,sy2,sy3,sy4)
```

NOTE: THE EXACT METHOD FOR COMPUTING RESIDUALS IS USED

10 FORECASTS, BEGINNING AT 48

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
49	5.6030	.2223	5.4270
50	5.6030	.4217	5.5937
51	5.6030	.5533	5.5918
52	5.6030	.6592	5.6362
53	5.6030	.7502	5.6600
54	5.6030	.8314	5.6682
55	5.6030	.9053	5.6923
56	5.6030	.9736	6.0362
57	5.6030	1.0374	6.3981
58	5.6030	1.0976	6.1333

10 FORECASTS, BEGINNING AT 72

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
73	8.1169	.2223	8.2681
74	8.1169	.4217	8.5279
75	8.1169	.5533	8.8164
76	8.1169	.6592	8.6445
77	8.1169	.7502	8.4282
78	8.1169	.8314	8.1496
79	8.1169	.9053	7.8705
80	8.1169	.9736	7.8991
81	8.1169	1.0374	7.7435
82	8.1169	1.0976	7.6227

 10 FORECASTS, BEGINNING AT 96

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
97	6.6044	.2223	6.2223
98	6.6044	.4217	5.9468
99	6.6044	.5533	5.9025
100	6.6044	.6592	5.6468
101	6.6044	.7502	5.4668
102	6.6044	.8314	5.5790
103	6.6044	.9053	5.5764
104	6.6044	.9736	5.3441
105	6.6044	1.0374	5.2170
106	6.6044	1.0976	4.9970

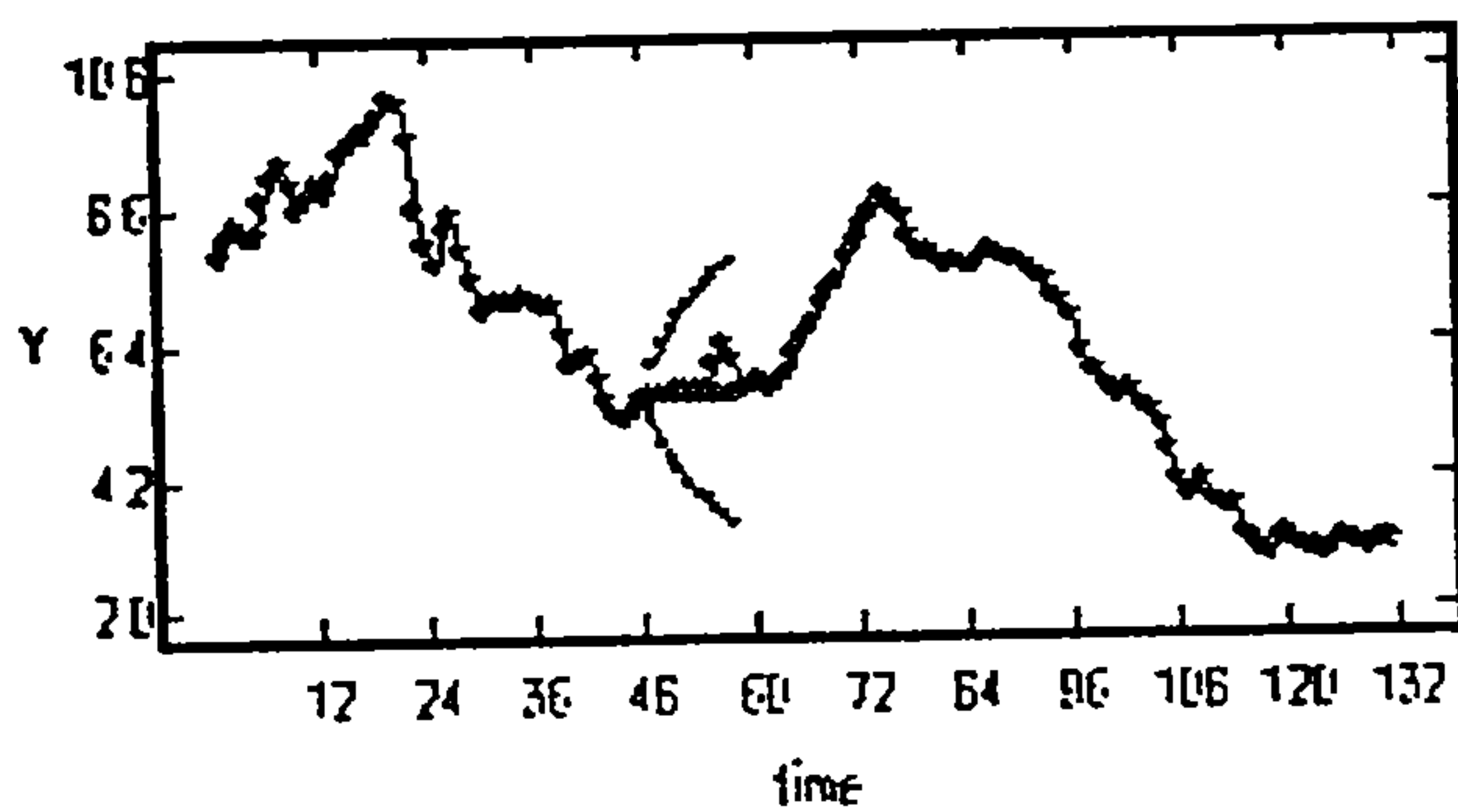
 10 FORECASTS, BEGINNING AT 120

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
121	3.2116	.2223	3.0105
122	3.2116	.4217	2.9390
123	3.2116	.5533	2.9574
124	3.2116	.6592	2.8733
125	3.2116	.7502	2.9620
126	3.2116	.8314	3.0800
127	3.2116	.9053	3.0438
128	3.2116	.9736	3.0291
129	3.2116	1.0374	2.9514
130	3.2116	1.0976	3.0270

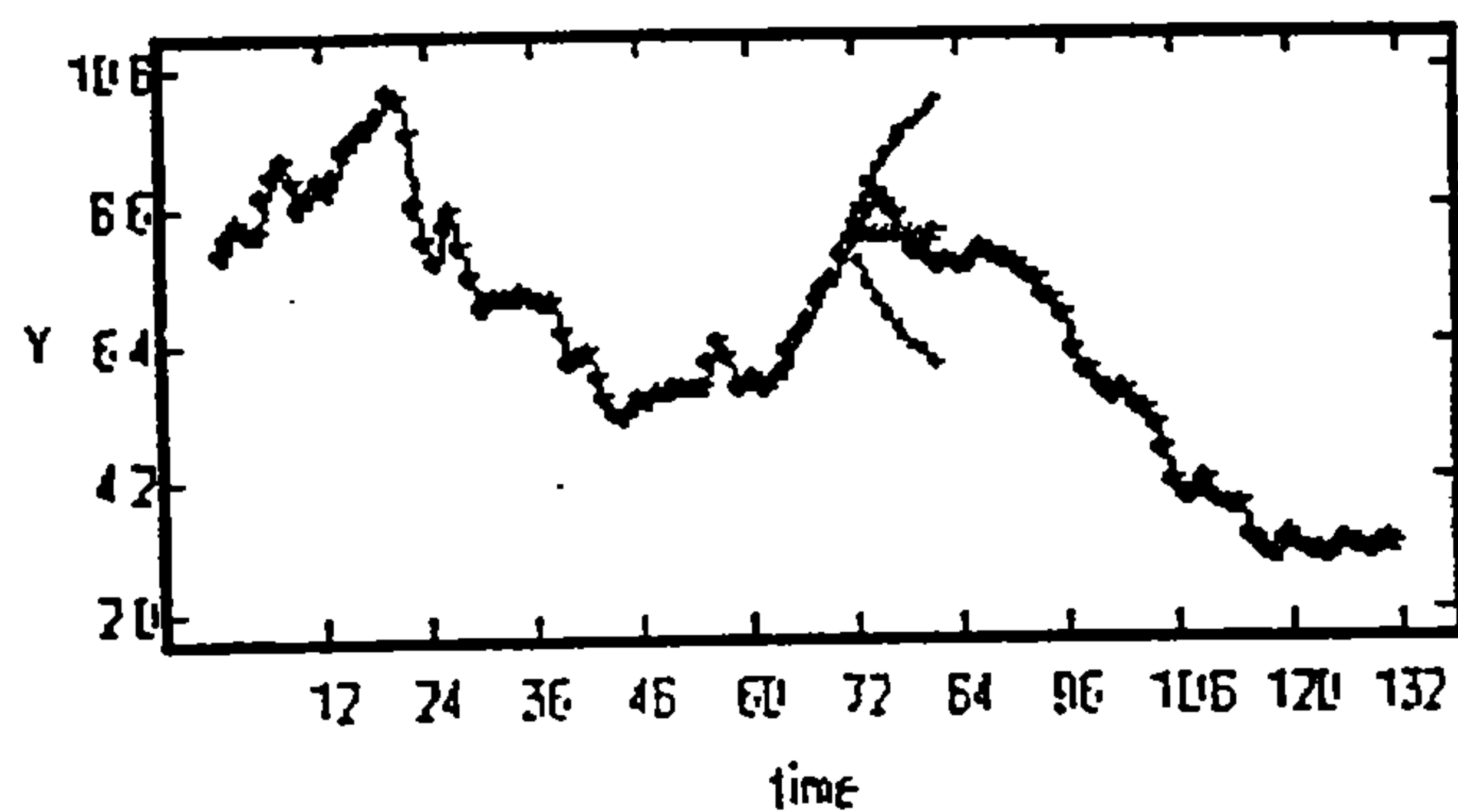
--
 fsave y48.y72.y96.y120.sy1.sy2.sy3.sy4.file'yfor.fmt'

** THE SPECIFIED FILE DOES NOT EXIST. A NEW FILE IS OPENED.
 THE FILE NAME IS YFOR.FMT
 THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE 7

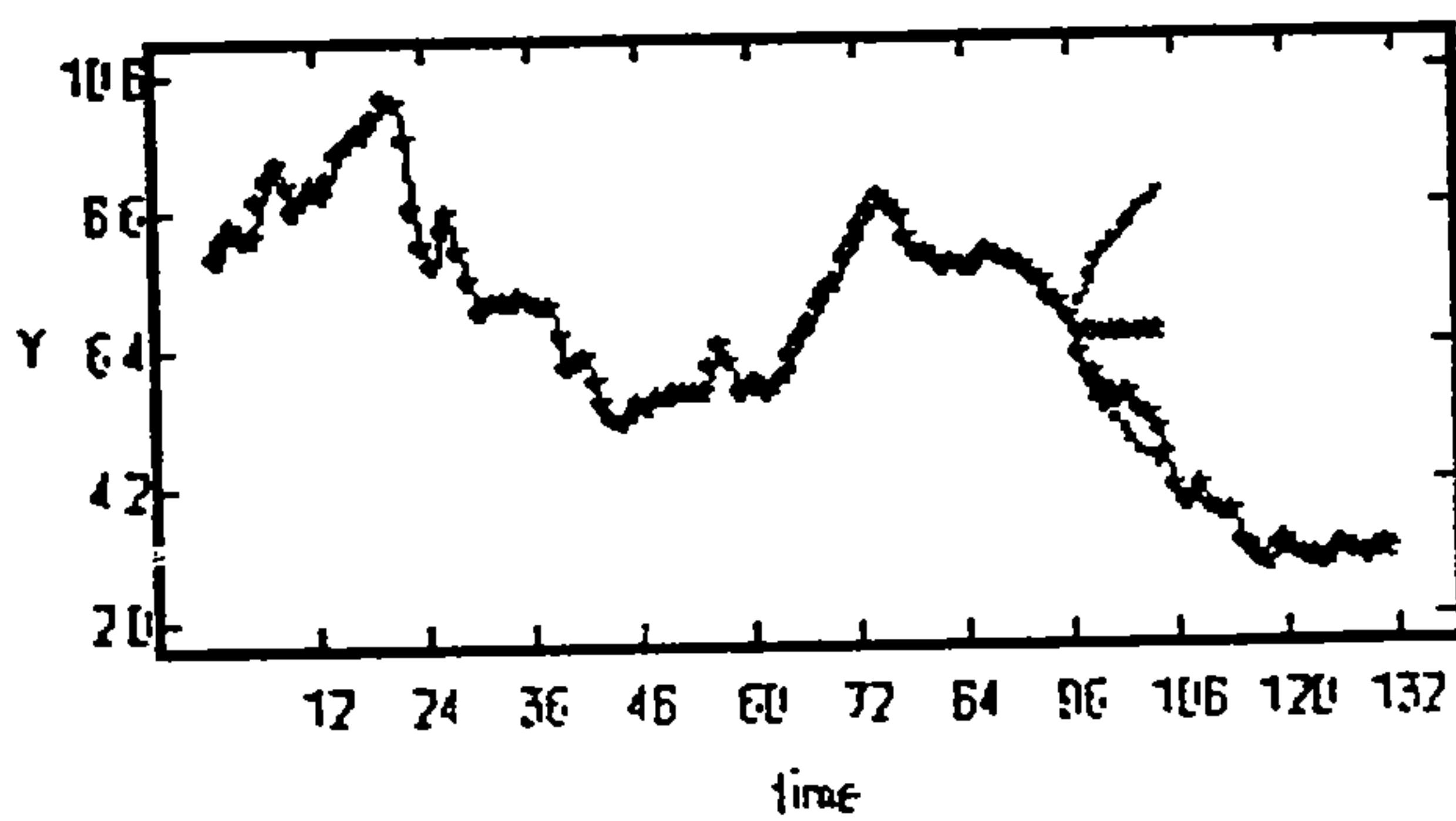
Forecast Plot of Y



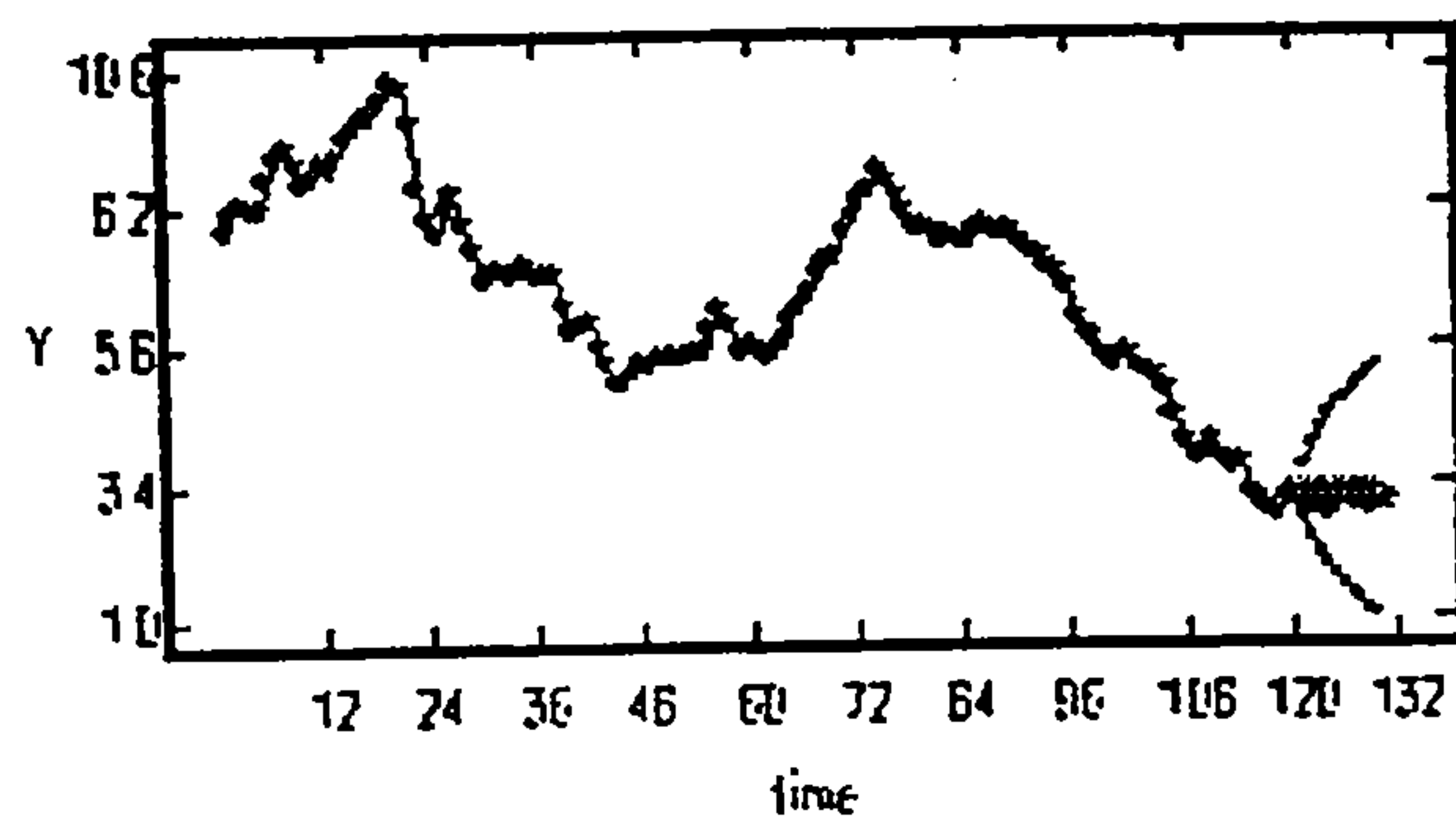
Forecast Plot of Y



Forecast Plot of Y



Forecast Plot of Y



#From the EACF of y, we may alternatively start with an ARMA(1,1) model as follows:

```
tsm my1.model (1)y=c+(1)noise
```

```
estim my1.method exact.hold resi(ry1)
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MY1

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
----------	------------------	----------------------	--------------

Y	RANDOM	ORIGINAL	NONE
---	--------	----------	------

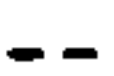
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONSTRAINT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	.0020	.1056	.02
2	Y	MA	1	1	NONE	-.6088	.0689	-8.84
3	Y	AR	1	1	NONE	.9942	.0155	64.16

TOTAL SUM OF SQUARES 533696E+03
TOTAL NUMBER OF OBSERVATIONS 132
RESIDUAL SUM OF SQUARES. 635519E+01
R-SQUARE 988
EFFECTIVE NUMBER OF OBSERVATIONS 130
RESIDUAL VARIANCE ESTIMATE 488861E-01
RESIDUAL STANDARD ERROR. 221102E+00

--
acf ry1.maxlag 12

TIME PERIOD ANALYZED 3 TO 132
NAME OF THE SERIES RY1
EFFECTIVE NUMBER OF OBSERVATIONS 130
STANDARD DEVIATION OF THE SERIES 2217
MEAN OF THE (DIFFERENCED) SERIES -.0005
STANDARD DEVIATION OF THE MEAN 0194
T-VALUE OF MEAN (AGAINST ZERO) -.0256

1- 12	.05	.06	-.04	.02	.09	.09	.07	-.05	.09	-.05	-.00	.06
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09
Q	.3	.7	.9	1.0	2.1	3.2	3.9	4.3	5.4	5.8	5.8	6.4



outlier model my1

TIME	ESTIMATE	T-VALUE	TYPE
23	-.79	-3.74	IO
26	.70	3.47	IO

— —

302

#EXAMPLE VIII. QUARTERLY U.S. REAL GNP DATA 1/46 TO 1/91

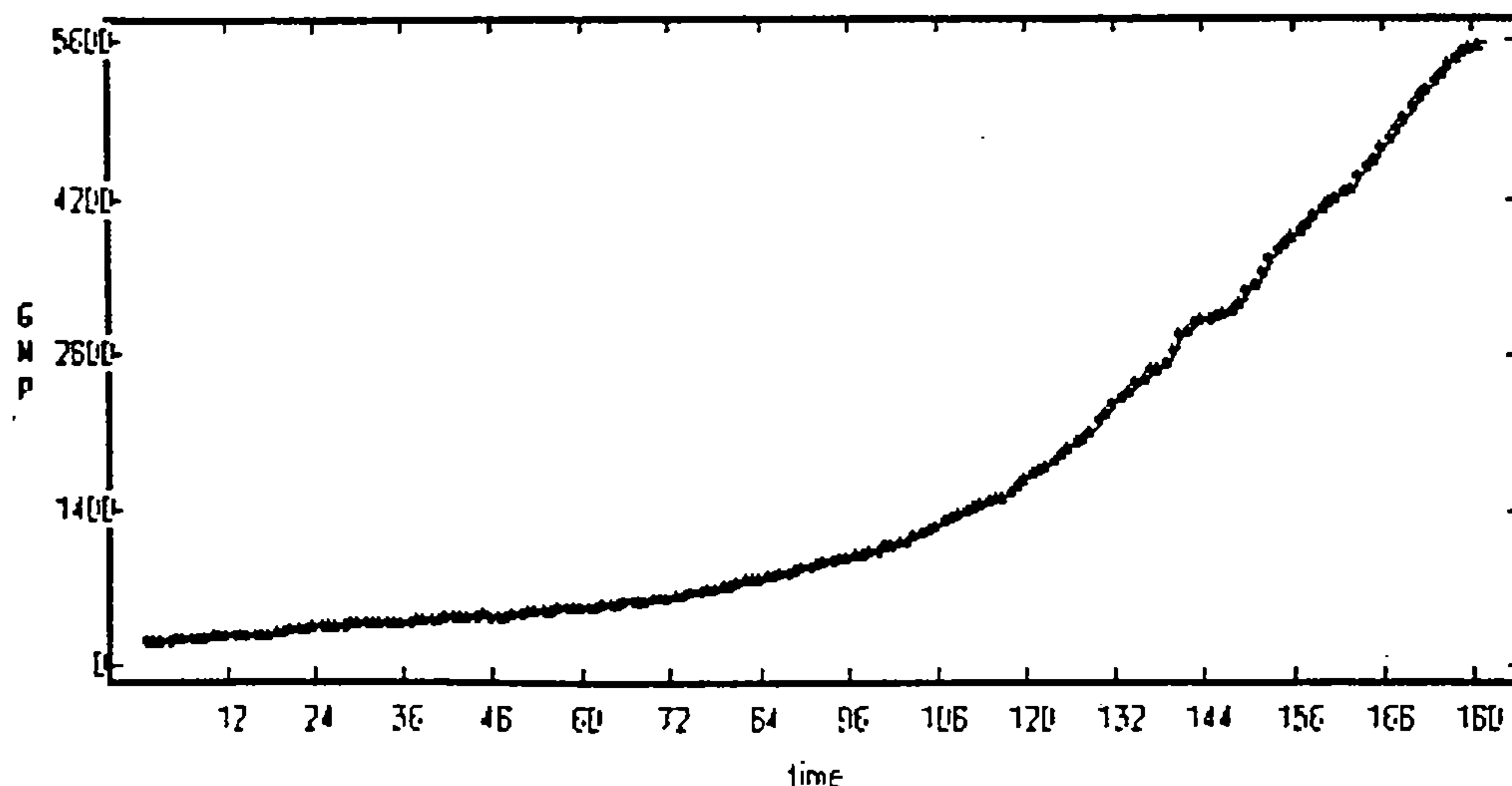
input yr.qr.gnp.file'c:\b325\qgnp.dat'

YR , A 181 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
 QR , A 181 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
 GNP , A 181 BY 1 VARIABLE, IS STORED IN THE WORKSPACE

--
 print yr.qr.gnp

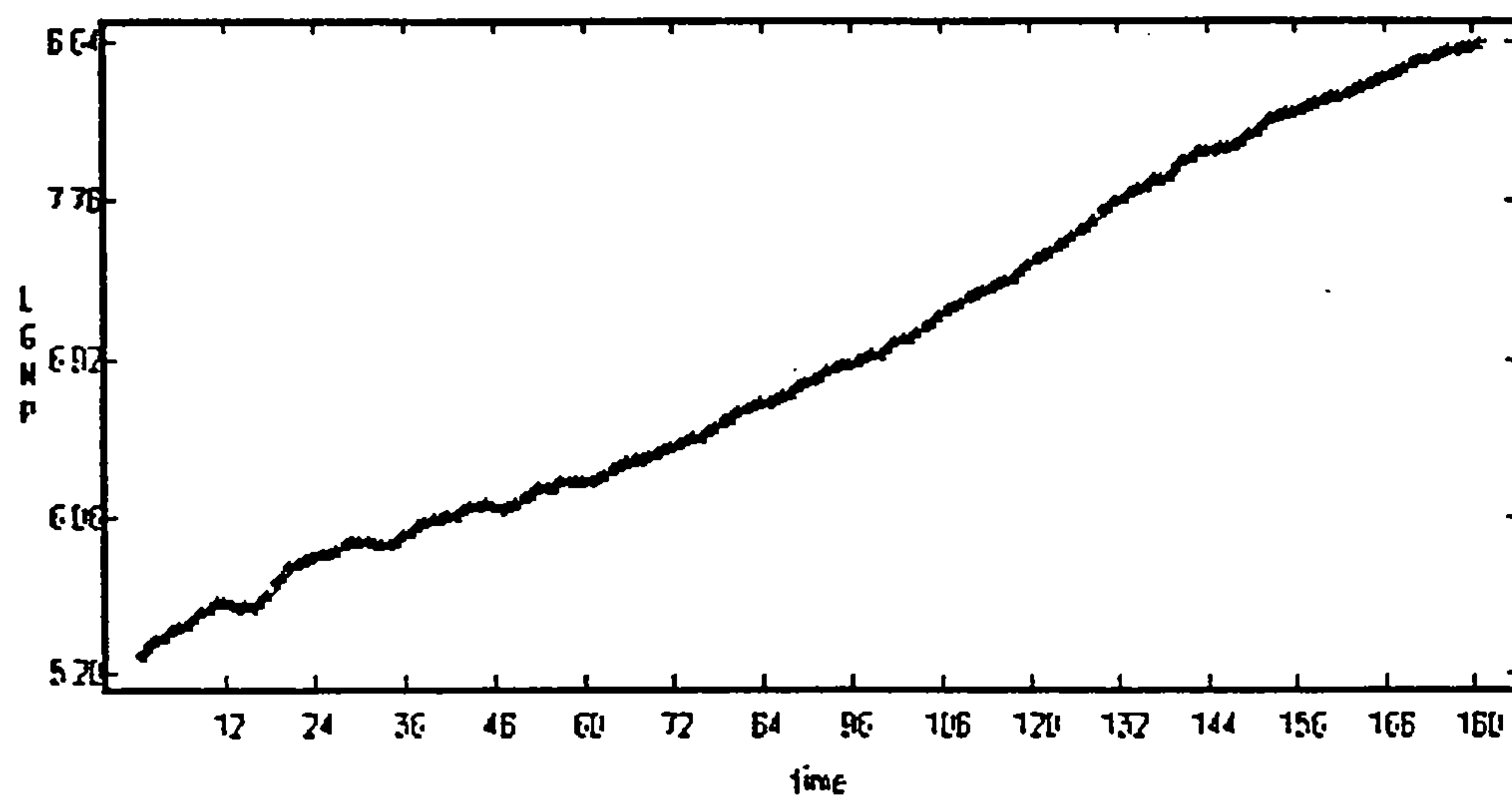
VARIABLE COLUMN-->	YR 1	QR 1	GNP 1
ROW			
1	1946.000	1.000	200.100
2	1946.000	2.000	208.200
3	1946.000	3.000	218.400
4	1946.000	4.000	222.800
5	1947.000	1.000	227.500
.			
150	1983.000	2.000	3367.400
151	1983.000	3.000	3443.900
152	1983.000	4.000	3545.800
.			
181	1991.000	1.000	5561.700

--
 graph gnp



C-79

lgnp=ln(gnp)
graph lgnp



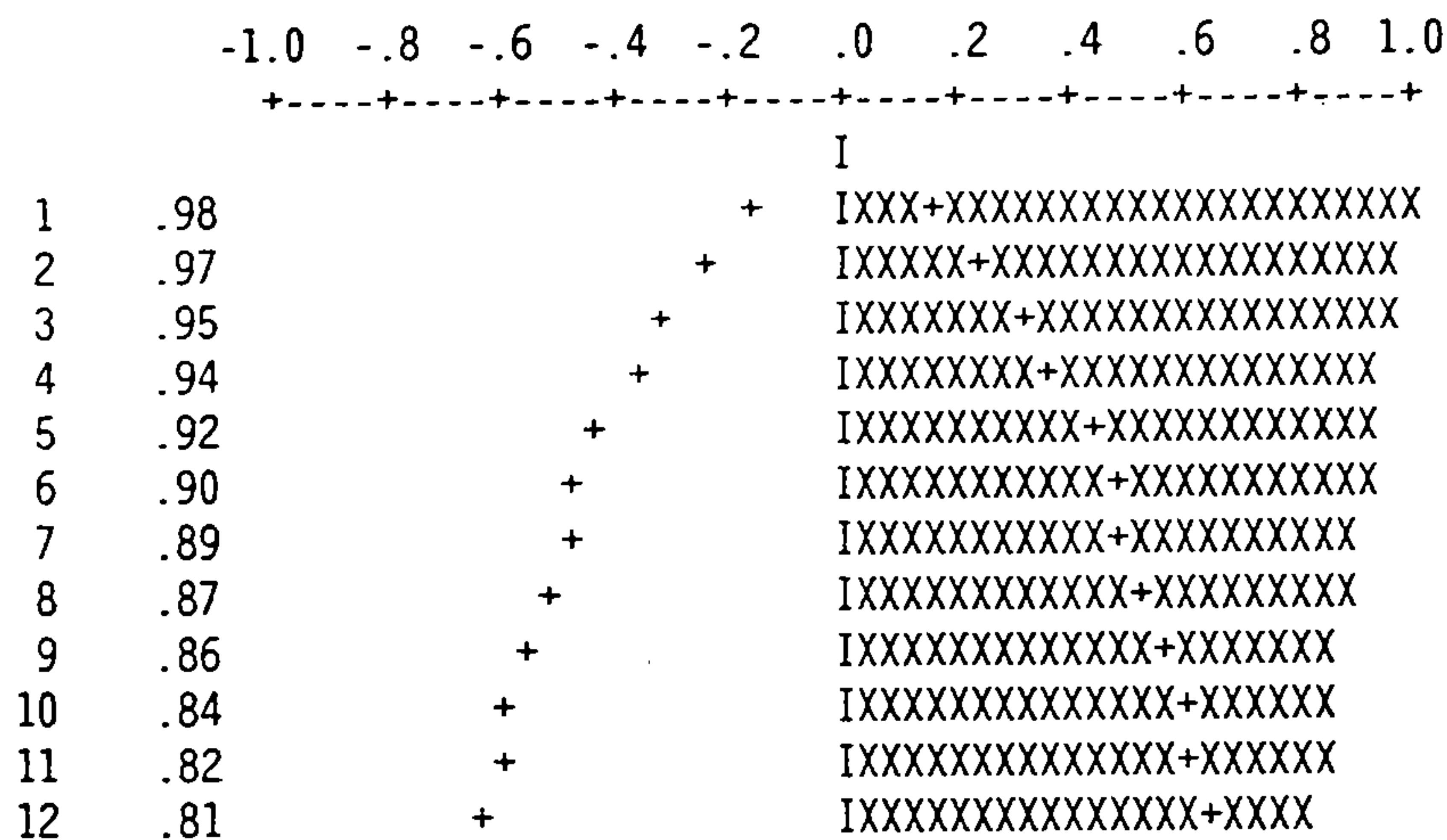
--

iden lgnp.maxlag 12

TIME PERIOD ANALYZED	1 TO 181
NAME OF THE SERIES	LGNP
EFFECTIVE NUMBER OF OBSERVATIONS	181
STANDARD DEVIATION OF THE SERIES	.9838
MEAN OF THE (DIFFERENCED) SERIES	6.9335
STANDARD DEVIATION OF THE MEAN	.0731
T-VALUE OF MEAN (AGAINST ZERO)	94.8143

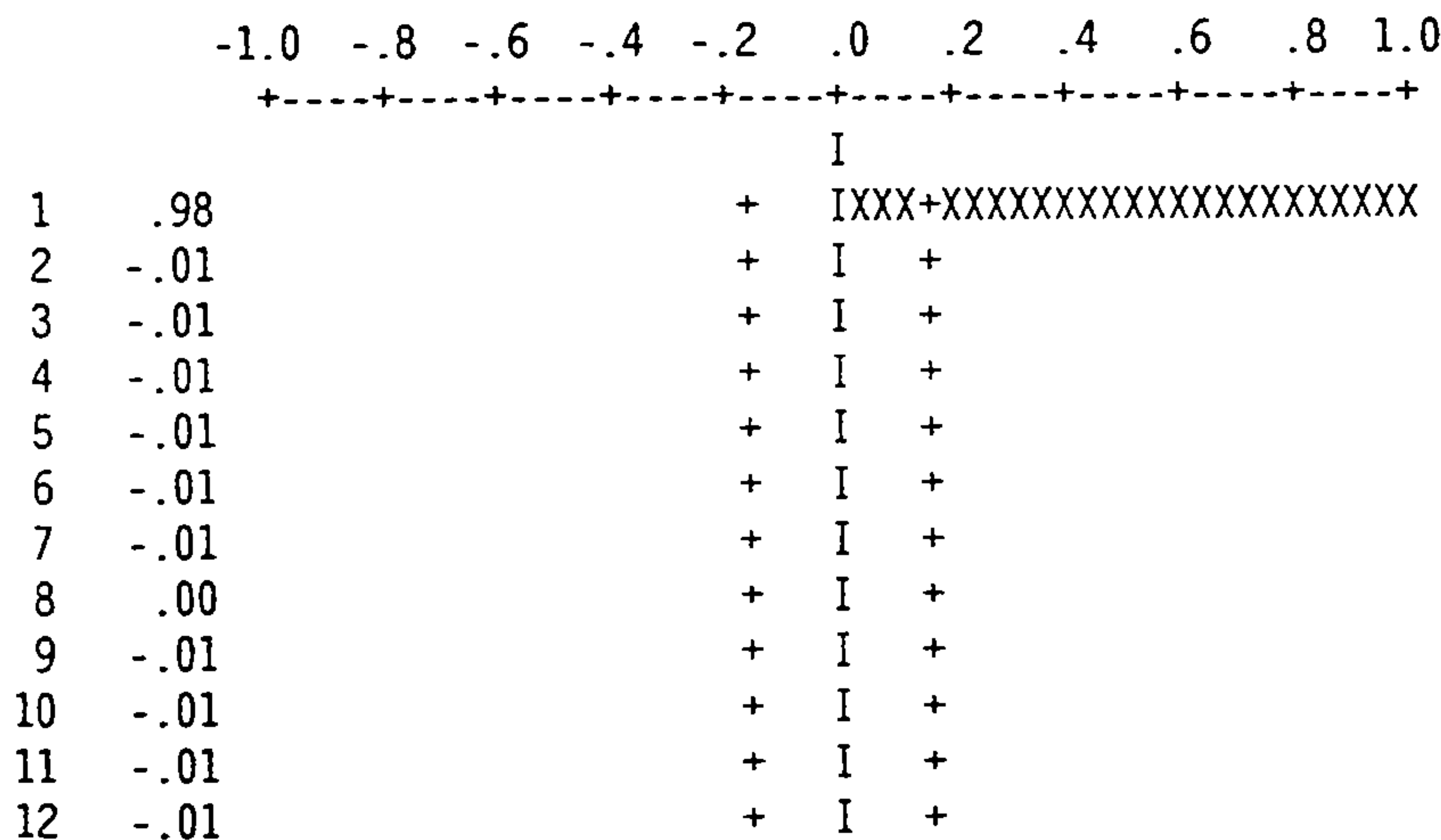
AUTOCORRELATIONS

1- 12	.98	.97	.95	.94	.92	.90	.89	.87	.86	.84	.82	.81
ST.E.	.07	.13	.16	.19	.22	.24	.25	.27	.29	.30	.31	.32
Q	178	352	520	684	843	998	1148	1294	1435	1572	1704	1833



PARTIAL AUTOCORRELATIONS

1- 12	.98	-.01	-.01	-.01	-.01	-.01	-.01	-.00	-.01	-.01	-.01	-.01
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07



iden lgnp.dfor 1.maxlag 12

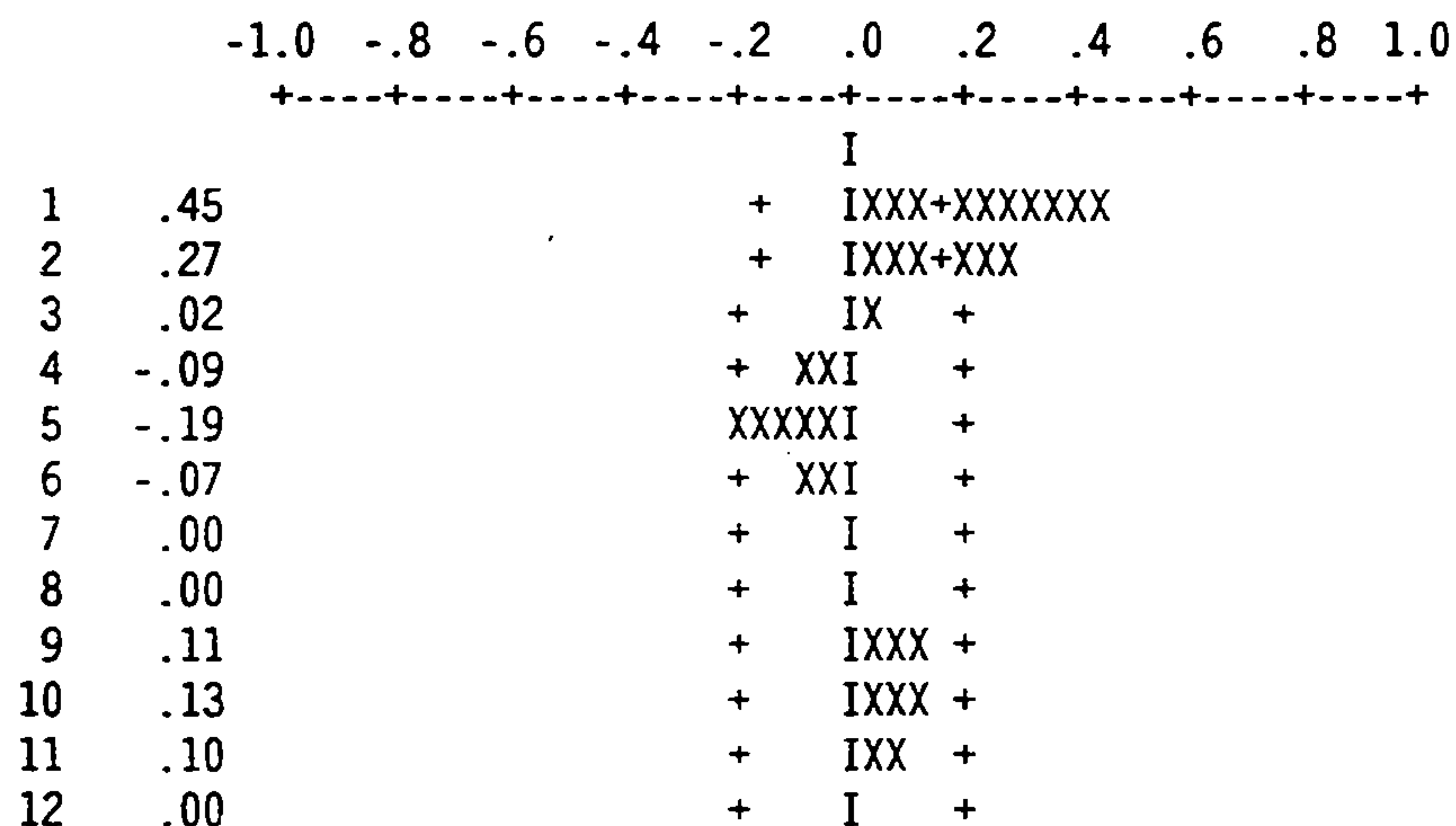
```

1
DIFFERENCE ORDERS. . . . . (1-B )
TIME PERIOD ANALYZED . . . . . 1 TO 181
NAME OF THE SERIES . . . . . LGNP
EFFECTIVE NUMBER OF OBSERVATIONS . . . 180
STANDARD DEVIATION OF THE SERIES . . . .0127
MEAN OF THE (DIFFERENCED) SERIES . . . .0185
STANDARD DEVIATION OF THE MEAN . . . .0009
T-VALUE OF MEAN (AGAINST ZERO) . . . .19.5371

```

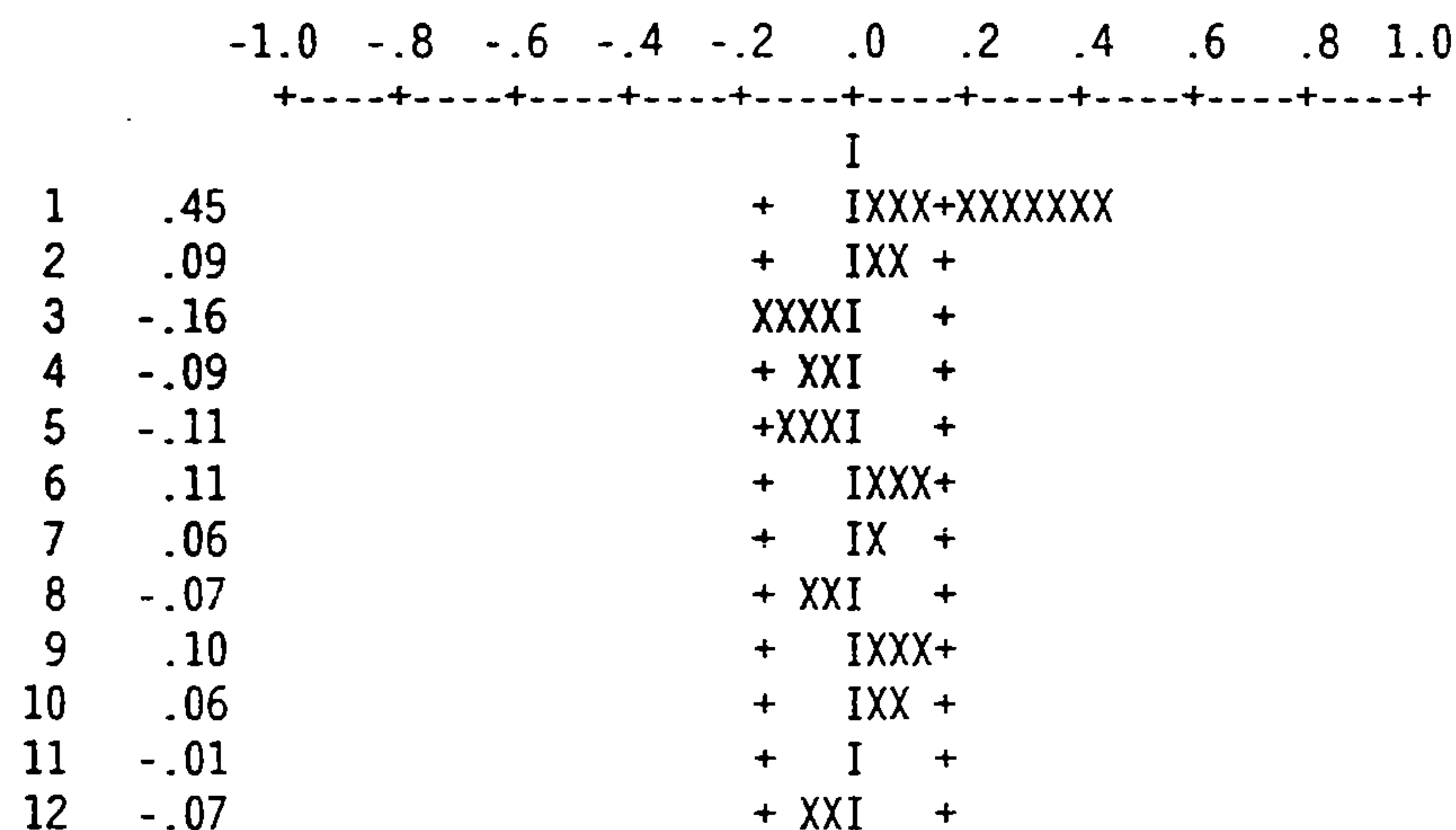
AUTOCORRELATIONS

1- 12	.45	.27	.02	-.09	-.19	-.07	-.00	.00	.11	.13	.10	.00
ST.E.	.07	.09	.09	.09	.09	.10	.10	.10	.10	.10	.10	.10
Q	37.2	50.9	51.1	52.4	59.3	60.3	60.3	60.3	62.4	65.8	67.6	67.6



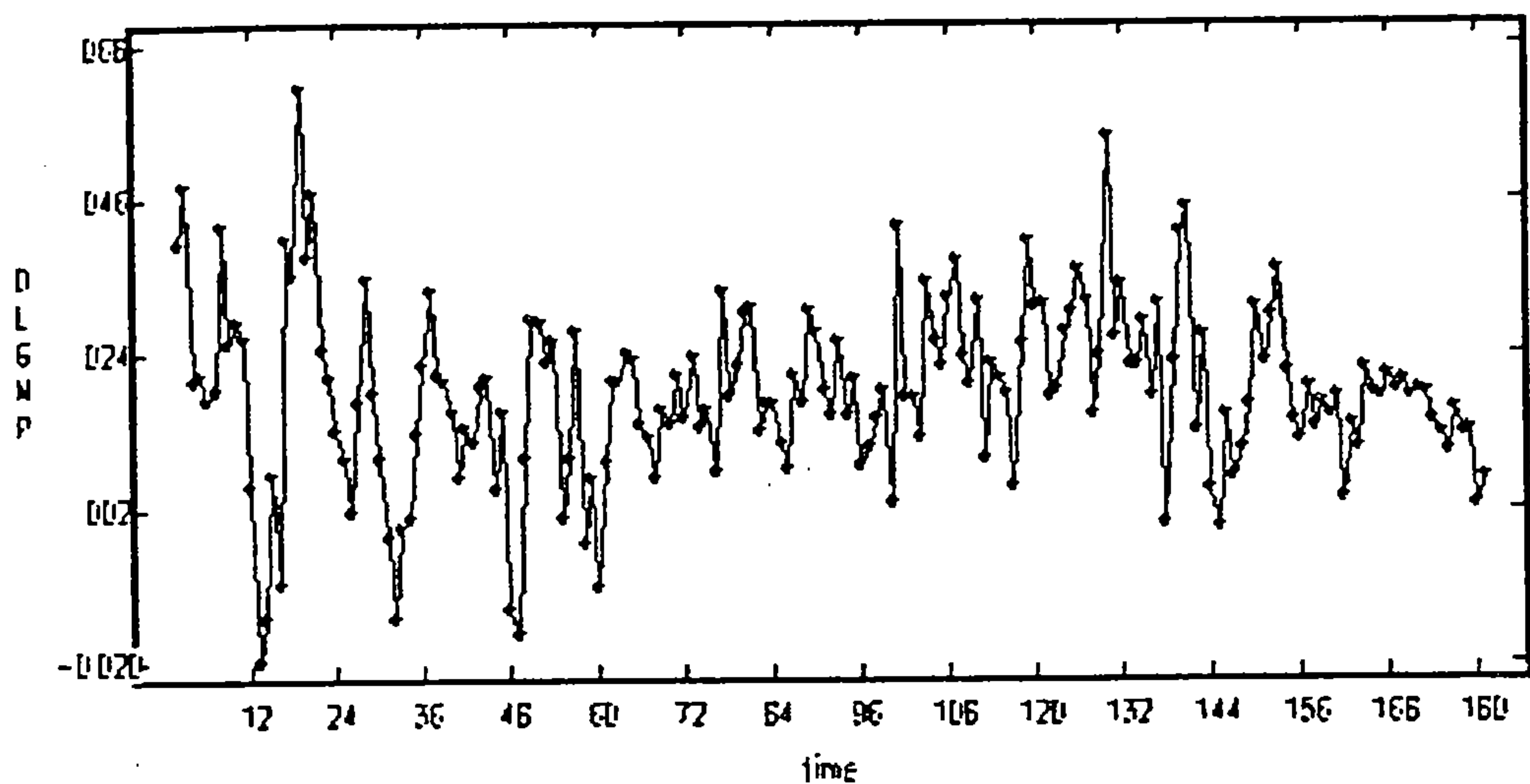
PARTIAL AUTOCORRELATIONS

1- 12	.45	.09	-.16	-.09	-.11	.11	.06	-.07	.10	.06	-.01	-.07
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07



diff lgnp.new dlnp

graph dlnp



--
eacf lgnp

TIME PERIOD ANALYZED 1 TO 181
 NAME OF THE SERIES LGNP
 EFFECTIVE NUMBER OF OBSERVATIONS . . . 181
 STANDARD DEVIATION OF THE SERIES 9838
 MEAN OF THE (DIFFERENCED) SERIES . . . 6.9335
 STANDARD DEVIATION OF THE MEAN 0731
 T-VALUE OF MEAN (AGAINST ZERO) 94.8143

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.98	.97	.95	.94	.92	.90	.89	.87	.86	.84	.82	.81	.79
(P= 1)	.45	.27	.02	-.09	-.19	-.07	-.00	.00	.10	.13	.10	.01	-.01
(P= 2)	-.35	.09	-.13	-.00	-.21	.03	.07	-.09	.07	.07	.07	-.06	-.10
(P= 3)	.02	-.22	-.13	-.01	-.24	.08	.07	-.03	.06	.00	.12	-.04	-.14
(P= 4)	.14	-.21	.04	-.05	-.27	-.15	.03	.01	.06	.00	.04	.00	-.07
(P= 5)	-.49	-.35	-.04	-.07	-.26	-.17	.03	.02	.06	.07	.05	.00	-.11
(P= 6)	-.27	-.33	.10	-.26	-.33	.11	.03	-.03	.11	-.02	.05	-.02	-.07

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	X	X	X	X	X	X	X	X	X
(P= 1)	X	X	0	0	X	0	0	0	0	0	0	0	0
(P= 2)	X	0	0	0	X	0	0	0	0	0	0	0	0
(P= 3)	0	X	0	0	X	0	0	0	0	0	0	0	0
(P= 4)	0	X	0	0	X	0	0	0	0	0	0	0	0
(P= 5)	X	X	0	0	X	0	0	0	0	0	0	0	0
(P= 6)	X	X	0	X	X	0	0	0	0	0	0	0	0

```
--
tsm m1.model (1)lgnp=c+(1,2,3,4,5)noise
```

```
estim m1
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE		TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
LGNP		RANDOM	ORIGINAL	NONE					
PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAIT	VALUE	STD. ERROR	T VALUE
1	C		CNST	1	0	NONE	.0187	.0096	1.94
2		LGNP	MA	1	1	NONE	-.4173	.0731	-5.71
3		LGNP	MA	1	2	NONE	-.3655	.0802	-4.56
4		LGNP	MA	1	3	NONE	-.0909	.0847	-1.07
5		LGNP	MA	1	4	NONE	-.0048	.0804	-.06
6		LGNP	MA	1	5	NONE	.2117	.0735	2.88
7		LGNP	AR	1	1	NONE	1.0000	.0014	725.19

```
TOTAL SUM OF SQUARES . . . . . .175192E+03
TOTAL NUMBER OF OBSERVATIONS . . . . .181
RESIDUAL SUM OF SQUARES. . . . . .213303E-01
R-SQUARE . . . . . .1.000
EFFECTIVE NUMBER OF OBSERVATIONS . . .180
RESIDUAL VARIANCE ESTIMATE . . . . . .118502E-03
RESIDUAL STANDARD ERROR. . . . . .108859E-01
```

```
--
tsm m2.model lgnp(1)=c+(1,2,5)noise
estim m2.method exact.hold resi(rlg)
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M2

VARIABLE		TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
LGNP		RANDOM	ORIGINAL	1 (1-B)					
PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAIT	VALUE	STD ERROR	T VALUE
1	C		CNST	1	0	NONE	.0185	.0012	15.25
2		LGNP	MA	1	1	NONE	-.3896	.0688	-5.66
3		LGNP	MA	1	2	NONE	-.3535	.0685	-5.16
4		LGNP	MA	1	5	NONE	.2444	.0653	3.75

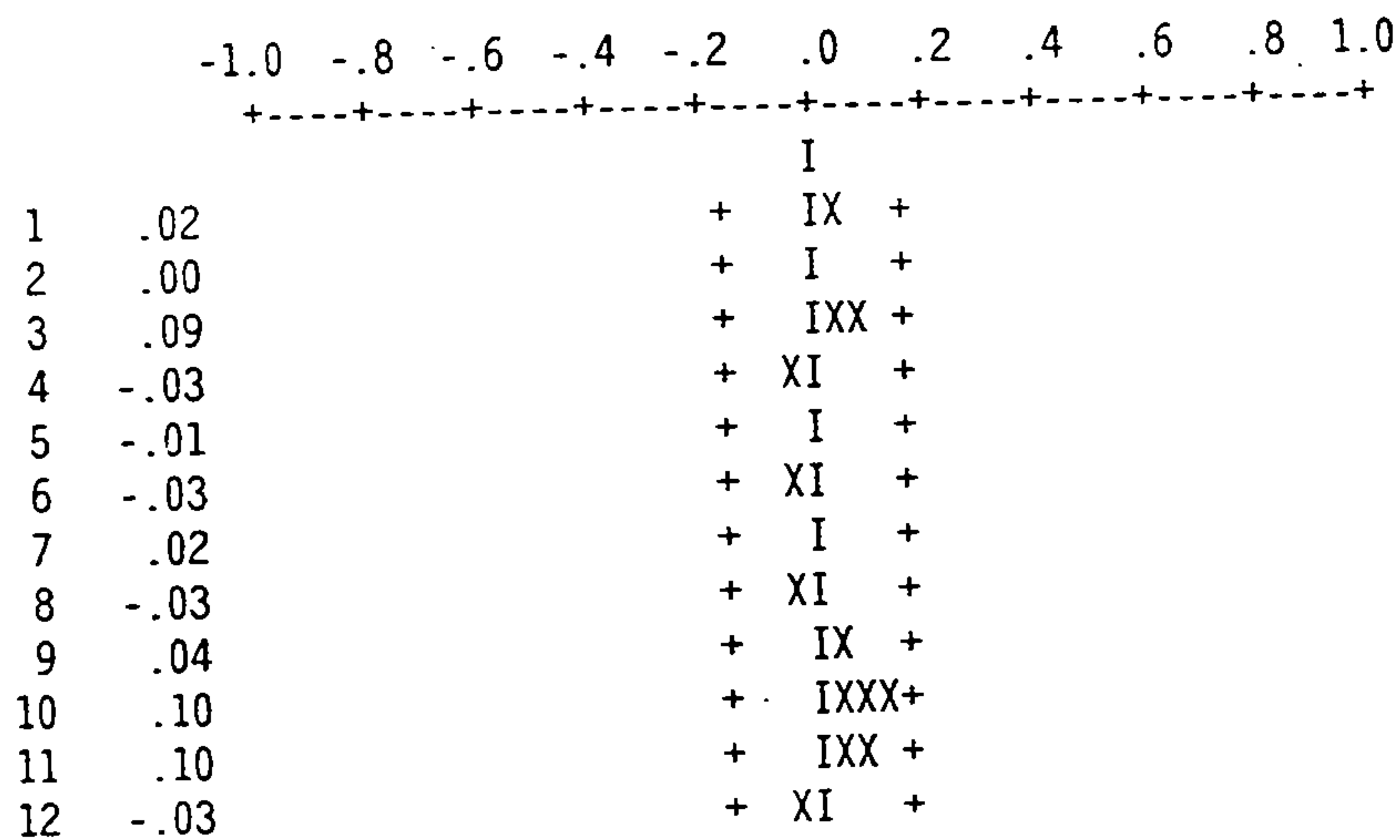
TOTAL SUM OF SQUARES 175192E+03
 TOTAL NUMBER OF OBSERVATIONS 181
 RESIDUAL SUM OF SQUARES. 210766E-01
 R-SQUARE 1.000
 EFFECTIVE NUMBER OF OBSERVATIONS . . .179
 RESIDUAL VARIANCE ESTIMATE 117746E-03
 RESIDUAL STANDARD ERROR. 108511E-01

acf rlg.maxlag 12

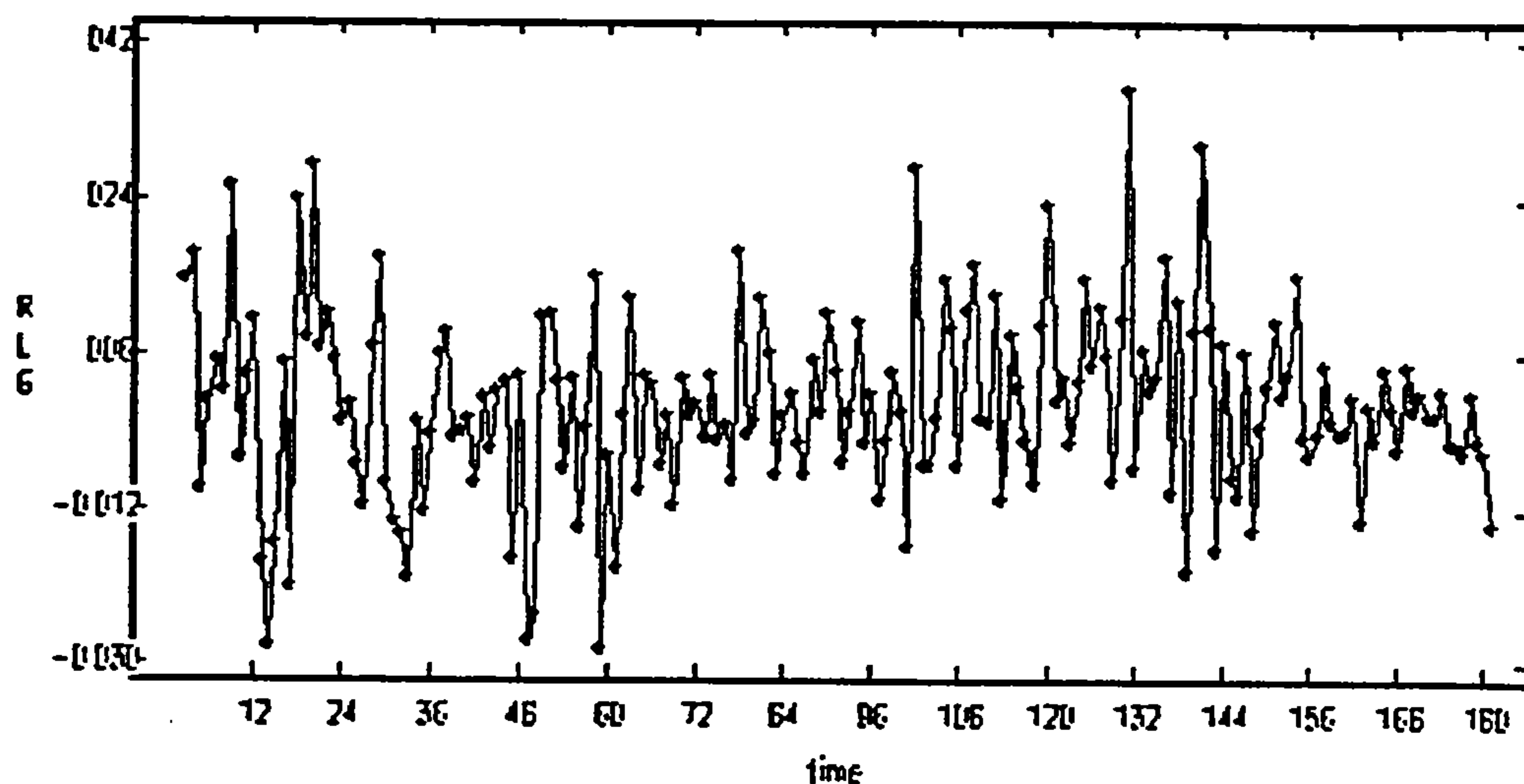
TIME PERIOD ANALYZED 3 TO 181
 NAME OF THE SERIES RLG
 EFFECTIVE NUMBER OF OBSERVATIONS . . .179
 STANDARD DEVIATION OF THE SERIES 0108
 MEAN OF THE (DIFFERENCED) SERIES 0000
 STANDARD DEVIATION OF THE MEAN 0008
 T-VALUE OF MEAN (AGAINST ZERO)-.0239

AUTOCORRELATIONS

1- 12	.02	.00	.09	-.03	-.01	-.03	.02	-.03	.04	.10	.10	-.03
ST.E.	.07	.07	.07	.08	.08	.08	.08	.08	.08	.08	.08	.08
Q	.1	.1	1.7	1.9	1.9	2.1	2.1	2.3	2.7	4.7	6.5	6.7



graph rlg



forecast m2.origins 150.181.nofs 12.hold forecasts(lgf1.lgf2).@
std_errs(sgf1.sgf2)

NOTE: THE EXACT METHOD FOR COMPUTING RESIDUALS IS USED

12 FORECASTS, BEGINNING AT 150

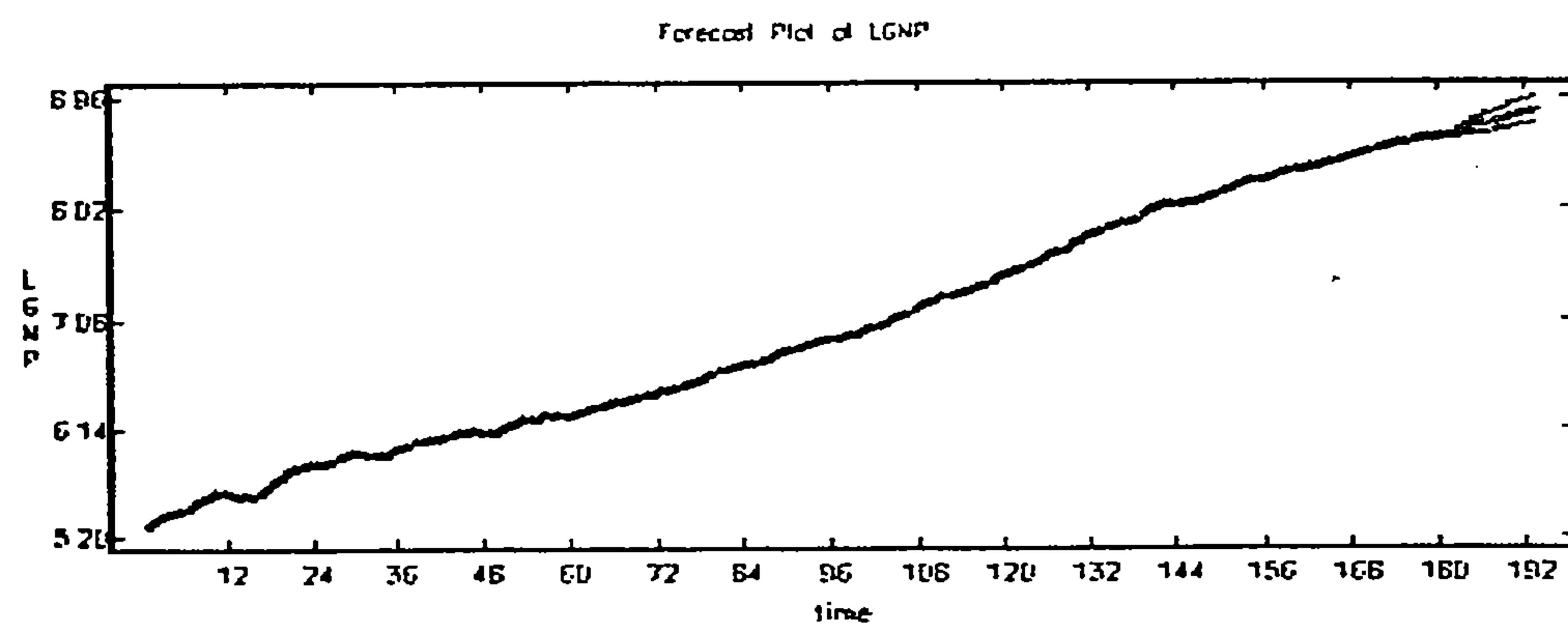
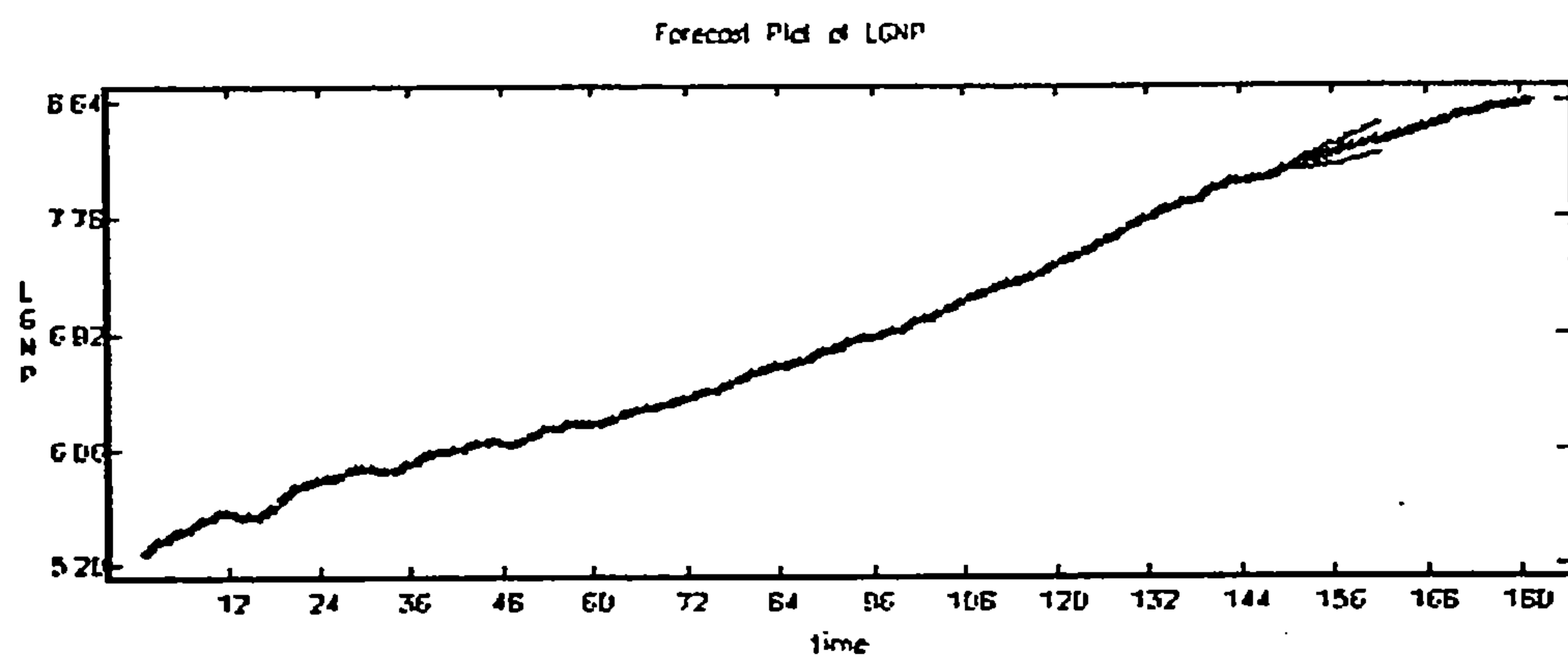
TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
151	8.1435	.0109	8.1444
152	8.1690	.0186	8.1735
153	8.1882	.0265	8.2093
154	8.2061	.0326	8.2306
155	8.2223	.0377	8.2448
156	8.2408	.0410	8.2563
157	8.2592	.0441	8.2753
158	8.2777	.0470	8.2888
159	8.2962	.0498	8.3057
160	8.3147	.0524	8.3207
161	8.3332	.0548	8.3384
162	8.3517	.0572	8.341

12 FORECASTS, BEGINNING AT 181

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
182	8.6345	.0109	
183	8.6517	.0186	
184	8.6716	.0265	
185	8.6934	.0326	
186	8.7135	.0377	
187	8.7320	.0410	

188	8.7505	.0441
189	8.7690	.0470
190	8.7875	.0498
191	8.8059	.0524
192	8.8244	.0548
193	8.8429	.0572

--
fsave lgf1.sgf1.lgf2.sgf2.file'lgfor.fmt'



diff lgf1.new lgfld

print lgf1.lgfld

LGFI IS A 12 BY 1 VARIABLE
LGFID IS A 12 BY 1 VARIABLE

VARIABLE COLUMN--> ROW	LGFI 1	LGFID 1
1	8.143	***
2	8.169	.026
3	8.188	.019
4	8.206	.018
5	8.222	.016
6	8.241	.018
7	8.259	.018
8	8.278	.018
9	8.296	.018
10	8.315	.018
11	8.333	.018
12	8.352	.018

--

tsm m3.model dlgnp=c+(1,2,5)noise

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M3

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
----------	---------------------	-------------------------	--------------

DLGNP	RANDOM	ORIGINAL	NONE
-------	--------	----------	------

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRRAINT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	.0185	.0012	15.25
2	DLGNP	MA	1	1	NONE	-.3896	.0688	-5.66
3	DLGNP	MA	1	2	NONE	-.3535	.0685	-5.16
4	DLGNP	MA	1	5	NONE	.2444	.0653	3.75

TOTAL SUM OF SQUARES 289616E-01
TOTAL NUMBER OF OBSERVATIONS 180
RESIDUAL SUM OF SQUARES. 210766E-01
R-SQUARE 268
EFFECTIVE NUMBER OF OBSERVATIONS . . 179
RESIDUAL VARIANCE ESTIMATE 117746E-03
RESIDUAL STANDARD ERROR. 108511E-01

```
--
forecast m3.origin 150.181.nofs 12.hold forecasts@
(dlgf1,dlg2).std_errs(sdlg1,sdlg2)
```

```
-----
12 FORECASTS, BEGINNING AT 150
-----
```

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
151	.0216	.0109	.0225
152	.0255	.0116	.0292
153	.0192	.0123	.0358
154	.0180	.0123	.0213
155	.0161	.0123	.0142
156	.0185	.0125	.0115
157	.0185	.0125	.0190
158	.0185	.0125	.0135
159	.0185	.0125	.0169
160	.0185	.0125	.0149
161	.0185	.0125	.0177
162	.0185	.0125	.0032

```
-----
12 FORECASTS, BEGINNING AT 181
-----
```

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
182	.0109	.0109	
183	.0172	.0116	
184	.0198	.0123	
185	.0219	.0123	
186	.0201	.0123	
187	.0185	.0125	
188	.0185	.0125	
189	.0185	.0125	
190	.0185	.0125	
191	.0185	.0125	
192	.0185	.0125	
193	.0185	.0125	

```
--
fsave dlgf1,sdlg1,dlg2,sdlg2.file'dlgfor.fmt'
```

```
fsave gnp.lgnp.dlgnp.rlg.file'gnp.fmt'
```

```
workspace memory saved
```

```
THE WORKSPACE IS SAVED ON FILE 9
```

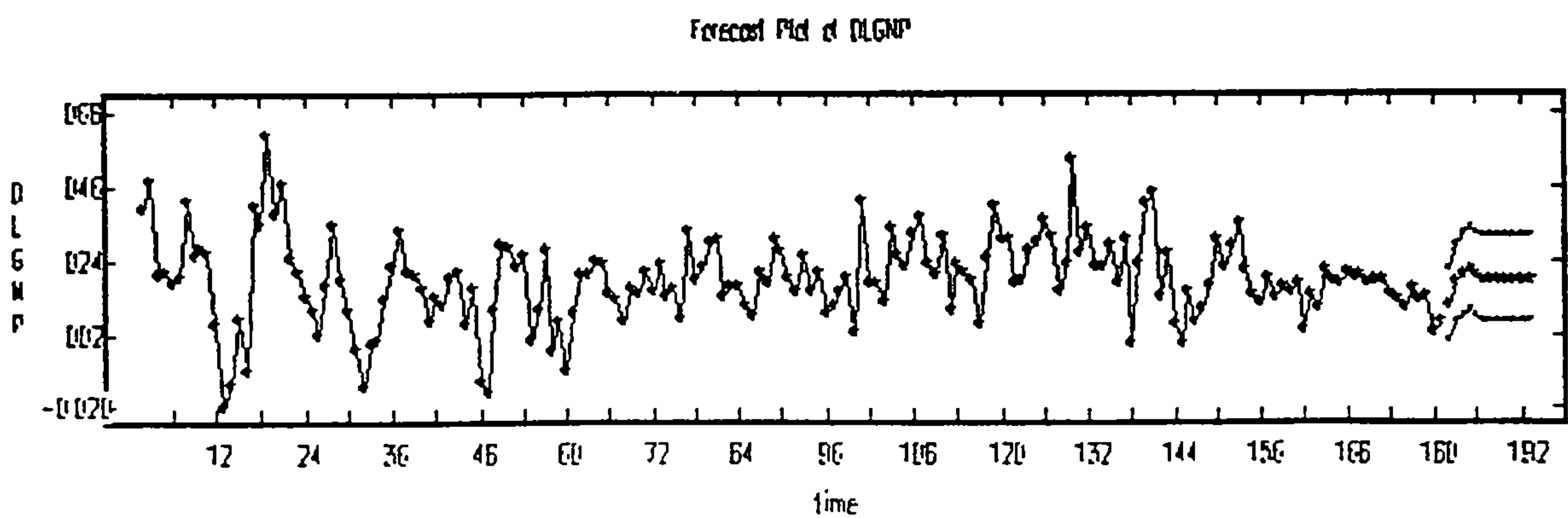
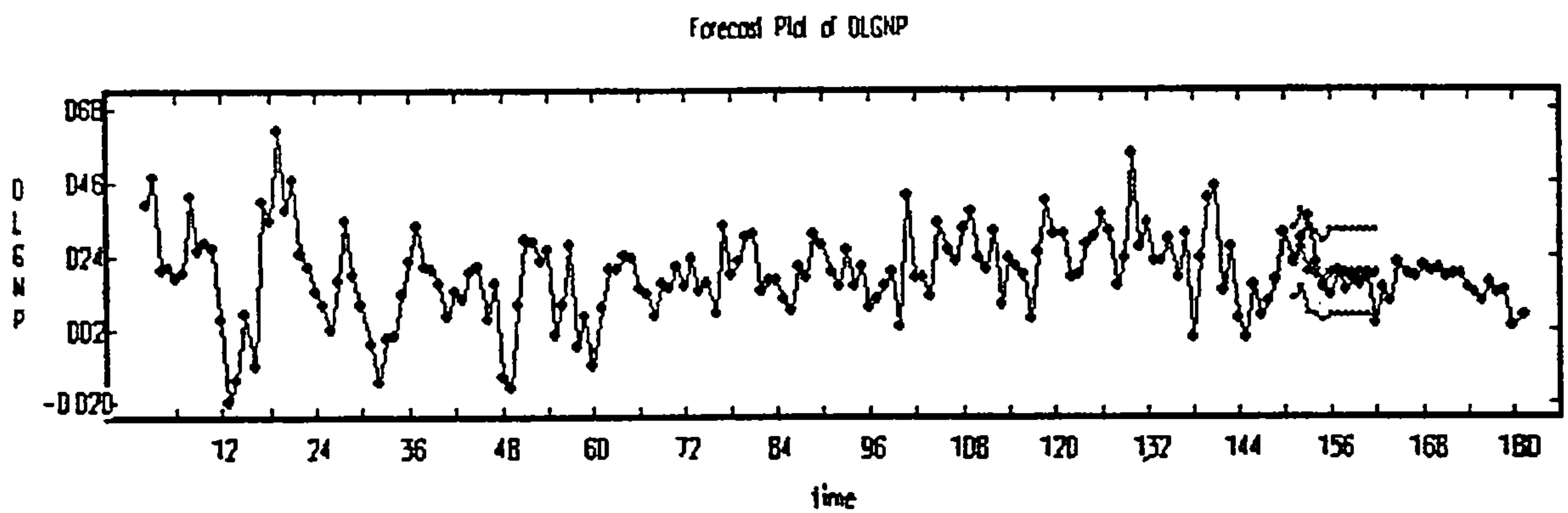
```
--
```

```
stop
```

#This example has illustrated the usefulness of models of the form

$$\phi(B)(1-B)Z_t = c + \theta(B)a_t$$

and the relationship between forecasts of the original series Z_t and those of the differenced series $w_t = (1-B)Z_t$.



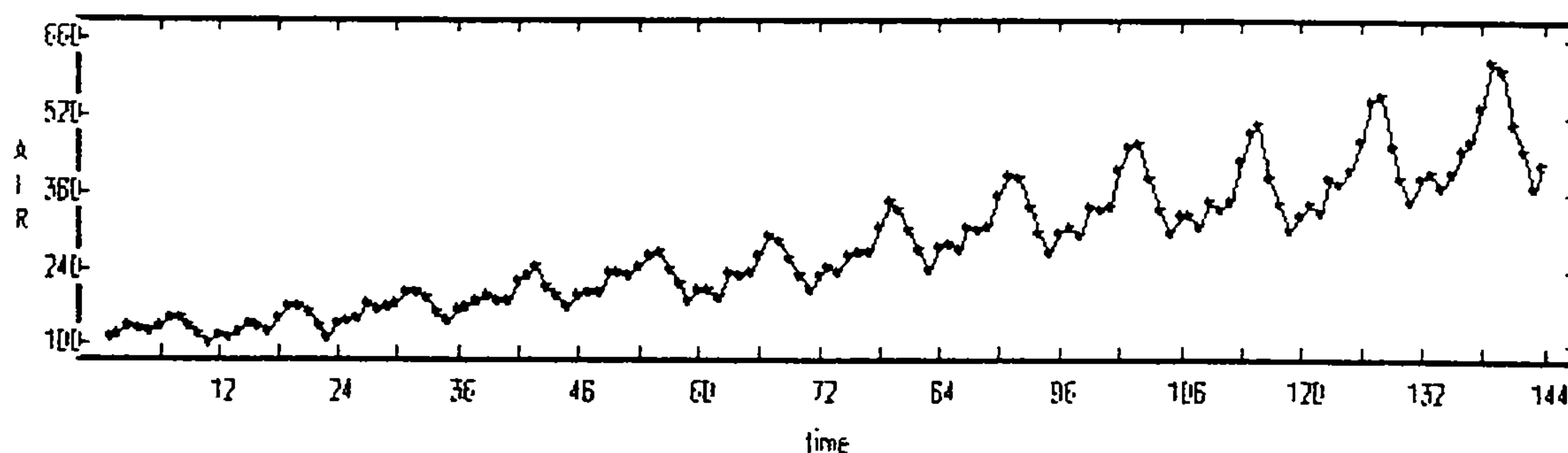
C-90

#EXAMPLE IX. THE AIRLINE PASSENGER DATA

```
input air.file'c:\b325\bjg_air1.dat'
```

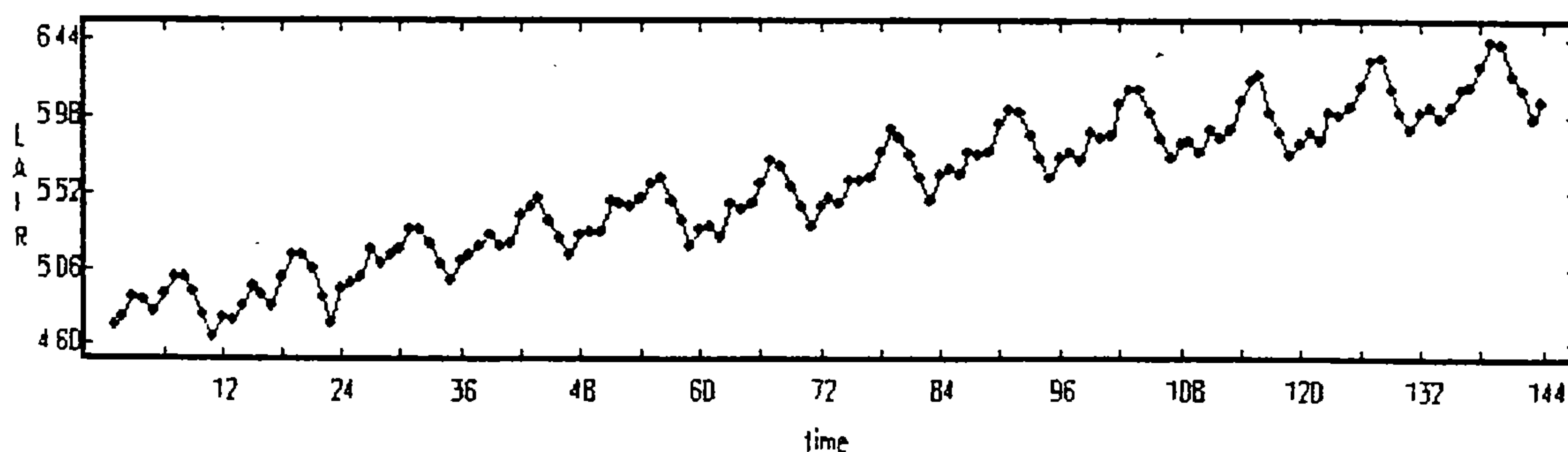
```
AIR      , A 144 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
```

```
--  
graph air
```



```
--  
lair=ln(air)
```

```
--  
graph lair
```



```
uiden lair.maxlag 36
```

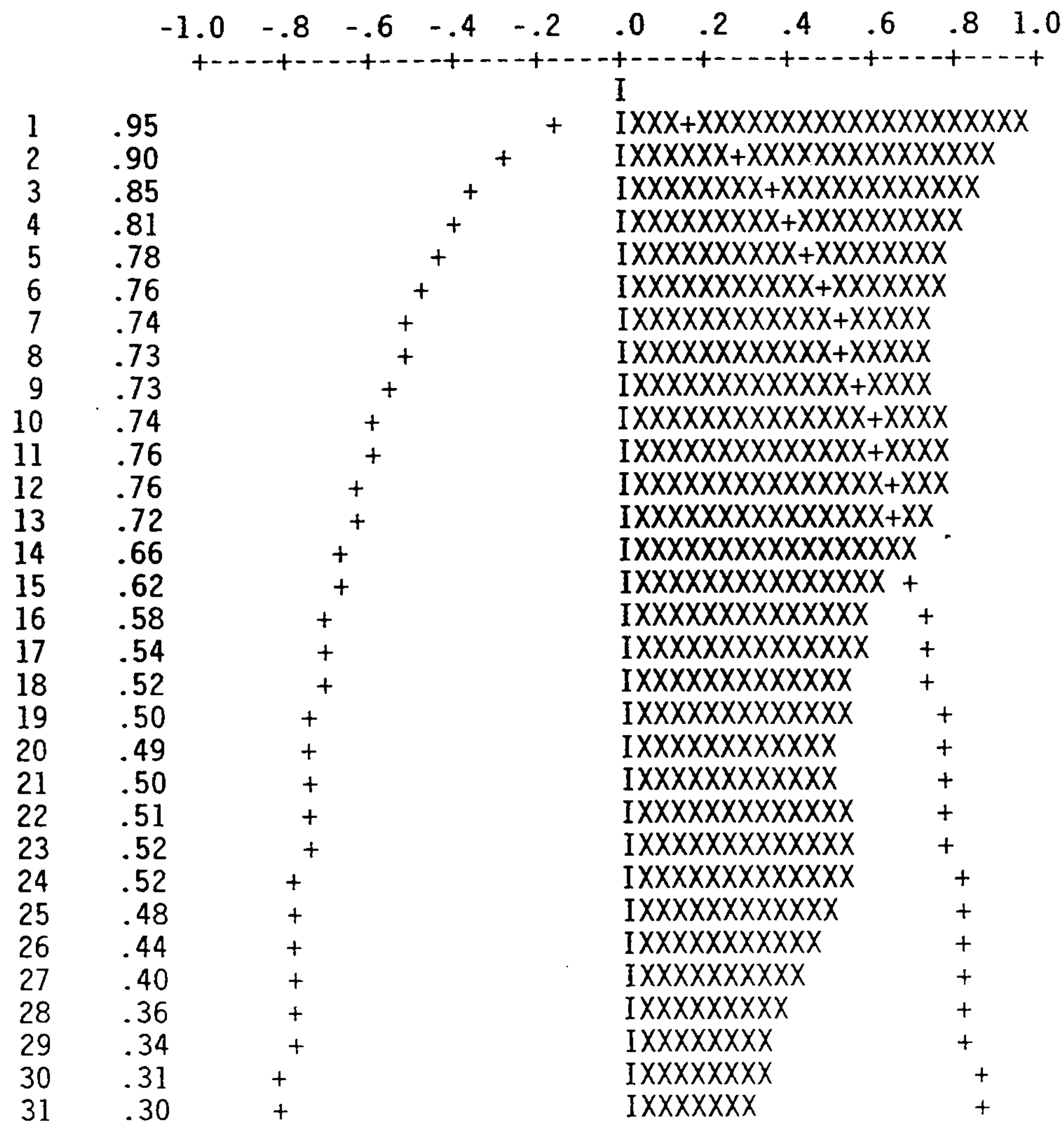
```
TIME PERIOD ANALYZED . . . . . 1 TO 144  
NAME OF THE SERIES . . . . . LAIR  
EFFECTIVE NUMBER OF OBSERVATIONS . . . 144  
STANDARD DEVIATION OF THE SERIES . . . .4399
```

C-91

MEAN OF THE (DIFFERENCED) SERIES . . . 5.5422
 STANDARD DEVIATION OF THE MEAN 0367
 T-VALUE OF MEAN (AGAINST ZERO) 151.1774

AUTOCORRELATIONS

1- 12	.95	.90	.85	.81	.78	.76	.74	.73	.73	.74	.76	.76
ST.E.	.08	.14	.18	.20	.22	.24	.26	.27	.28	.30	.31	.32
Q	134	253	361	459	551	638	722	804	887	974	1065	1158
13- 24	.72	.66	.62	.58	.54	.52	.50	.49	.50	.51	.52	.52
ST.E.	.34	.35	.35	.36	.37	.37	.38	.38	.39	.39	.40	.40
Q	1240	1311	1373	1428	1477	1522	1564	1605	1647	1691	1738	1785
25- 36	.48	.44	.40	.36	.34	.31	.30	.29	.30	.30	.32	.32
ST.E.	.41	.41	.41	.42	.42	.42	.42	.42	.42	.43	.43	.43
Q	1827	1861	1890	1914	1934	1953	1969	1985	2001	2019	2038	2058



PARTIAL AUTOCORRELATIONS

-1.0 -0.8 -0.6 -0.4 -0.2 0.0 0.2 0.4 0.6 0.8 1.0
+-----+-----+-----+-----+-----+-----+-----+-----+-----+-----+

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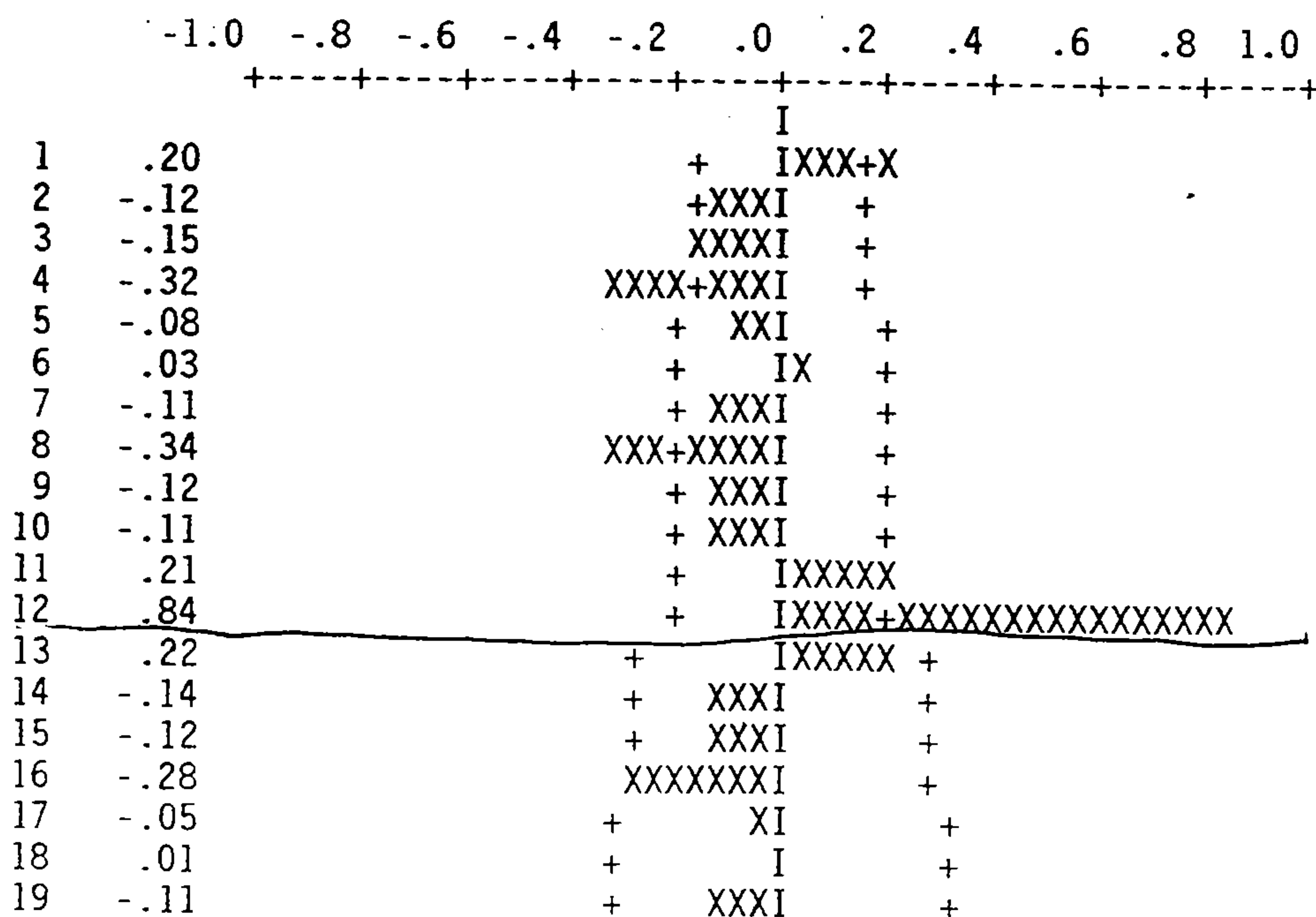
33	-.02	+	I	+
34	.03	+	IX	+
35	.03	+	IX	+
36	.00	+	I	+

--
 uiden lair.dfor 1.maxlag 36

DIFFERENCE ORDERS.	(1-B ¹)
TIME PERIOD ANALYZED	1 TO 144
NAME OF THE SERIES	LAIR
EFFECTIVE NUMBER OF OBSERVATIONS	143
STANDARD DEVIATION OF THE SERIES	.1062
MEAN OF THE (DIFFERENCED) SERIES	.0094
STANDARD DEVIATION OF THE MEAN	.0089
T-VALUE OF MEAN (AGAINST ZERO)	1.0631

AUTOCORRELATIONS

1- 12	.20	-.12	-.15	-.32	-.08	.03	-.11	-.34	-.12	-.11	.21	.84
ST.E.	.08	.09	.09	.09	.10	.10	.10	.10	.11	.11	.11	.11
Q	5.8	7.9	11.3	26.8	27.8	27.9	29.8	47.2	49.3	51.2	57.8	170
13- 24	.22	-.14	-.12	-.28	-.05	.01	-.11	-.34	-.11	-.08	.20	.74
ST.E.	.15	.15	.15	.15	.16	.16	.16	.16	.16	.16	.16	.16
Q	177	180	183	195	196	196	198	217	219	220	227	322
25- 36	.20	-.12	-.10	-.21	-.07	.02	-.12	-.29	-.13	-.04	.15	.66
ST.E.	.19	.19	.19	.19	.19	.19	.19	.19	.19	.19	.19	.20
Q	328	331	333	341	342	342	344	360	363	363	367	451



20	-.34	XXXXXXXXXI	+
21	-.11	+ XXXI	+
22	-.08	+ XXI	+
23	.20	+ IXXXXX	+
24	.74	+ IXXXXXXXX+XXXXXXXXXX	
25	.20	+ IXXXXX	+
26	-.12	+ XXXI	+
27	-.10	+ XXXI	+
28	-.21	+ XXXXI	+
29	-.07	+ XXI	+
30	.02	+ I	+
31	-.12	+ XXXI	+
32	-.29	+ XXXXXXI	+
33	-.13	+ XXXI	+
34	-.04	+ XI	+
35	.15	+ IXXXX	+
36	.66	+ IXXXXXXXXXX+XXXXXX	

PARTIAL AUTOCORRELATIONS

1- 12	.20	-.17	-.10	-.31	.01	-.07	-.21	-.49	-.19	-.53	-.30	.59
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08
13- 24	.03	-.18	.12	.00	.03	-.12	.09	-.05	-.06	-.03	.03	-.01
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08
25- 36	-.05	.02	.03	.02	-.09	.01	.07	.11	-.09	.06	-.09	.06
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08

		-1.0	-.8	-.6	-.4	-.2	.0	.2	.4	.6	.8	1.0	
		+-----+-----+-----+-----+-----+-----+-----+-----+											
							I						
1	.20						+ IXXX+X						
2	-.17						XXXXI						
3	-.10						+ XXI						
4	-.31						XXXX+XXXI						
5	.01						+ I						
6	-.07						+ XXI						
7	-.21						X+XXXI						
8	-.49						XXXXXXXX+XXXI						
9	-.19						X+XXXI						
10	-.53						XXXXXXXX+XXXI						
11	-.30						XXXX+XXXI						
12	.59						+ IXXX+XXXXXXXXXXXX						
13	.03						+ IX						
14	-.18						X+XXXI						
15	.12						+ IXXX+						
16	.00						+ I						
17	.03						+ IX						
18	-.12						+XXXI						
19	.09						+ IXX						

20	-.05	+	XI	+
21	-.06	+	XXI	+
22	-.03	+	XI	+
23	.03	+	IX	+
24	-.01	+	I	+
25	-.05	+	XI	+
26	.02	+	I	+
27	.03	+	IX	+
28	.02	+	I	+
29	-.09	+	XXI	+
30	.01	+	I	+
31	.07	+	IXX	+
32	.11	+	IXXX	+
33	-.09	+	XXI	+
34	.06	+	IXX	+
35	-.09	+	XXI	+
36	.06	+	IX	+

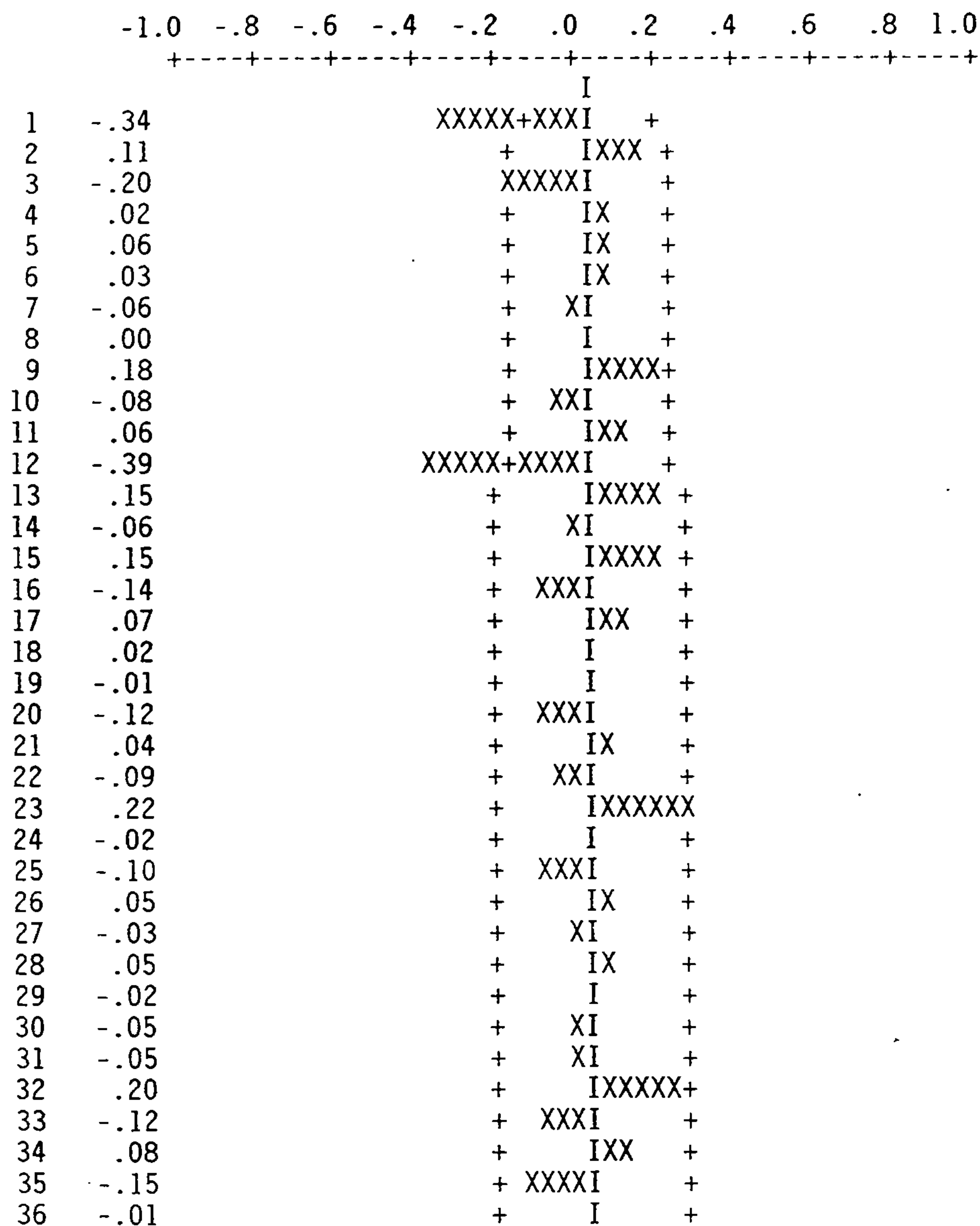
--

uident lair.dfor 1,12.maxlag 36

	1	12
DIFFERENCE ORDERS.	(1-B)	(1-B)
TIME PERIOD ANALYZED	1 TO	144
NAME OF THE SERIES	LAIR	
EFFECTIVE NUMBER OF OBSERVATIONS	131	
STANDARD DEVIATION OF THE SERIES	.0457	
MEAN OF THE (DIFFERENCED) SERIES	.0003	
STANDARD DEVIATION OF THE MEAN	.0040	
T-VALUE OF MEAN (AGAINST ZERO)	.0729	

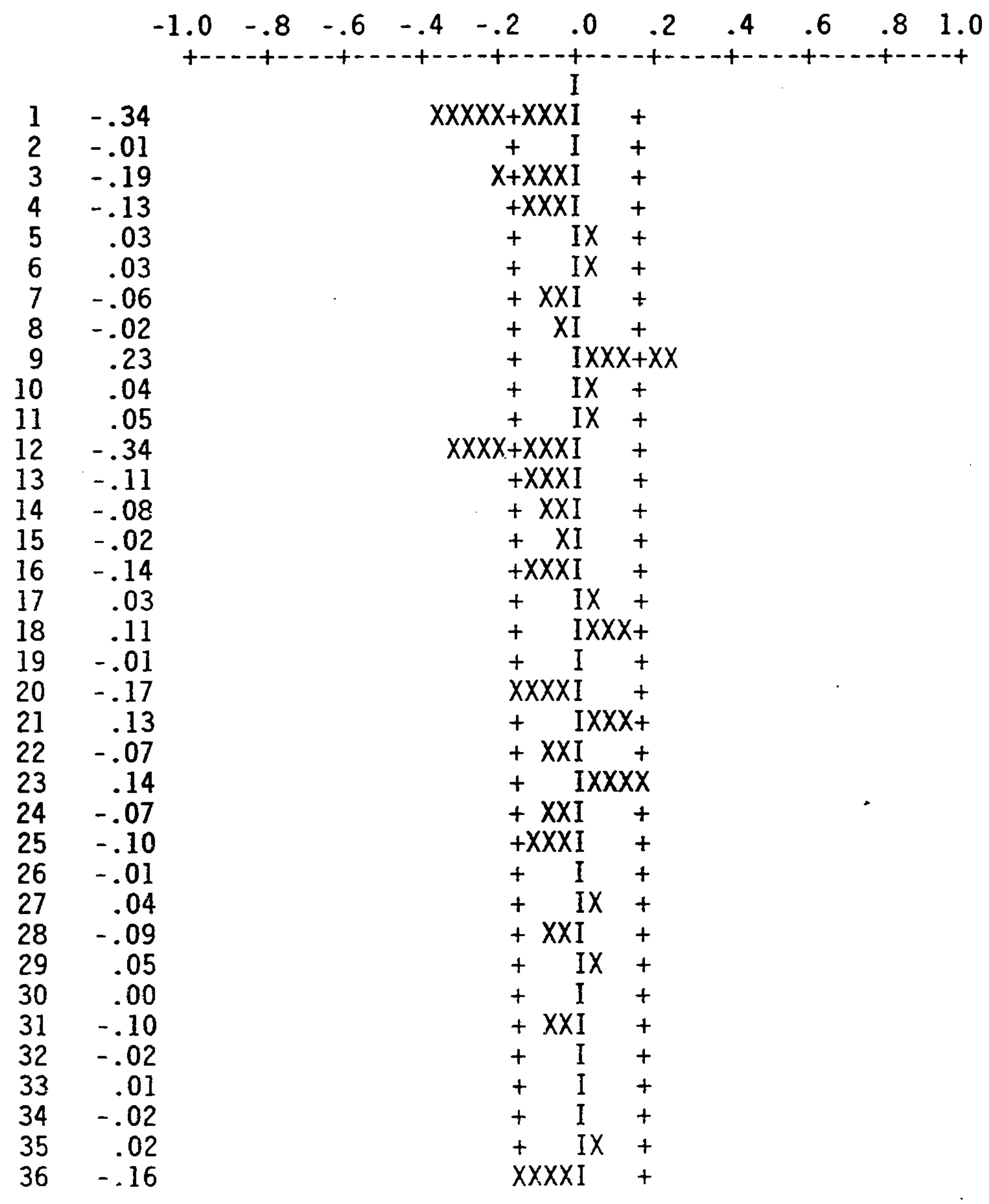
AUTOCORRELATIONS

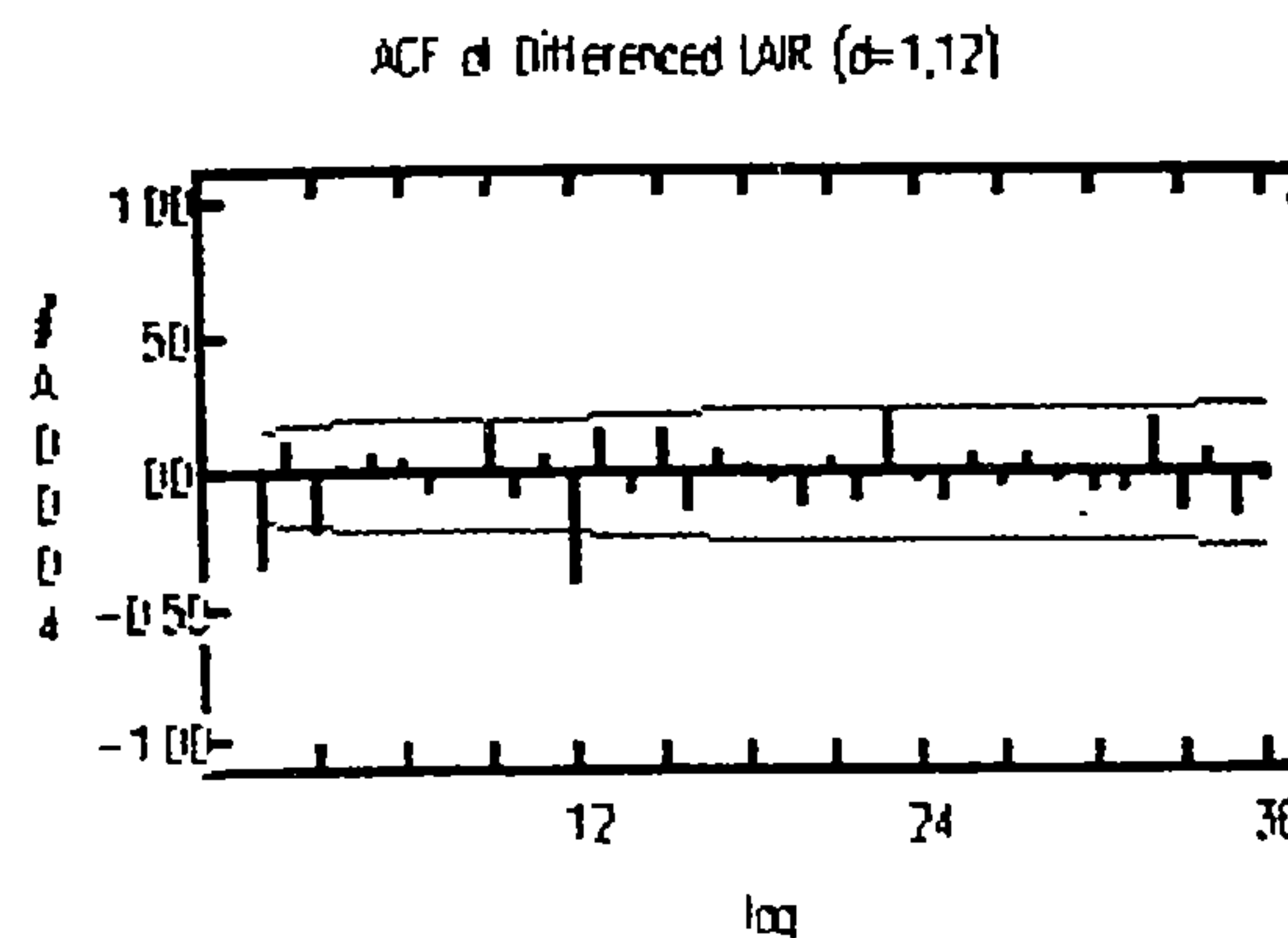
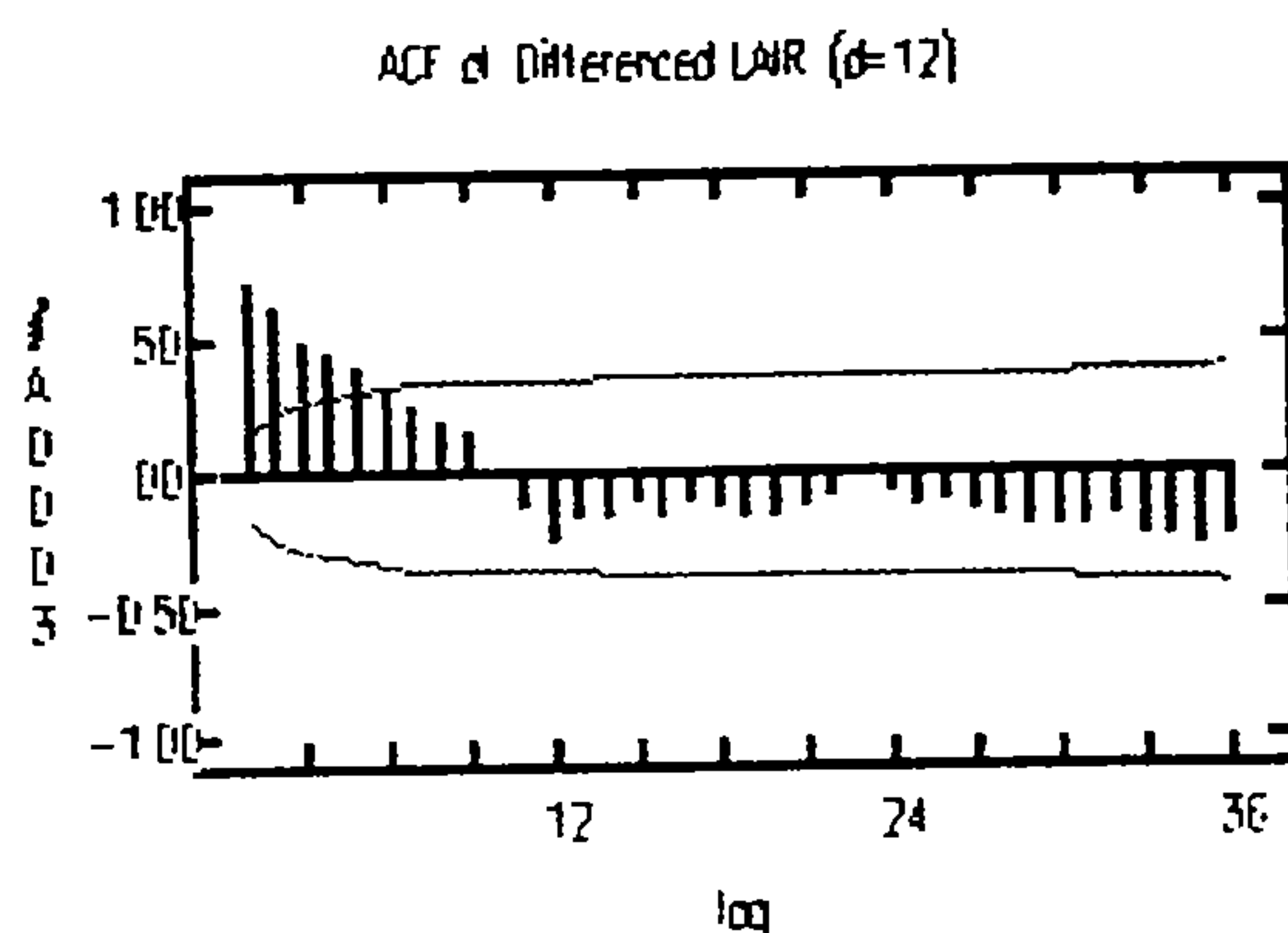
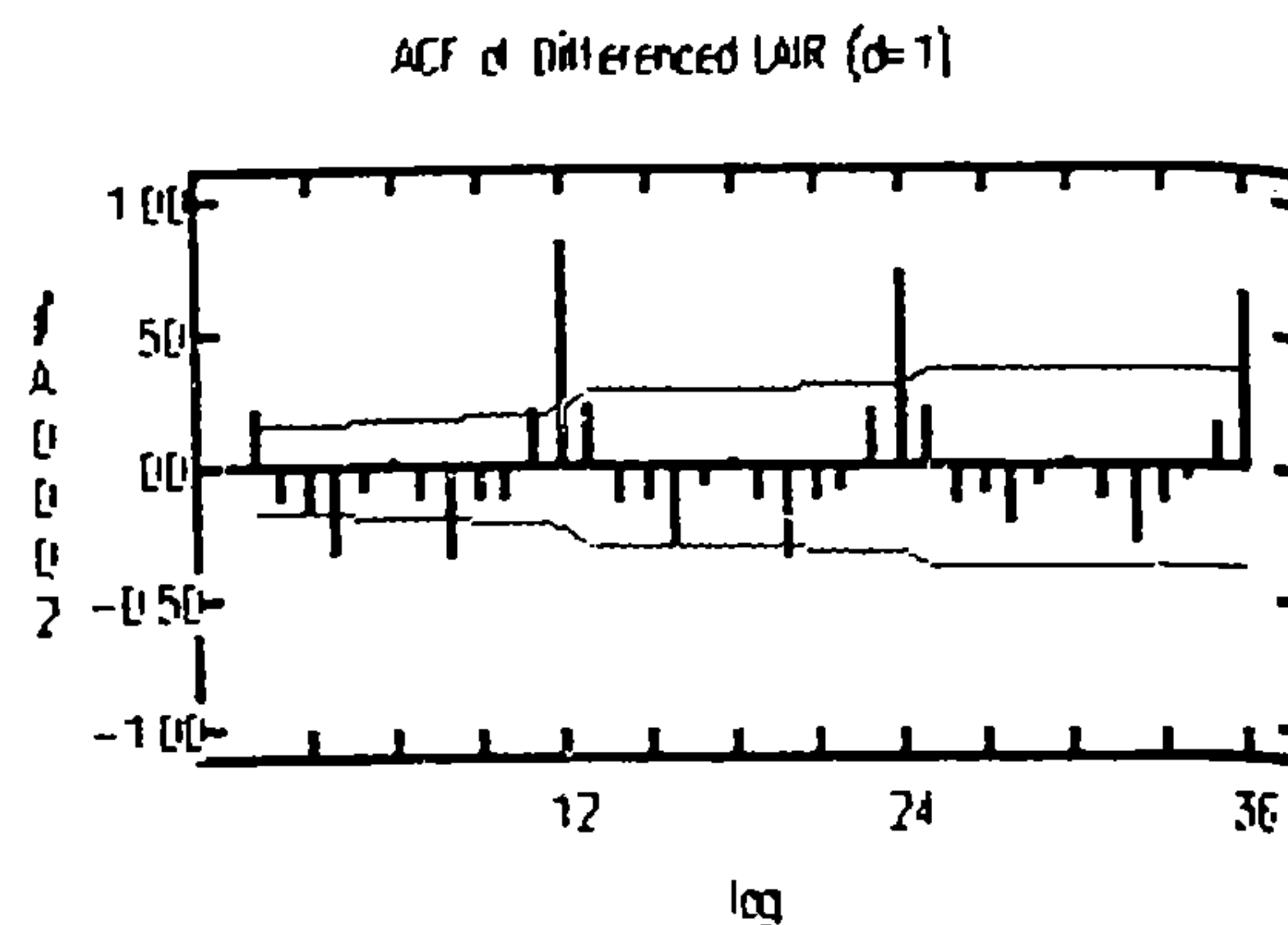
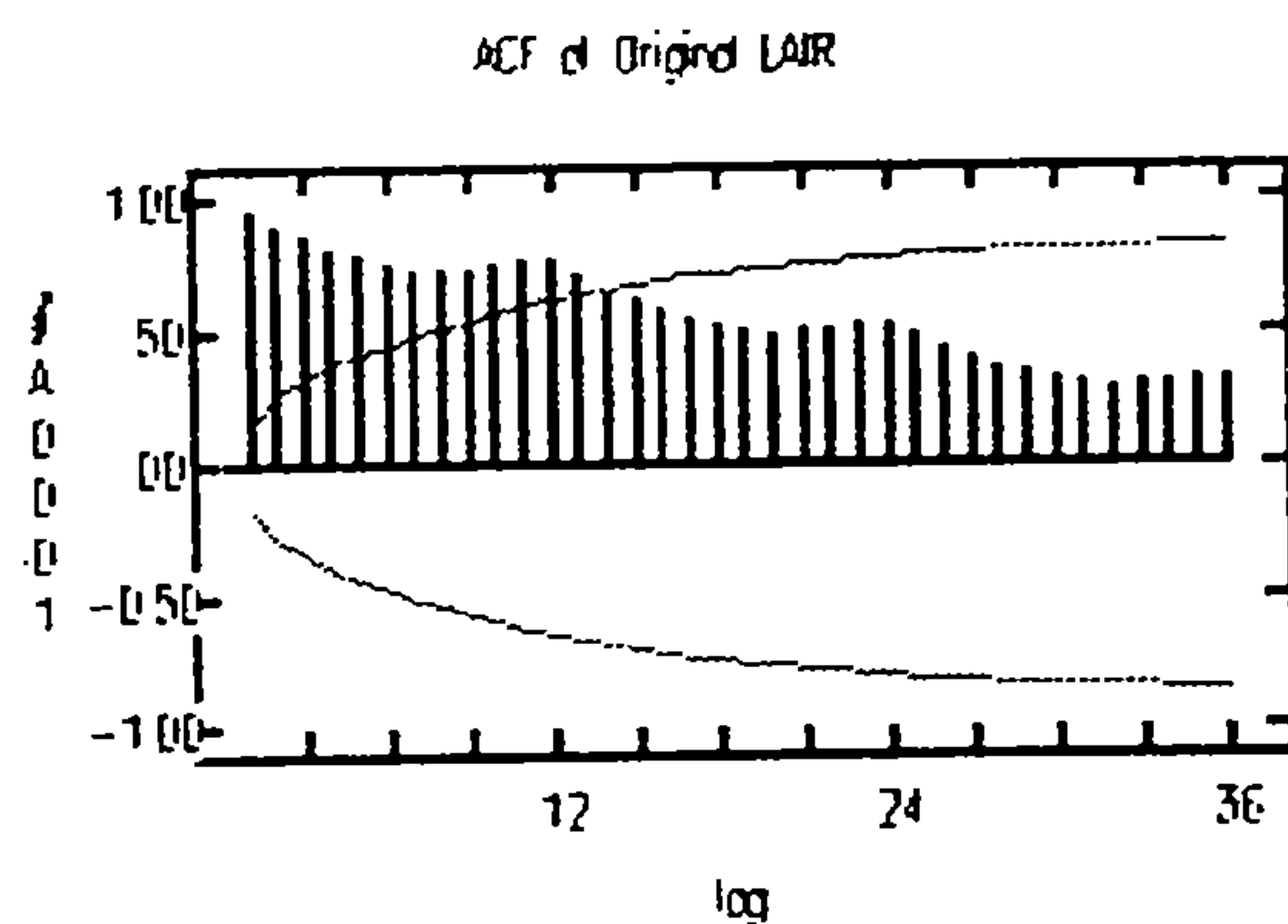
1- 12	-.34	.11	-.20	.02	.06	.03	-.06	-.00	.18	-.08	.06	-.39
ST.E.	.09	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10
Q	15.6	17.1	22.6	22.7	23.1	23.3	23.7	23.7	28.1	29.0	29.6	51.5
13- 24	.15	-.06	.15	-.14	.07	.02	-.01	-.12	.04	-.09	.22	-.02
ST.E.	.12	.12	.12	.12	.12	.12	.12	.12	.12	.12	.12	.12
Q	54.9	55.4	58.7	61.6	62.4	62.4	62.5	64.6	64.8	66.2	74.2	74.3
25- 36	-.10	.05	-.03	.05	-.02	-.05	-.05	.20	-.12	.08	-.15	-.01
ST.E.	.12	.12	.13	.13	.13	.13	.13	.13	.13	.13	.13	.13
Q	75.9	76.3	76.5	76.8	76.9	77.3	77.8	84.6	87.3	88.3	92.6	92.6



PARTIAL AUTOCORRELATIONS

1- 12	-.34	-.01	-.19	-.13	.03	.03	-.06	-.02	.23	.04	.05	-.34
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09
13- 24	-.11	-.08	-.02	-.14	.03	.11	-.01	-.17	.13	-.07	.14	-.07
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09
25- 36	-.10	-.01	.04	-.09	.05	-.00	-.10	-.02	.01	-.02	.02	-.16
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09





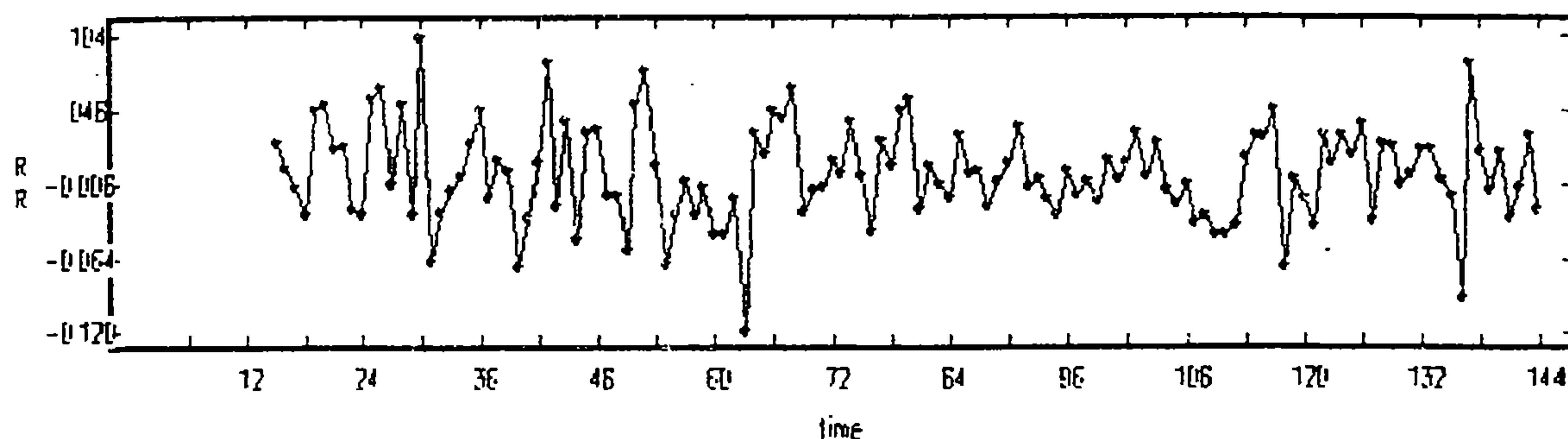
```
tsm ml.model lair(1,12)=(1-t1*b)(1-t12*b**12)noise
estim ml.method exact.hold resi(rr)
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE		TYPE OF	ORIGINAL		DIFFERENCING				
		VARIABLE	OR CENTERED						
					1	12			
LAIR		RANDOM	ORIGINAL		(1-B)	(1-B)			

PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	T1	LAIR	MA	1	1	NONE	.4023	.0801	5.02
2	T12	LAIR	MA	2	12	NONE	.5568	.0729	7.64
TOTAL SUM OF SQUARES							.278684E+02		
TOTAL NUMBER OF OBSERVATIONS							144		
RESIDUAL SUM OF SQUARES.							.175255E+00		

--
graph rr



--
uforecast ml.origins 103,144.nofs 36.hold forecasts(laf1,laf2)@
,std_errs(sla1,sla2)

NOTE: THE EXACT METHOD FOR COMPUTING RESIDUALS IS USED

36 FORECASTS, BEGINNING AT 103

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
104	6.1229	.0367	6.1463
105	6.0003	.0428	6.0014
106	5.8652	.0481	5.8493
107	5.7357	.0528	5.7203
108	5.8670	.0572	5.8171
109	5.8953	.0613	5.8289
110	5.8604	.0651	5.7621
111	6.0176	.0687	5.8916
112	5.9993	.0721	5.8522
113	6.0137	.0754	5.8944
114	6.1702	.0785	6.0753
115	6.2760	.0815	6.1964
116	6.2563	.0900	6.2246
117	6.1337	.0954	6.0014
118	5.9986	.1006	5.8833
119	5.8691	.1054	5.7366
120	6.0004	.1101	5.8201
121	6.0287	.1145	5.8861
122	5.9938	.1188	5.8348
123	6.1510	.1230	6.0064
124	6.1327	.1270	5.9814
125	6.1471	.1309	6.0403
126	6.3036	.1347	6.1570

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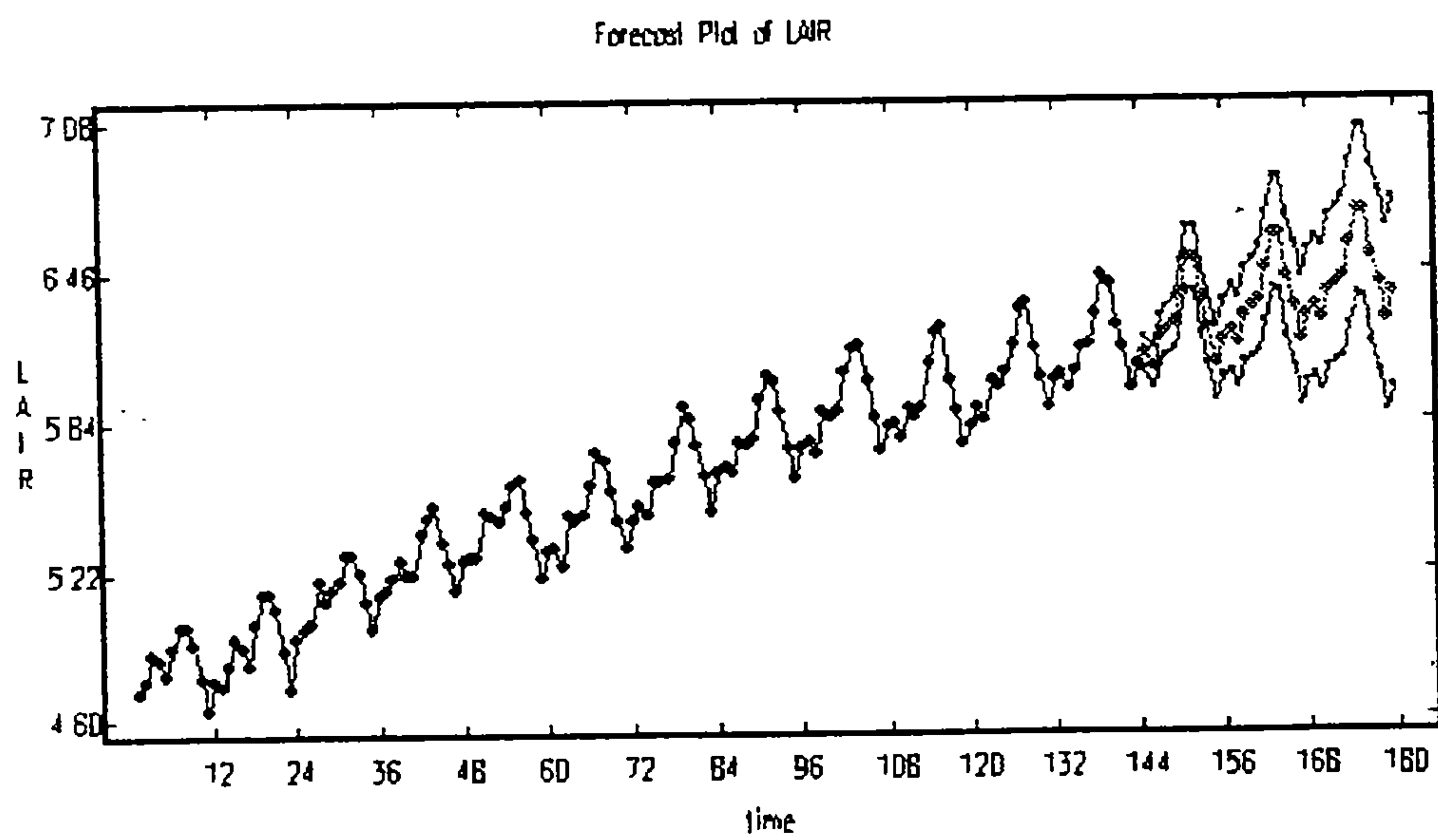
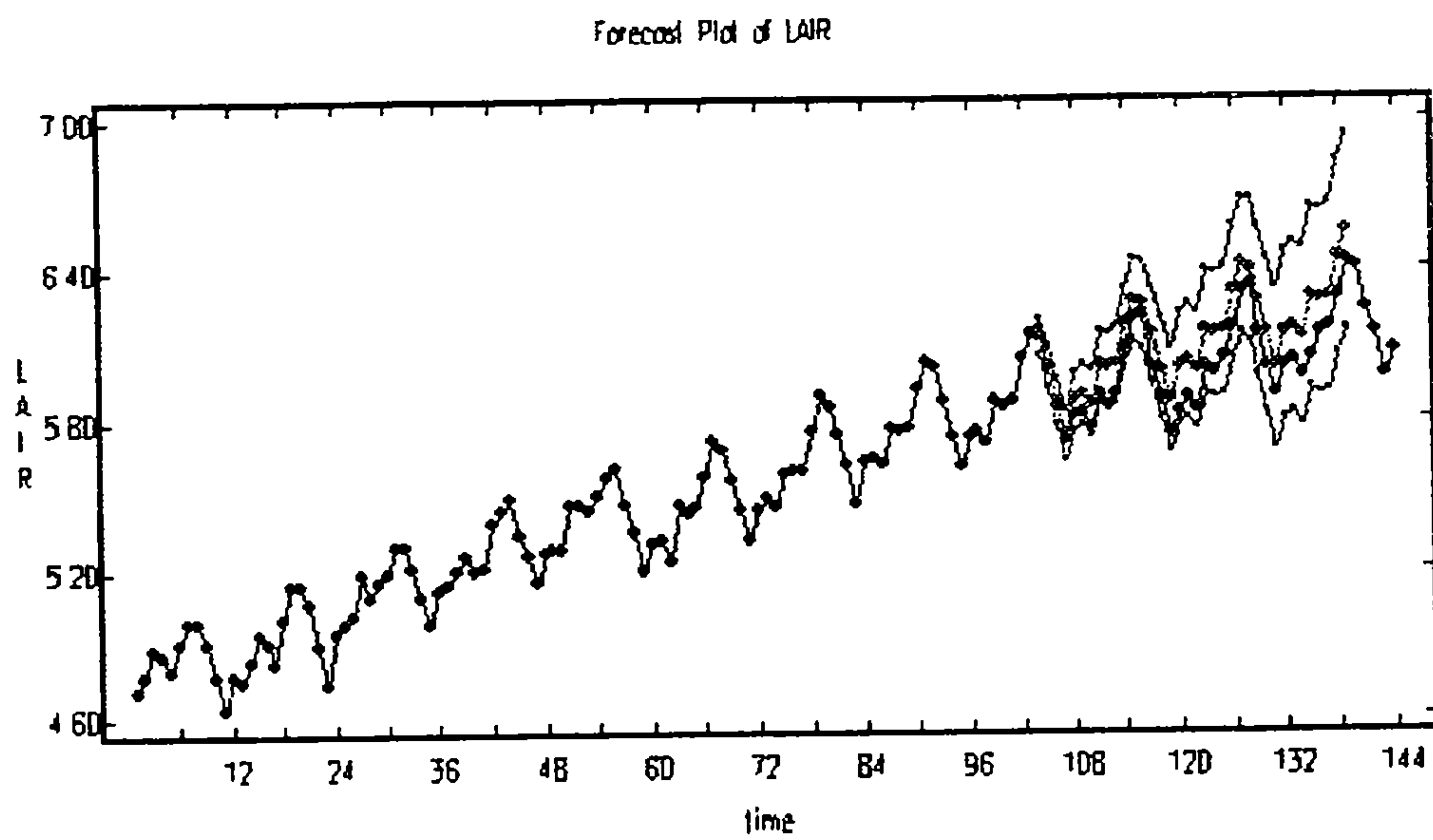
127	6.4094	.1383	6.3063
128	6.3897	.1464	6.3261
129	6.2671	.1521	6.1377
130	6.1320	.1577	6.0088
131	6.0025	.1630	5.8916
132	6.1338	.1682	6.0039
133	6.1622	.1732	6.0331
134	6.1272	.1781	5.9687
135	6.2844	.1828	6.0379
136	6.2661	.1875	6.1334
137	6.2805	.1920	6.1570
138	6.4370	.1964	6.2823
139	6.5428	.2007	6.4329

 36 FORECASTS, BEGINNING AT 144

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
145	6.1102	.0367	
146	6.0538	.0428	
147	6.1717	.0481	
148	6.1993	.0528	
149	6.2326	.0572	
150	6.3688	.0613	
151	6.5073	.0651	
152	6.5029	.0687	
153	6.3247	.0721	
154	6.2090	.0754	
155	6.0635	.0785	
156	6.1680	.0815	
157	6.2064	.0900	
158	6.1500	.0954	
159	6.2680	.1006	
160	6.2956	.1054	
161	6.3288	.1101	
162	6.4650	.1145	
163	6.6036	.1188	
164	6.5992	.1230	
165	6.4210	.1270	
166	6.3053	.1309	
167	6.1597	.1347	
168	6.2643	.1383	
169	6.3027	.1464	
170	6.2463	.1521	
171	6.3642	.1577	
172	6.3918	.1630	
173	6.4251	.1682	
174	6.5613	.1732	
175	6.6998	.1781	
176	6.6954	.1828	
177	6.5172	.1875	

178	6.4015	.1920
179	6.2560	.1964
180	6.3605	.2007

```
--
fsave laf1,slal,laf2,sla2,file'lafor.fmt'
```



C-103

**#EXAMPLE X. MONTHLY AVERAGES OF OZONE AT DOWNTOWN L.A.
1/55-12/72**

--

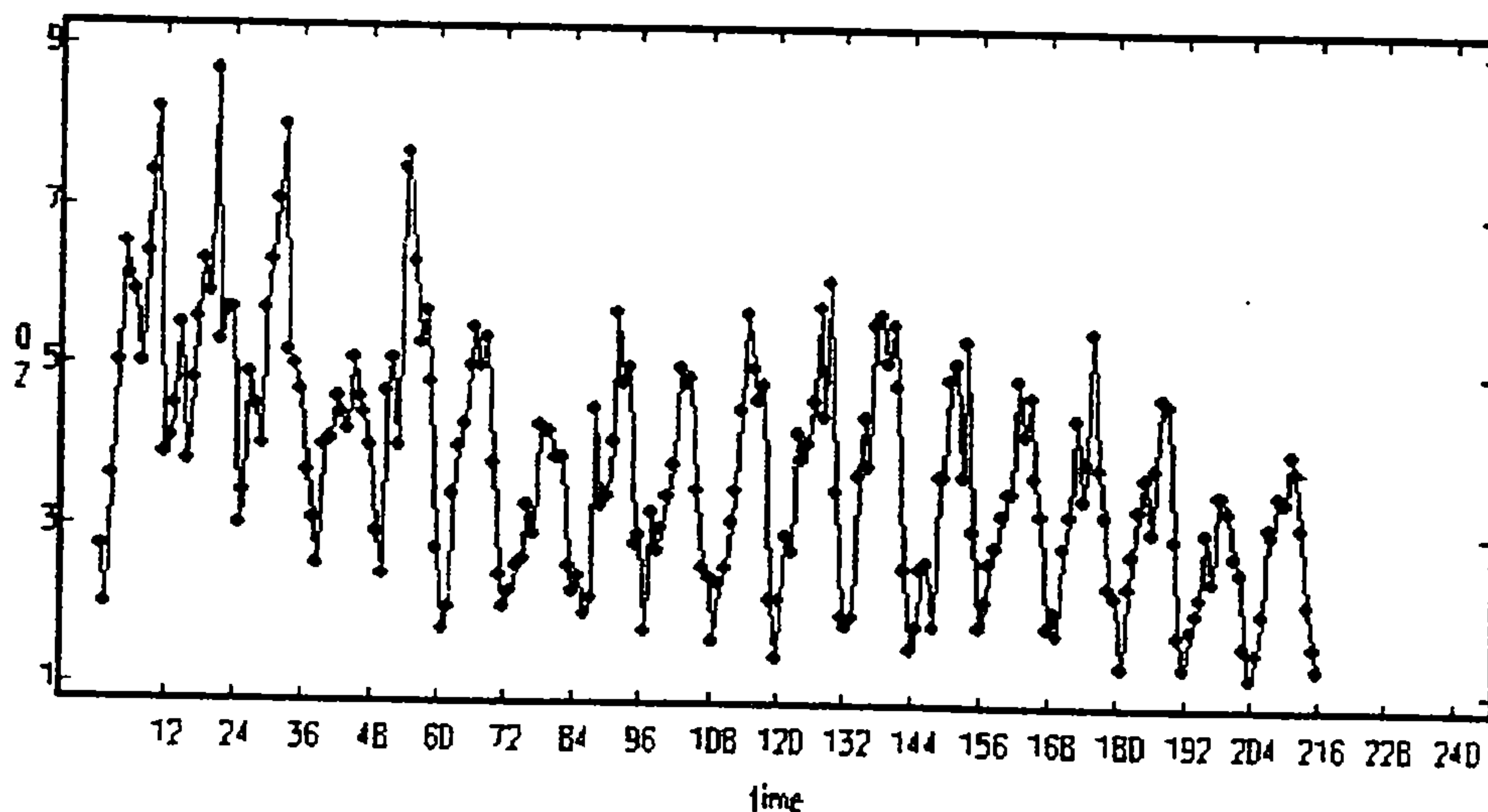
```
input oz,x1,x2,x3.file'c:\b325\btozone.dat'
```

```
OZ      , A 240 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
X1      , A 240 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
X2      , A 240 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
X3      , A 240 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
--
```

#The variable oz only contains 216 data points. The last 24 elements are reserved for forecasting purposes. The variables x1,x2 and x3 are indicator variables for intervention analysis.

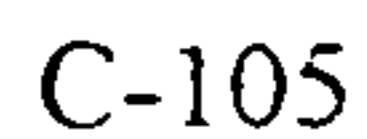
```
fsave oz,x1,x2,x3.file'ozone.fmt'
```

#We now go to SCA Graphics to plot oz.



TIME PERIOD ANALYZED	1	TO	216
NAME OF THE SERIES			0Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .			216
STANDARD DEVIATION OF THE SERIES . . .			1.4882
MEAN OF THE (DIFFERENCED) SERIES . . .			3.7727
STANDARD DEVIATION OF THE MEAN			.1013
T-VALUE OF MEAN (AGAINST ZERO)			37.2583

1- 12	.73	.50	.28	.08	-.09	-.12	-.12	.01	.23	.46	.61	.67
ST.E.	.07	.10	.11	.11	.11	.11	.11	.11	.11	.12	.12	.14
Q	116	170	187	188	190	193	196	196	209	258	342	445
13- 24	.57	.37	.14	-.04	-.18	-.20	-.15	-.03	.19	.41	.52	.57
ST.E.	.15	.16	.16	.17	.17	.17	.17	.17	.17	.17	.17	.18
Q	520	552	557	557	565	575	580	580	589	629	696	775
25- 36	.44	.26	.08	-.13	-.25	-.24	-.19	-.08	.14	.32	.43	.48
ST.E.	.19	.19	.20	.20	.20	.20	.20	.20	.20	.20	.20	.21
Q	823	840	842	846	862	877	886	887	892	919	966	1026



15	.14	+	IXXXX	+
16	-.04	+	XI	+
17	-.18	+	XXXXI	+
18	-.20	+	XXXXXI	+
19	-.15	+	XXXXI	+
20	-.03	+	XI	+
21	.19	+	IXXXX	+
22	.41	+	IXXXXXXX+XX	
23	.52	+	IXXXXXXXX+XXXX	
24	.57	+	IXXXXXXXX+XXXXX	
25	.44	+	IXXXXXXXX+XX	
26	.26	+	IXXXXXXX	+
27	.08	+	IXX	+
28	-.13	+	XXXI	+
29	-.25	+	XXXXXXI	+
30	-.24	+	XXXXXXI	+
31	-.19	+	XXXXXI	+
32	-.08	+	XXI	+
33	.14	+	IXXX	+
34	.32	+	IXXXXXXXX	+
35	.43	+	IXXXXXXXX+X	
36	.48	+	IXXXXXXXX+XX	

PARTIAL AUTOCORRELATIONS

1- 12	.73	-.08	-.12	-.13	-.11	.11	.01	.22	.30	.28	.19	.14
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07
13- 24	-.07	-.13	-.15	-.06	-.02	.07	.03	-.05	.10	.07	-.01	.08
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07
25- 36	-.12	-.01	.01	-.13	.01	.09	-.06	-.09	.03	-.02	.01	.13
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07

-1.0	-.8	-.6	-.4	-.2	.0	.2	.4	.6	.8	1.0
+	+	+	+	+	+	+	+	+	+	+

1	.73	+	IXX+XXXXXXXXXXXXXXXXX
2	-.08	+XXI	+
3	-.12	XXXI	+
4	-.13	XXXI	+
5	-.11	XXXI	+
6	.11	+ IXXX	
7	.01	+ I	+
8	.22	+ IXX+XXX	
9	.30	+ IXX+XXXXX	
10	.28	+ IXX+XXXX	
11	.19	+ IXX+XX	
12	.14	+ IXXX	
13	-.07	+XXI	+
14	-.13	XXXI	+
15	-.15	X+XXI	+

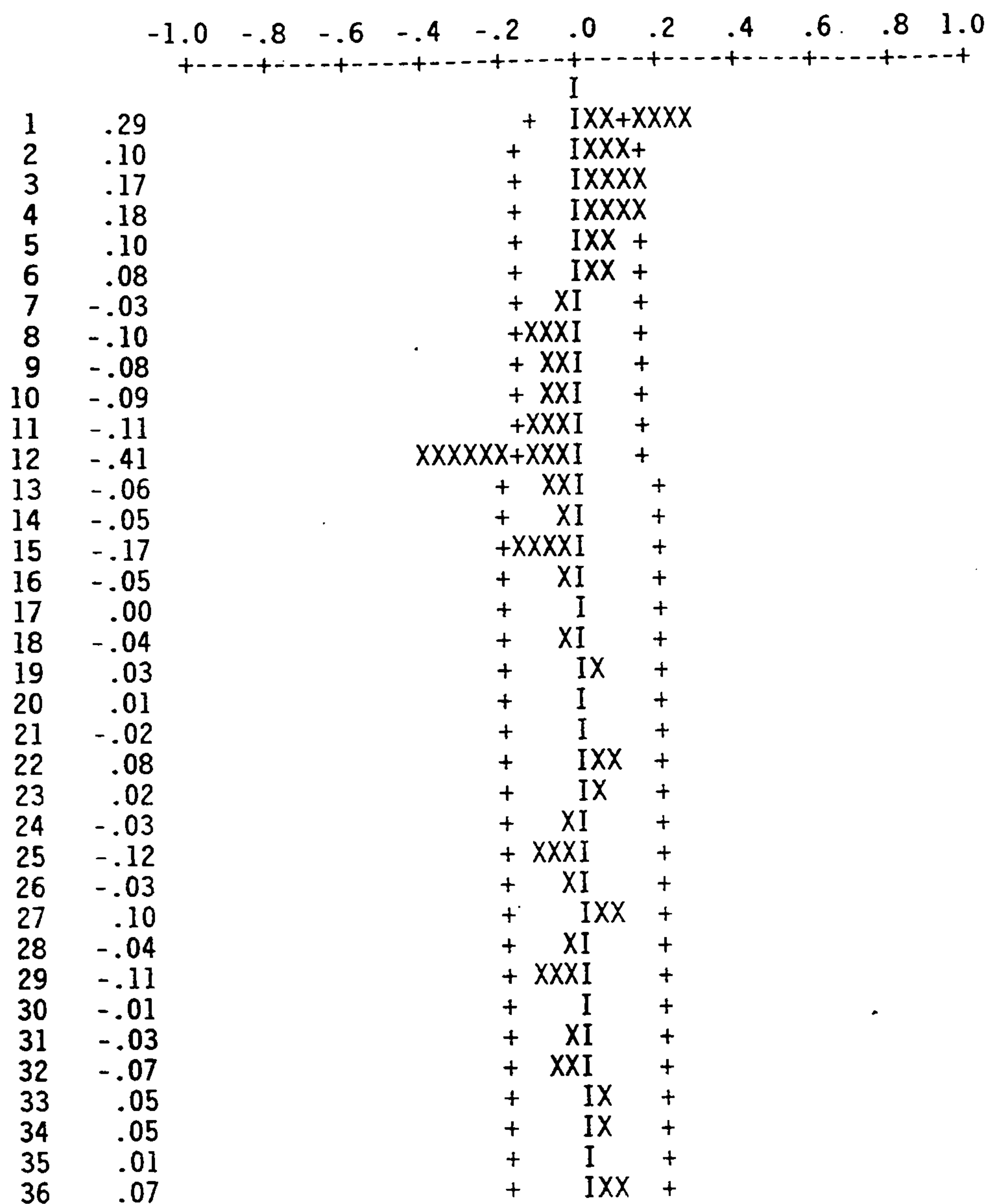
16	-.06	+XXI +
17	-.02	+ XI +
18	.07	+ IXX+
19	.03	+ IX +
20	-.05	+ XI +
21	.10	+ IXXX
22	.07	+ IXX+
23	-.01	+ I +
24	.08	+ IXX+
25	-.12	XXXI +
26	-.01	+ I +
27	.01	+ I +
28	-.13	XXXI +
29	.01	+ I +
30	.09	+ IXX+
31	-.06	+ XI +
32	-.09	+XXI +
33	.03	+ IX +
34	-.02	+ I +
35	.01	+ I +
36	.13	+ IXXX

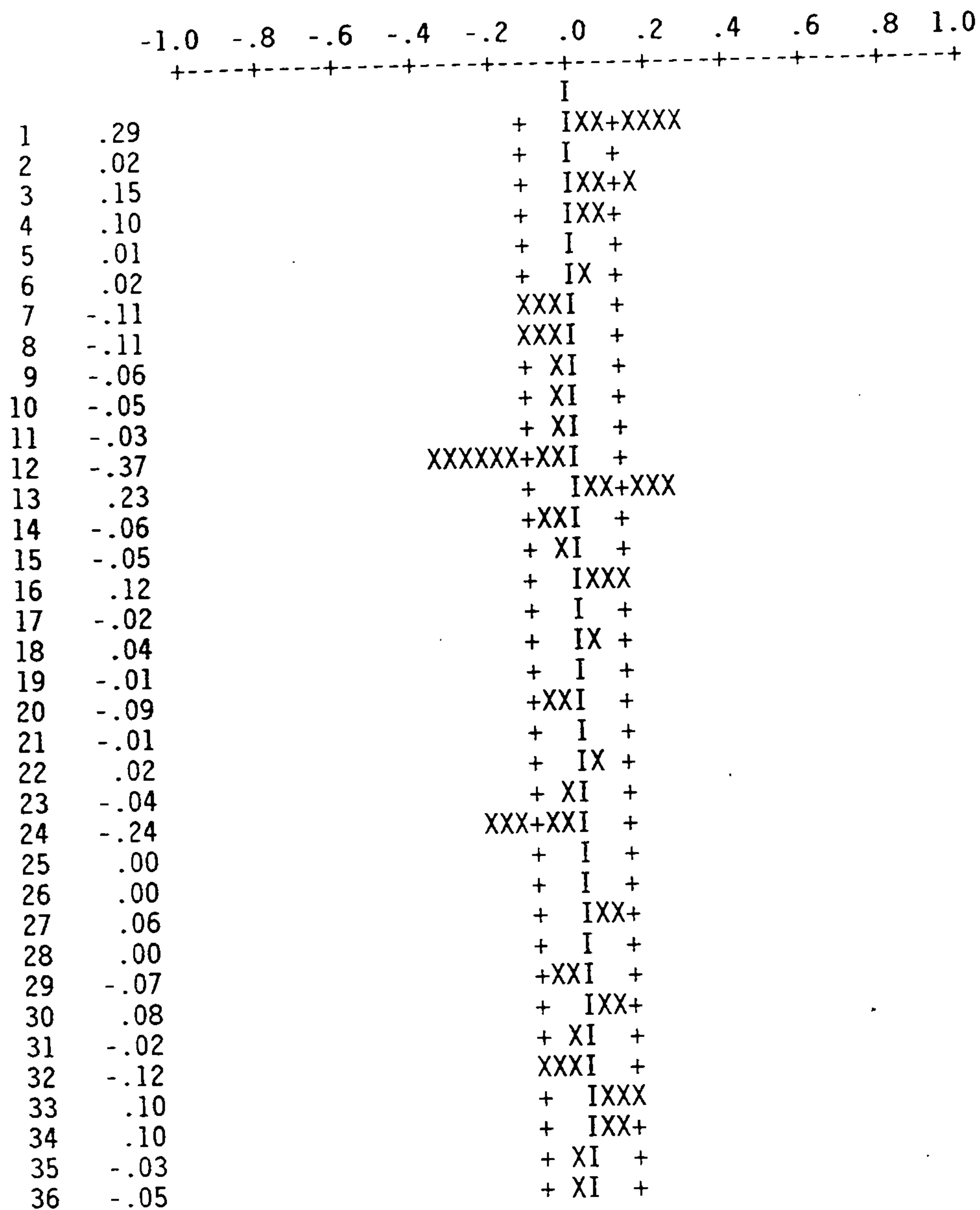
--
iden oz.dfor 12

		12	
DIFFERENCE ORDERS.		(1-B)	
TIME PERIOD ANALYZED		1 TO	216
NAME OF THE SERIES			OZ
EFFECTIVE NUMBER OF OBSERVATIONS			204
STANDARD DEVIATION OF THE SERIES			1.0497
MEAN OF THE (DIFFERENCED) SERIES			-.1485
STANDARD DEVIATION OF THE MEAN			.0735
T-VALUE OF MEAN (AGAINST ZERO)			-2.0210

AUTOCORRELATIONS

1- 12	.29	.10	.17	.18	.10	.08	-.03	-.10	-.08	-.09	-.11	-.41
ST.E.	.07	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08
Q	17.7	19.9	26.3	33.1	35.1	36.4	36.7	39.0	40.5	42.1	44.6	80.9
13- 24	-.06	-.05	-.17	-.05	-.00	-.04	.03	.01	-.02	.08	.02	-.03
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.10	.10
Q	81.8	82.4	88.6	89.2	89.2	89.5	89.7	89.7	89.8	91.2	91.3	91.5
25- 36	-.12	-.03	.10	-.04	-.11	-.01	-.03	-.07	.05	.05	.01	.07
ST.E.	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10	.10
Q	94.7	94.9	97.2	97.7	100	100	101	102	102	103	103	104





```
--
tsm moz.model oz(12)=(1-t1*b)(1-t12*b**12)noise
estim moz.
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MOZ

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
0Z	RANDOM	ORIGINAL	12 (1-B)

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 T1	OZ	MA	1	1	NONE	-.3620	.0651	-5.56
2 T12	OZ	MA	2	12	NONE	.4877	.0613	7.95

TOTAL SUM OF SQUARES 478369E+03
 TOTAL NUMBER OF OBSERVATIONS 216
 RESIDUAL SUM OF SQUARES. 166394E+03
 R-SQUARE 632
 EFFECTIVE NUMBER OF OBSERVATIONS . . .204
 RESIDUAL VARIANCE ESTIMATE 815656E+00
 RESIDUAL STANDARD ERROR. 903137E+00

--
estim moz.method exact.hold resi(r1)

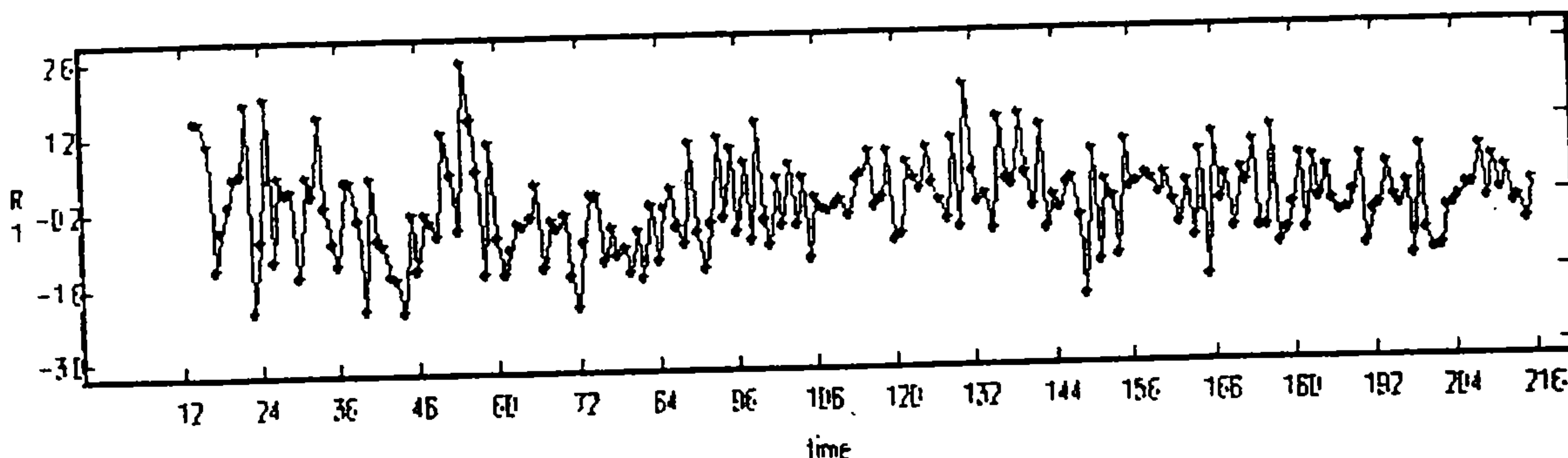
SUMMARY FOR UNIVARIATE TIME SERIES MODEL --- MOZ

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
OZ	RANDOM	ORIGINAL	12 (1-B)

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 T1	OZ	MA	1	1	NONE	-.3605	.0653	-5.52
2 T12	OZ	MA	2	12	NONE	.5466	.0544	10.04

TOTAL SUM OF SQUARES 478369E+03
 TOTAL NUMBER OF OBSERVATIONS 216
 RESIDUAL SUM OF SQUARES. 156976E+03
 R-SQUARE 651
 EFFECTIVE NUMBER OF OBSERVATIONS . . .203
 RESIDUAL VARIANCE ESTIMATE 773282E+00
 RESIDUAL STANDARD ERROR. 879365E+00

--
graph r1



--
acf r1.maxlag 24

TIME PERIOD ANALYZED 14 TO 216
NAME OF THE SERIES R1
EFFECTIVE NUMBER OF OBSERVATIONS 203
STANDARD DEVIATION OF THE SERIES 8326
MEAN OF THE (DIFFERENCED) SERIES -.2256
STANDARD DEVIATION OF THE MEAN 0584
T-VALUE OF MEAN (AGAINST ZERO) -3.8610

AUTOCORRELATIONS

1- 12	-.01	.14	.09	.17	.02	.13	-.01	-.02	.01	-.02	.08	-.10
ST.E.	.07	.07	.07	.07	.07	.07	.08	.08	.08	.08	.08	.08
Q	.0	4.0	5.5	11.7	11.8	15.2	15.2	15.2	15.2	15.3	16.6	18.7
13- 24	.07	.01	-.08	.04	-.01	.02	.05	-.03	-.05	.11	-.00	-.02
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08
Q	19.9	19.9	21.4	21.7	21.8	21.9	22.5	22.7	23.2	25.8	25.8	25.9

-1.0 - .8 - .6 - .4 - .2 .0 .2 .4 .6 .8 1.0
+-----+-----+-----+-----+-----+-----+-----+-----+

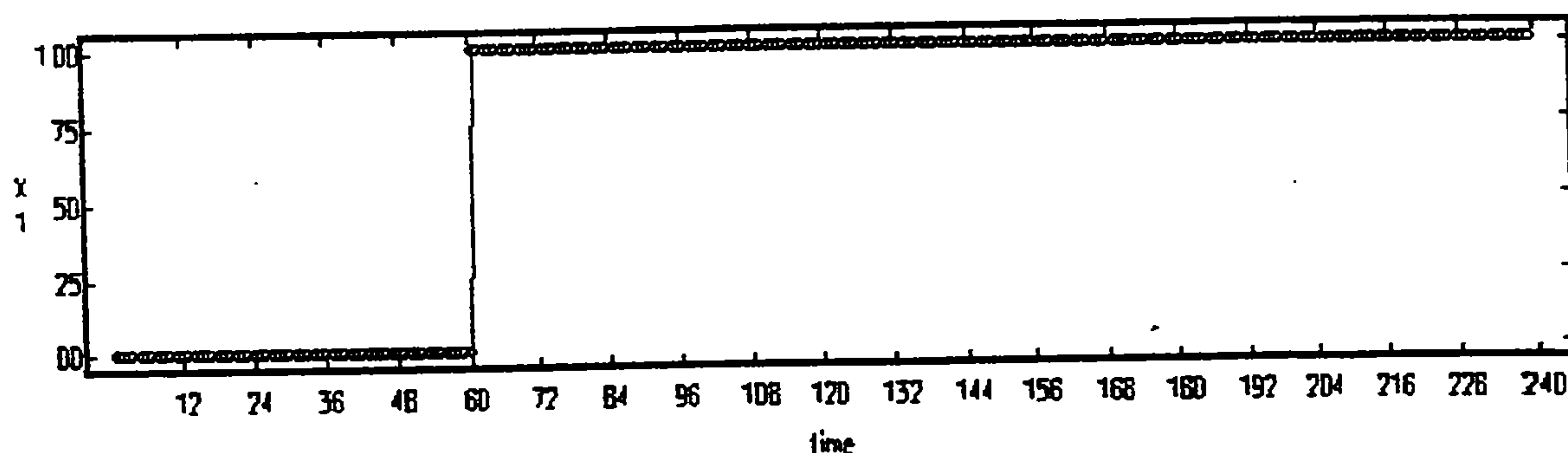
1	-.01											
2	.14											
3	.09											
4	.17											
5	.02											
6	.13											
7	-.01											
8	-.02											
9	.01											
10	-.02											
11	.08											

C-111

12	-.10	+ XXI +
13	.07	+ IXX +
14	.01	+ I +
15	-.08	+ XXI +
16	.04	+ IX +
17	-.01	+ I +
18	.02	+ IX +
19	.05	+ IX +
20	-.03	+ XI +
21	-.05	+ XI +
22	.11	+ IXXX+
23	.00	+ I +
24	-.02	+ I +

--

#We now begin to apply intervention analysis techniques to this data set. The first is to estimate the possible level shift at the beginning of 1960 due to changes in the composition of gasolin sold in the L.A. county and the opening of a new freeway.



#The variable x1 shown above is an indicator (or dummy) variable which has the value 0 for months prior to January, 1960 and 1 since. Using this, the model becomes

$$Z_t = c_1 x_{1t} + \frac{(1-\theta_1 B)(1-\theta_{12} B^{12})}{(1-B^{12})} a_t$$

where Z_t is the ozone observation for the t-th month.

--

$$Z_t(12) = c_1 X_{1t}(12) + (1 - \theta B)(1 - \phi B^{12}) a_t$$

tsm mozz.model oz(12)=(c1)x1(12)+(1-t1*b)(1-t12*b**12)noise

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MOZZ

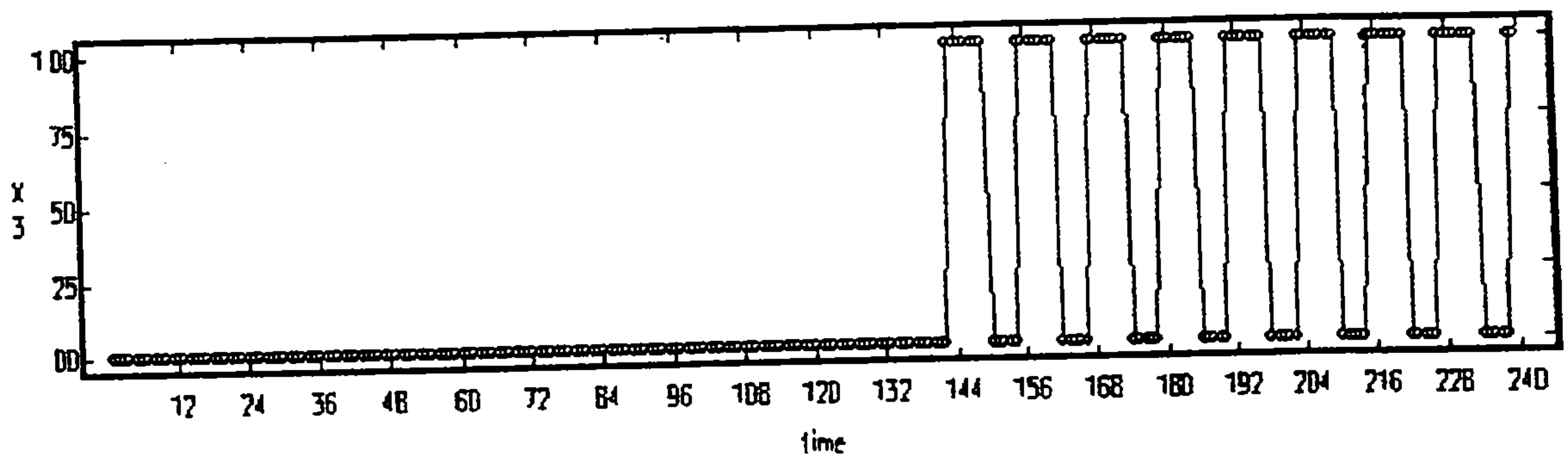
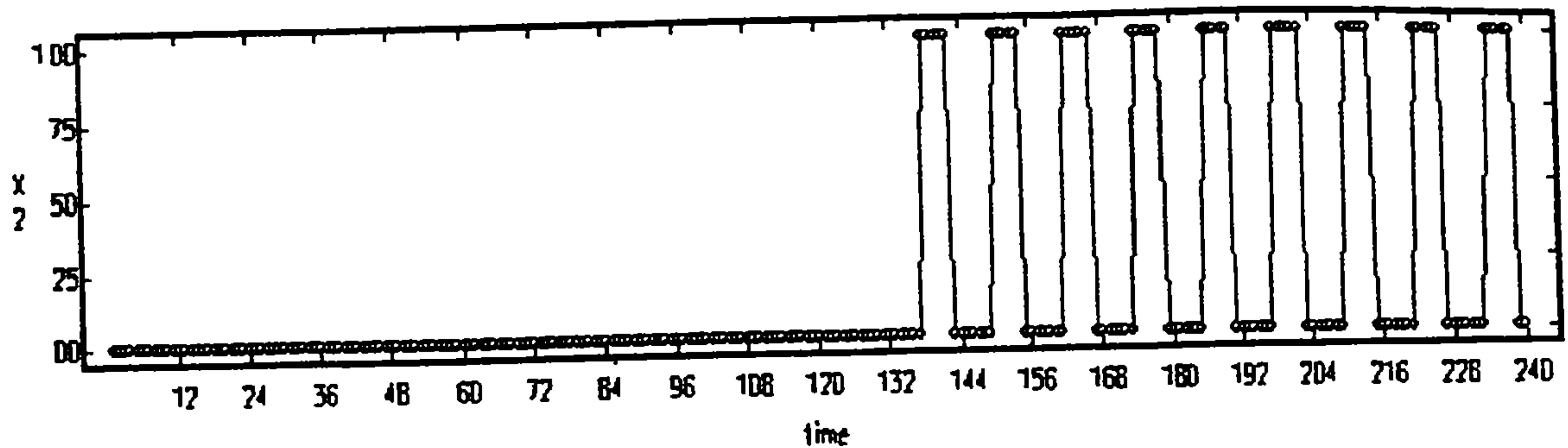
VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
OZ	RANDOM	ORIGINAL			12	(1-B)		
X1	RANDOM	ORIGINAL			12	(1-B)		
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 C1	X1	NUM.	1	0	NONE	.1000		
2 T1	OZ	MA	1	1	NONE	-.3605		
3 T12	OZ	MA	2	12	NONE	.5466		

-- estim mozz.method exact

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MOZZ

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 C1	X1	NUM.	1	0	NONE	-1.3388	.2294	-5.84
2 T1	OZ	MA	1	1	NONE	-.3051	.0667	-4.57
3 T12	OZ	MA	2	12	NONE	.6825	.0493	13.84

TOTAL SUM OF SQUARES 478369E+03
TOTAL NUMBER OF OBSERVATIONS 216
RESIDUAL SUM OF SQUARES. 137306E+03
R-SQUARE 695
EFFECTIVE NUMBER OF OBSERVATIONS 203
RESIDUAL VARIANCE ESTIMATE 676384E+00
RESIDUAL STANDARD ERROR. 822426E+00



#We now discuss the second intervention due to the progressively more stringent air quality standard enforced since 1966. Because the possible trends in ozone due to this intervention may be different between the summer months and the rest of the year, we use the above two indicators. The first one, x_2 , has the value 1 for June thru October (summer) beginning in 1966 and 0 otherwise, while x_3 has the value 1 for November thru May (winter) also beginning in 1966 and 0 otherwise. Now the model with both interventions takes the form,

$$Z_t = c_1 x_{1t} + \overset{\text{Trend summer}}{c_2 (1-B^{12})^{-1} x_2} + \overset{\text{Trend winter}}{c_3 (1-B^{12})^{-1} x_3} + \overset{\text{Time Series Coefficient}}{\frac{(1-\theta_1 B)(1-\theta_{12} B^{12})}{(1-B^{12})} a_t}$$

$$\overset{\text{c1 step change}}{\uparrow}$$

$$Z_t(12) + c_1 x_{1t}(12) + c_2 x_{2t} + c_3 x_{3t} + (1)(12) a_t$$

tsm mozz.add (c2)x2+(c3)x3
estim mozz.method exact.hold resi(r2) ←

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MOZZ

PARAMETER LABEL	VARIABLE NAME	NUM./DENOM.	FACTOR	ORDER	CONSTRAINT	VALUE	STD ERROR	T VALUE
1 C1	X1	NUM.	1	0	NONE	-1.3358	.1912	-6.99
2 C2	X2	NUM.	1	0	NONE	-.2382	.0584	-4.08
3 C3	X3	NUM.	1	0	NONE	-.0959	.0544	-1.76
4 T1	OZ	MA	1	1	NONE	-.2650	.0679	-3.90
5 T12	OZ	MA	2	12	NONE	.7777	.0405	19.22

TOTAL SUM OF SQUARES 478369E+03
 TOTAL NUMBER OF OBSERVATIONS 216
 RESIDUAL SUM OF SQUARES. 125070E+03
 R-SQUARE 722
 EFFECTIVE NUMBER OF OBSERVATIONS . . .203
 RESIDUAL VARIANCE ESTIMATE 616111E+00
 RESIDUAL STANDARD ERROR. 784927E+00

--

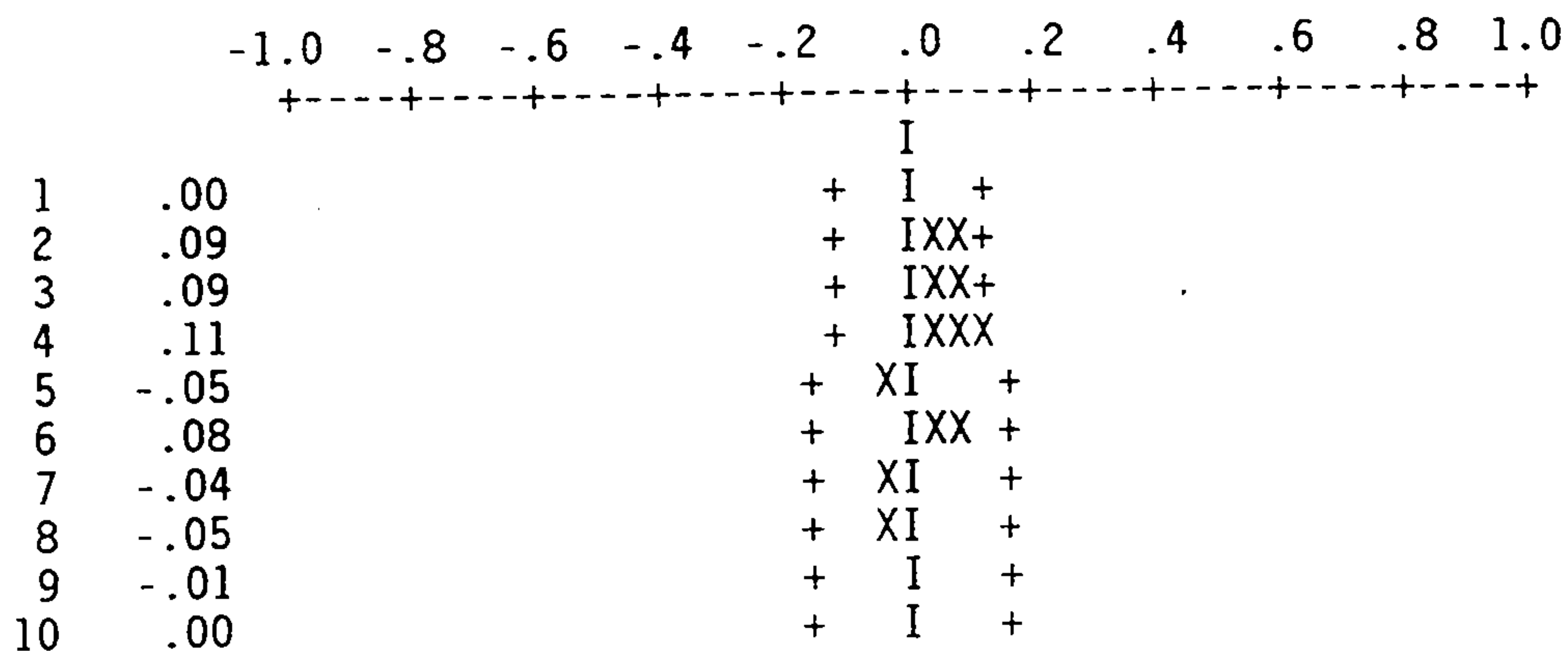
acf r2.maxlag 24

TIME PERIOD ANALYZED 14 TO 216
 NAME OF THE SERIES R2
 EFFECTIVE NUMBER OF OBSERVATIONS . . .203
 STANDARD DEVIATION OF THE SERIES 7445
 MEAN OF THE (DIFFERENCED) SERIES . . .-.0162
 STANDARD DEVIATION OF THE MEAN 0523
 T-VALUE OF MEAN (AGAINST ZERO) . . .-.3091

AUTOCORRELATIONS

1- 12	-.00	.09	.09	.11	-.05	.08	-.04	-.05	-.01	-.00	.02	.01
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07
Q	.0	1.6	3.1	5.5	5.9	7.3	7.6	8.2	8.2	8.2	8.3	8.4

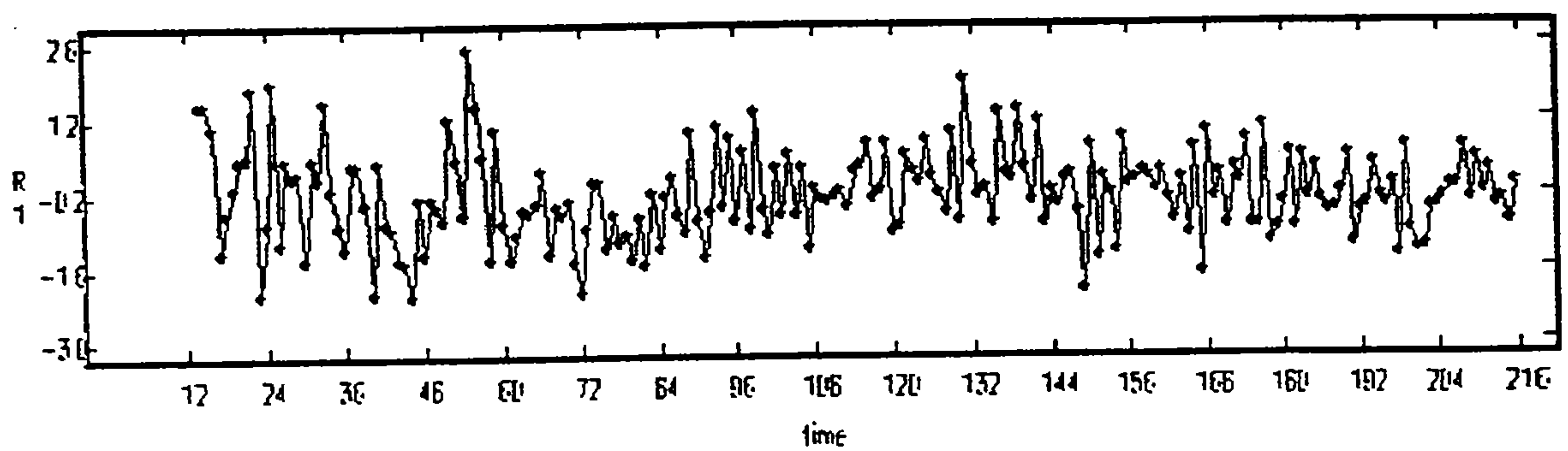
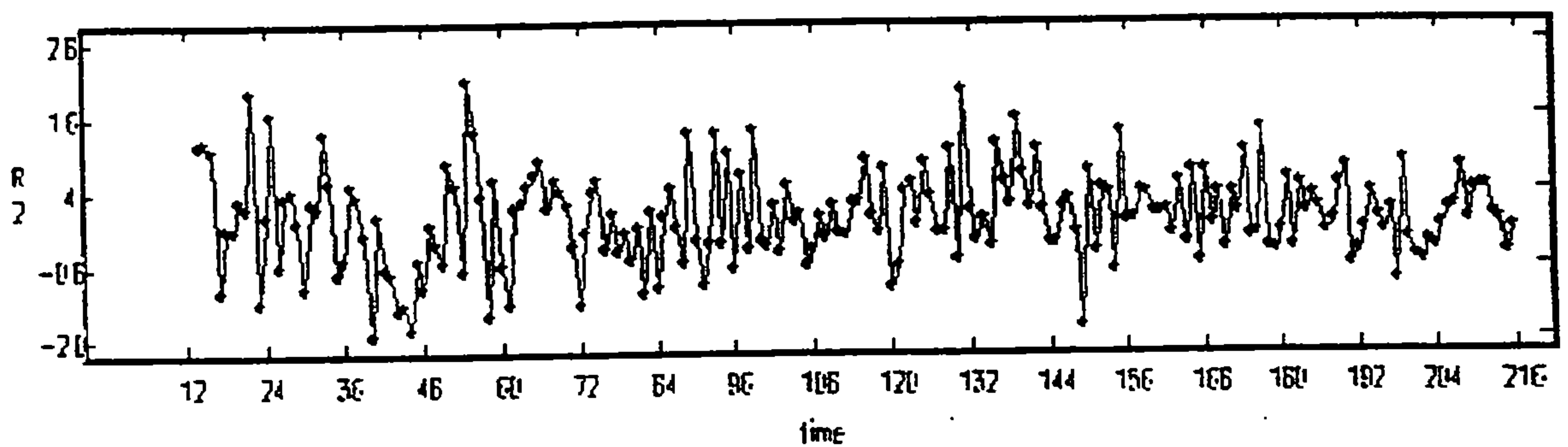
13- 24	.04	.01	-.09	.00	-.06	.01	.01	-.08	-.08	.06	-.06	-.00
ST.E.	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.07	.08
Q	8.6	8.6	10.4	10.4	11.3	11.3	11.3	12.9	14.2	14.9	15.7	15.7



11	.02
12	.01
13	.04
14	.01
15	-.09
16	.00
17	-.06
18	.01
19	.01
20	-.08
21	-.08
22	.06
23	-.06
24	.00

+	IX	+
+	I	+
+	IX	+
+	I	+
+	XXI	+
+	I	+
+	XXI	+
+	I	+
+	I	+
+	XXI	+
+	XXI	+
+	IX	+
+	XI	+
+	I	+

--
graph r2



#For easy comparison, we also give above the plot of r1 again.

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```
--
forecast mozz. nofs 24. hold forecasts(ozzf),std_errs(sozzf)
```

```
-----
24 FORECASTS, BEGINNING AT 216
-----
```

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
217	1.3831	.7849	
218	1.7942	.8120	
219	2.4030	.8120	
220	2.8126	.8120	
221	3.1013	.8120	
222	2.7161	.8120	
223	3.3123	.8120	
224	3.4752	.8120	
225	2.9490	.8120	
226	2.3782	.8120	
227	1.8179	.8120	
228	1.2376	.8120	
229	1.2832	.8306	
230	1.6983	.8318	
231	2.3072	.8318	
232	2.7167	.8318	
233	3.0055	.8318	
234	2.4779	.8318	
235	3.0741	.8318	
236	3.2370	.8318	
237	2.7108	.8318	
238	2.1400	.8318	
239	1.7221	.8318	
240	1.1417	.8318	

```
--
fsave ozzf,sozzf.file'ozzfor.fmt'
```

```
#It is interesting to compare the forecasts from the model moz without the
interventions with those from the model mozz with the interventions.
```

```
--
t1=-.3605
```

```
--
t12=.5466
```

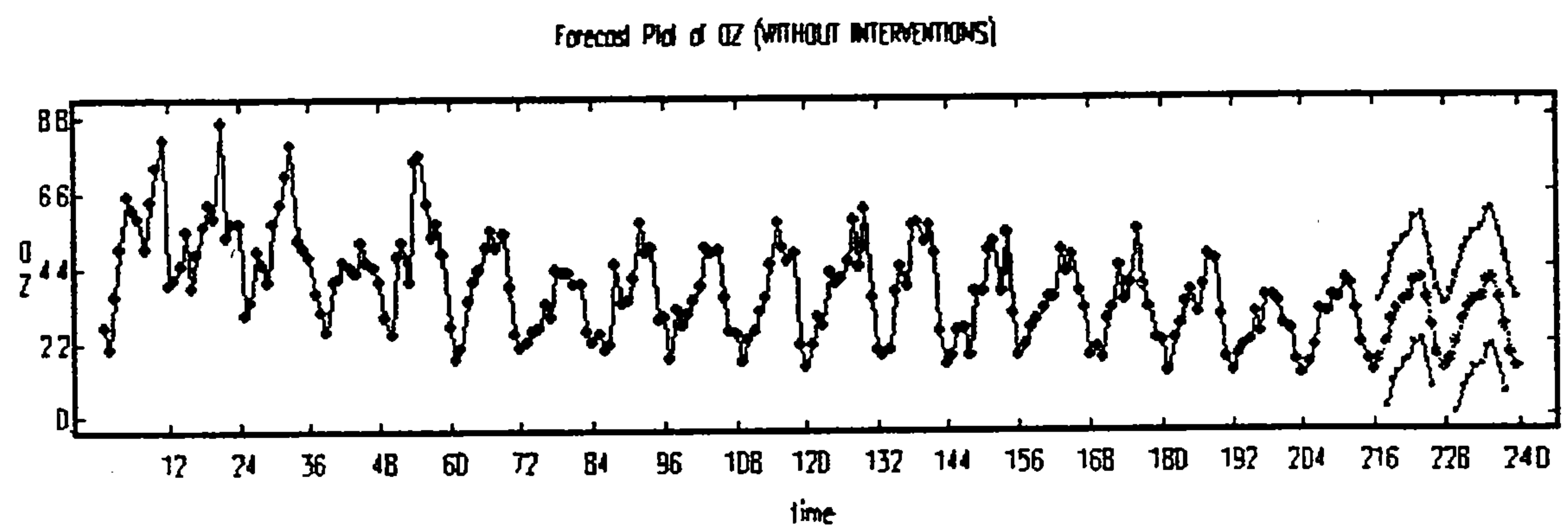
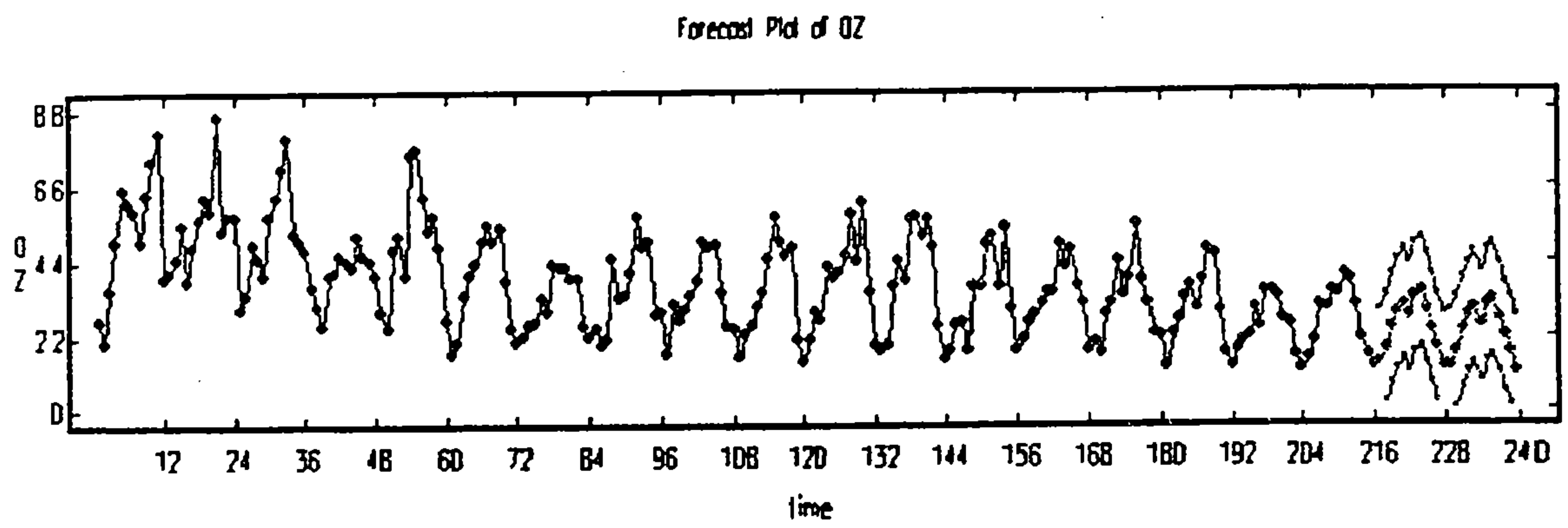
```
--
forecast moz.nofs 24.hold forecasts(ozf),std_errs(sozf)
```

 24 FORECASTS, BEGINNING AT 216

TIME	FORECAST	STD. ERROR	ACTUAL IF KNOWN
217	1.5777	.8794	
218	2.0625	.9348	
219	2.7781	.9348	
220	3.0650	.9348	
221	3.3316	.9348	
222	3.4194	.9348	
223	3.9322	.9348	
224	4.0075	.9348	
225	3.3859	.9348	
226	2.6043	.9348	
227	1.7915	.9348	
228	1.3855	.9348	
229	1.6039	1.0162	
230	2.0625	1.0264	
231	2.7781	1.0264	
232	3.0650	1.0264	
233	3.3316	1.0264	
234	3.4194	1.0264	
235	3.9322	1.0264	
236	4.0075	1.0264	
237	3.3859	1.0264	
238	2.6043	1.0264	
239	1.7915	1.0264	
240	1.3855	1.0264	

#Conclusions?

--
 fsave ozf,sozf.file'ozfor.fmt'



stop

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#EXAMPLE XI. THE CREST MARKET SHARE DATA 1/58 TO 4/63

This data set consists of 276 weekly market shares of the crest tooth paste from January, 1958 to April, 1963. In August, 1960 the American Dental Association made an announcement endorsing the crest brand which led to a substantial increase in its market share. Here we first apply interventional analysis techniques to estimate the level shift in the data associated with the known intervention. We then use outlier detection techniques to show how this shift can be detected if the intervention was not known in advance.

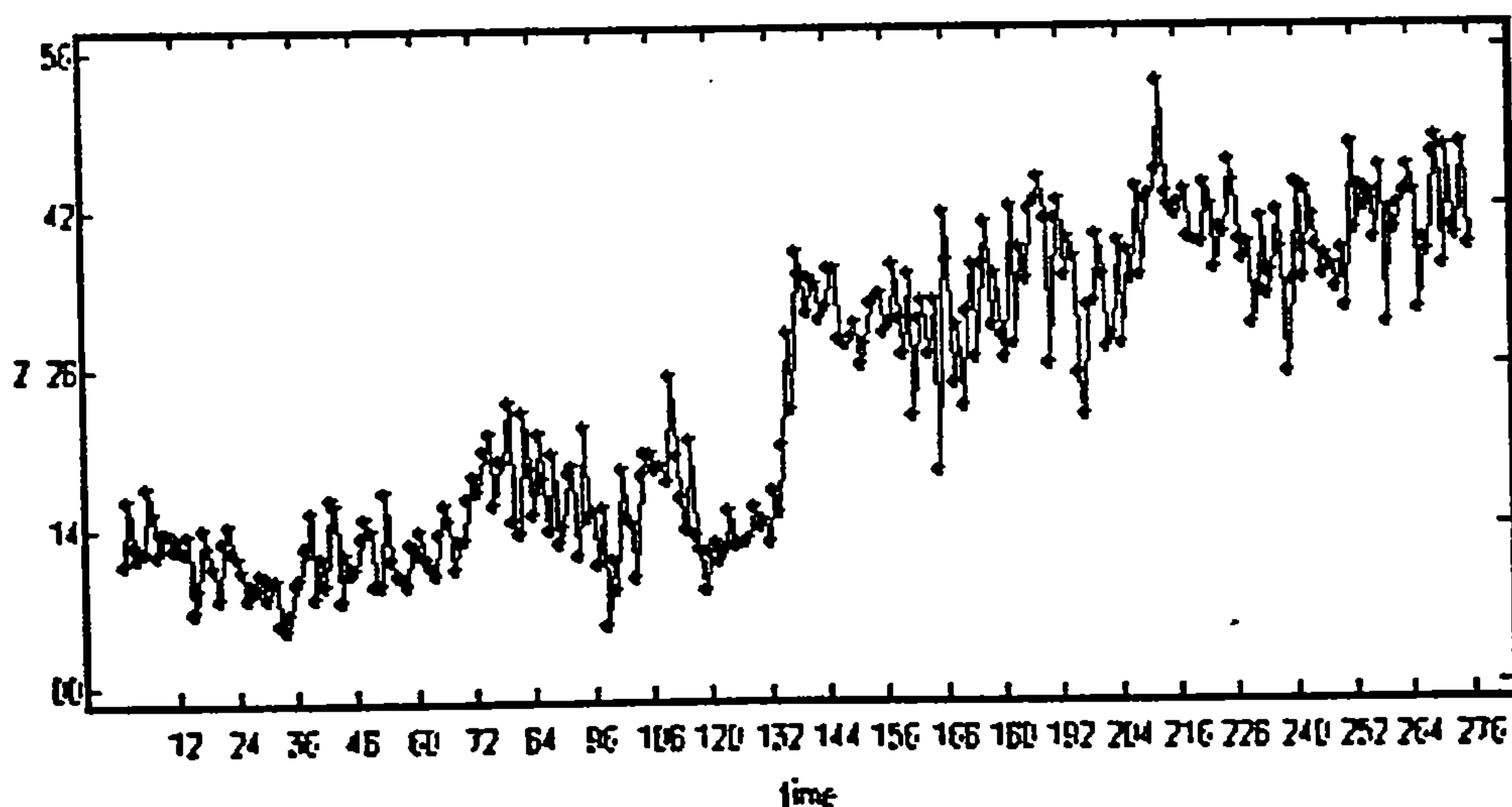
--

```
input z.file'c:\b325\tpaste.dat'
```

Z , A 276 BY 1 VARIABLE, IS STORED IN THE WORKSPACE

--

```
graph z
```



#The data clearly show a level increase beginning at about $t=135$ corresponding to the August announcement. We first build a model using data after the shift.

```
select z.new z1.span(138,276)
```

VARIABLE Z IS EDITED, THE RESULT IS STORED IN VARIABLE Z1
VARIABLE Z1 IS A 139 BY 1 MATRIX

--

eacf z1

TIME PERIOD ANALYZED 1 TO 139
NAME OF THE SERIES Z1
EFFECTIVE NUMBER OF OBSERVATIONS . . . 139
STANDARD DEVIATION OF THE SERIES 0576
MEAN OF THE (DIFFERENCED) SERIES 3697
STANDARD DEVIATION OF THE MEAN 0049
T-VALUE OF MEAN (AGAINST ZERO) 75.6669

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.38	.39	.40	.27	.39	.31	.29	.28	.24	.22	.18	.18	.18
(P= 1)	-.51	-.00	.15	-.15	.16	-.04	-.01	.03	-.01	.01	-.03	-.00	.05
(P= 2)	-.51	-.00	.05	-.14	.13	-.06	-.01	.03	.03	-.02	-.02	.00	.03
(P= 3)	-.07	-.33	-.36	-.09	.10	.07	.07	.04	.03	-.01	-.01	-.04	.03
(P= 4)	-.08	-.48	-.38	-.29	.11	-.03	-.00	.06	-.01	.01	-.02	-.05	.03
(P= 5)	-.31	-.14	-.02	-.05	.17	-.03	.01	.06	.05	-.03	-.03	.01	-.01
(P= 6)	-.49	-.09	.04	-.00	.12	.06	.04	.03	.04	-.09	-.03	-.03	.02

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	X	X	X	0	0	0	0	0	0
(P= 1)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 2)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 3)	0	X	X	0	0	0	0	0	0	0	0	0	0
(P= 4)	0	X	X	X	0	0	0	0	0	0	0	0	0
(P= 5)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 6)	X	0	0	0	0	0	0	0	0	0	0	0	0

--
tsm ml.model (1)z1=c+(1)noise
estim ml.method exact

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
Z1	RANDOM	ORIGINAL	NONE

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	C	CNST	1	0	NONE	.0159	.0138	1.15
2	Z1	MA	1	1	NONE	.7706	.0762	10.12
3	Z1	AR	1	1	NONE	.9583	.0373	25.66

TOTAL SUM OF SQUARES 461115E+00
 TOTAL NUMBER OF OBSERVATIONS 139
 RESIDUAL SUM OF SQUARES. 331886E+00
 R-SQUARE 275
 EFFECTIVE NUMBER OF OBSERVATIONS . . .138
 RESIDUAL VARIANCE ESTIMATE 240497E-02
 RESIDUAL STANDARD ERROR. 490405E-01

--

tsm m2.model (1)z1=(1)noise
 estim m2.method exact.hold resi(rz1)

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M2

 VARIABLE TYPE OF ORIGINAL DIFFERENCING
 VARIABLE OR CENTERED

Z1 RANDOM ORIGINAL NONE

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	Z1	MA	1	1	NONE	.8227	.0473	17.39
2	Z1	AR	1	1	NONE	1.0012	.0021	476.25

TOTAL SUM OF SQUARES 461115E+00
 TOTAL NUMBER OF OBSERVATIONS 139
 RESIDUAL SUM OF SQUARES. 336469E+00
 R-SQUARE 265
 EFFECTIVE NUMBER OF OBSERVATIONS . . .138
 RESIDUAL VARIANCE ESTIMATE 243818E-02
 RESIDUAL STANDARD ERROR. 493779E-01

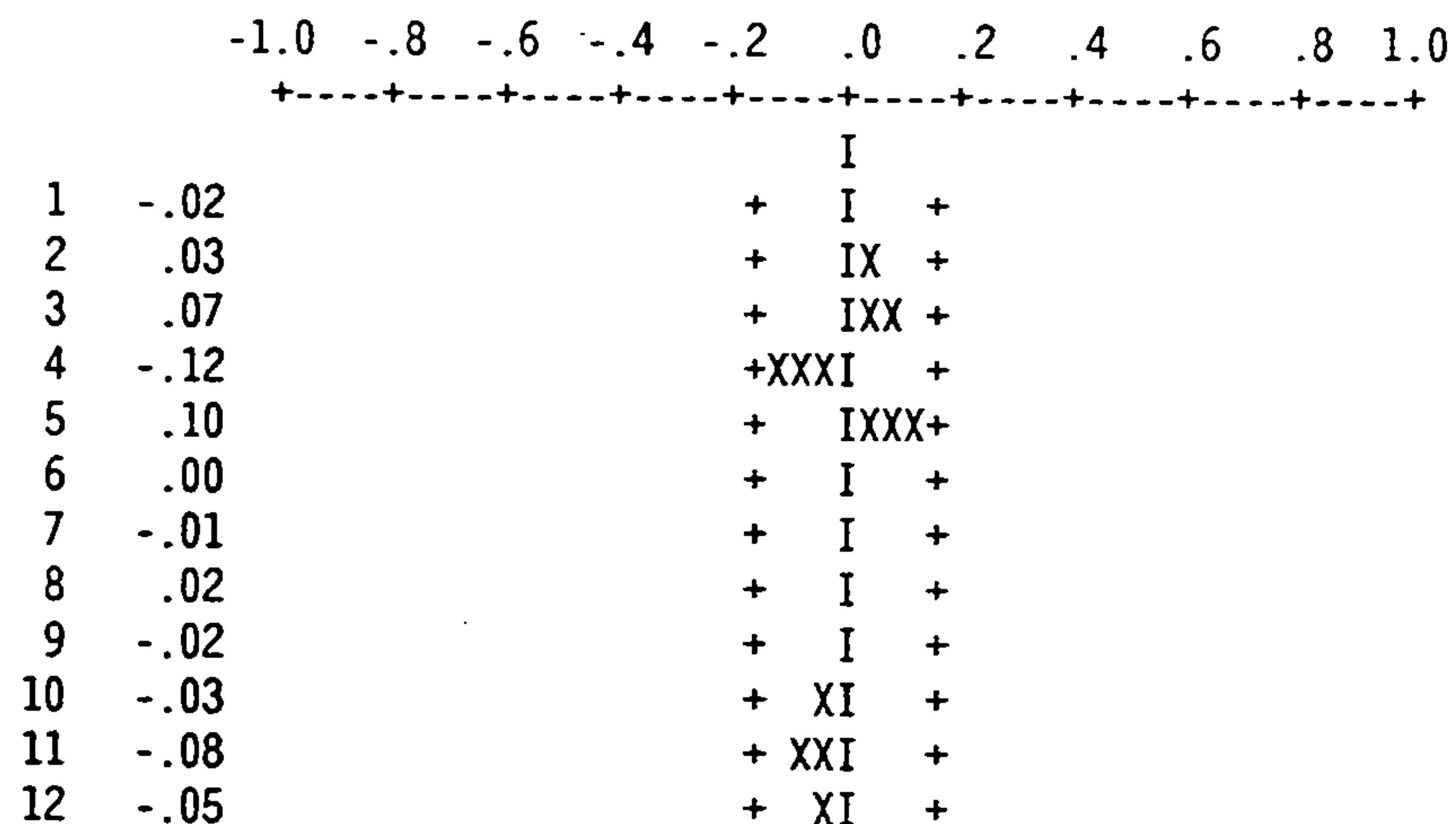
--

acf rz1.maxlag 12

TIME PERIOD ANALYZED 2 TO 139
 NAME OF THE SERIES RZ1
 EFFECTIVE NUMBER OF OBSERVATIONS . . .138
 STANDARD DEVIATION OF THE SERIES 0493
 MEAN OF THE (DIFFERENCED) SERIES 0003
 STANDARD DEVIATION OF THE MEAN 0042
 T-VALUE OF MEAN (AGAINST ZERO) 0618

AUTOCORRELATIONS

1- 12	-.02	.03	.07	-.12	.10	-.00	-.01	.02	-.02	-.03	-.08	-.05
ST.E.	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09	.09
Q	.0	.2	.9	3.1	4.7	4.7	4.7	4.7	4.8	5.0	6.0	6.4



--
 #Thus an (0.1.1) model seems adequate for the data after the intervention and we shall use this model for the entire data span with an first order dynamic model for the level shift intervention. The model is:

$$Z_t = \frac{\omega}{(1-\delta B)(1-B)} P_t^{(135)} + \frac{(1-\theta B)a_t}{(1-B)}$$

x=z-z

--

x(135)=1.

--

print x

X	IS	A	276	BY	1	VARIABLE		
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000
.000		.000		.000		.000	.000	.000

TOTAL SUM OF SQUARES 443206E+01
 TOTAL NUMBER OF OBSERVATIONS 276
 RESIDUAL SUM OF SQUARES. 512441E+00
 R-SQUARE 882
 EFFECTIVE NUMBER OF OBSERVATIONS . . . 270
 RESIDUAL VARIANCE ESTIMATE 189793E-02
 RESIDUAL STANDARD ERROR. 435652E-01

--
acf rz.maxlag 12

TIME PERIOD ANALYZED 7 TO 276
 NAME OF THE SERIES RZ
 EFFECTIVE NUMBER OF OBSERVATIONS . . . 270
 STANDARD DEVIATION OF THE SERIES 0435
 MEAN OF THE (DIFFERENCED) SERIES 0011
 STANDARD DEVIATION OF THE MEAN 0026
 T-VALUE OF MEAN (AGAINST ZERO) 4128

AUTOCORRELATIONS

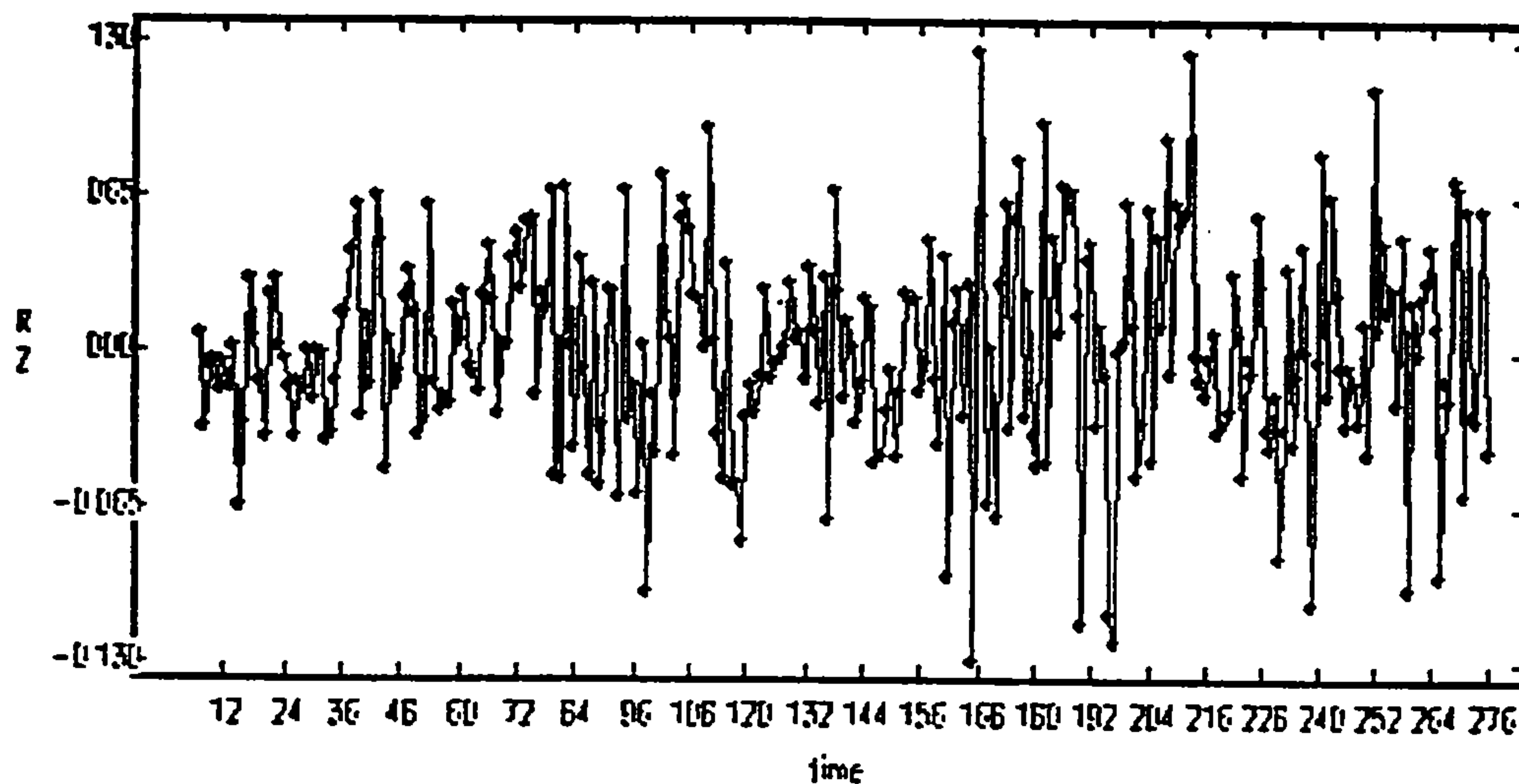
1- 12	-.02	.01	.05	-.04	.07	.03	-.05	.00	-.07	-.03	-.11	-.02
ST.E.	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06	.06
Q	.1	.2	.9	1.3	2.5	2.9	3.5	3.5	4.8	5.0	8.3	8.4

-1.0 - .8 - .6 - .4 - .2 .0 .2 .4 .6 .8 1.0
 +-----+-----+-----+-----+-----+-----+-----+-----+

		I
1	-.02	+ XI +
2	.01	+ I +
3	.05	+ IX +
4	-.04	+ XI +
5	.07	+ IXX+
6	.03	+ IX +
7	-.05	+ XI +
8	.00	+ I +
9	-.07	+XXI +
10	-.03	+ XI +
11	-.11	XXXI +
12	-.02	+ I +

--
#The fit seems adequate. Now the eventual gain is $\omega/(1-\delta)$ which is estimated to be $.0872/(1-.5670)=.2014$ representing an increase of market share by 20%.

graph rz



#We now apply the (0.1.1) model to the entire data and use outlier techniques to detect the level shift.

```
tsm mzz.model z(1)=(1)noise
estim mzz.method exact.hold resi(r)
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- MZZ

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
Z	RANDOM	ORIGINAL	1	(1-B)				
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	Z	MA	1	1	NONE	.6641	.0448	14.81

```
TOTAL SUM OF SQUARES . . . . . .443206E+01
TOTAL NUMBER OF OBSERVATIONS . . . . .276
RESIDUAL SUM OF SQUARES. . . . . .570477E+00
R-SQUARE . . . . . .871
EFFECTIVE NUMBER OF OBSERVATIONS . . . . .275
RESIDUAL VARIANCE ESTIMATE . . . . . .207446E-02
RESIDUAL STANDARD ERROR. . . . . .455462E-01
```

--

outlier mzz

INITIAL RESIDUAL STANDARD ERROR = .45416E-01

TIME	ESTIMATE	T-VALUE	TYPE
167	-.15	-3.63	A0
138	.15	3.56	I0
136	.14	3.26	I0
190	-.12	-3.03	A0

ADJUSTED RESIDUAL STANDARD ERROR = .41906E-01

--

#We see two I0's at t=136,138 implying level shifts occurring at these time points. Alternatively, we can directly detect level shifts as follows:

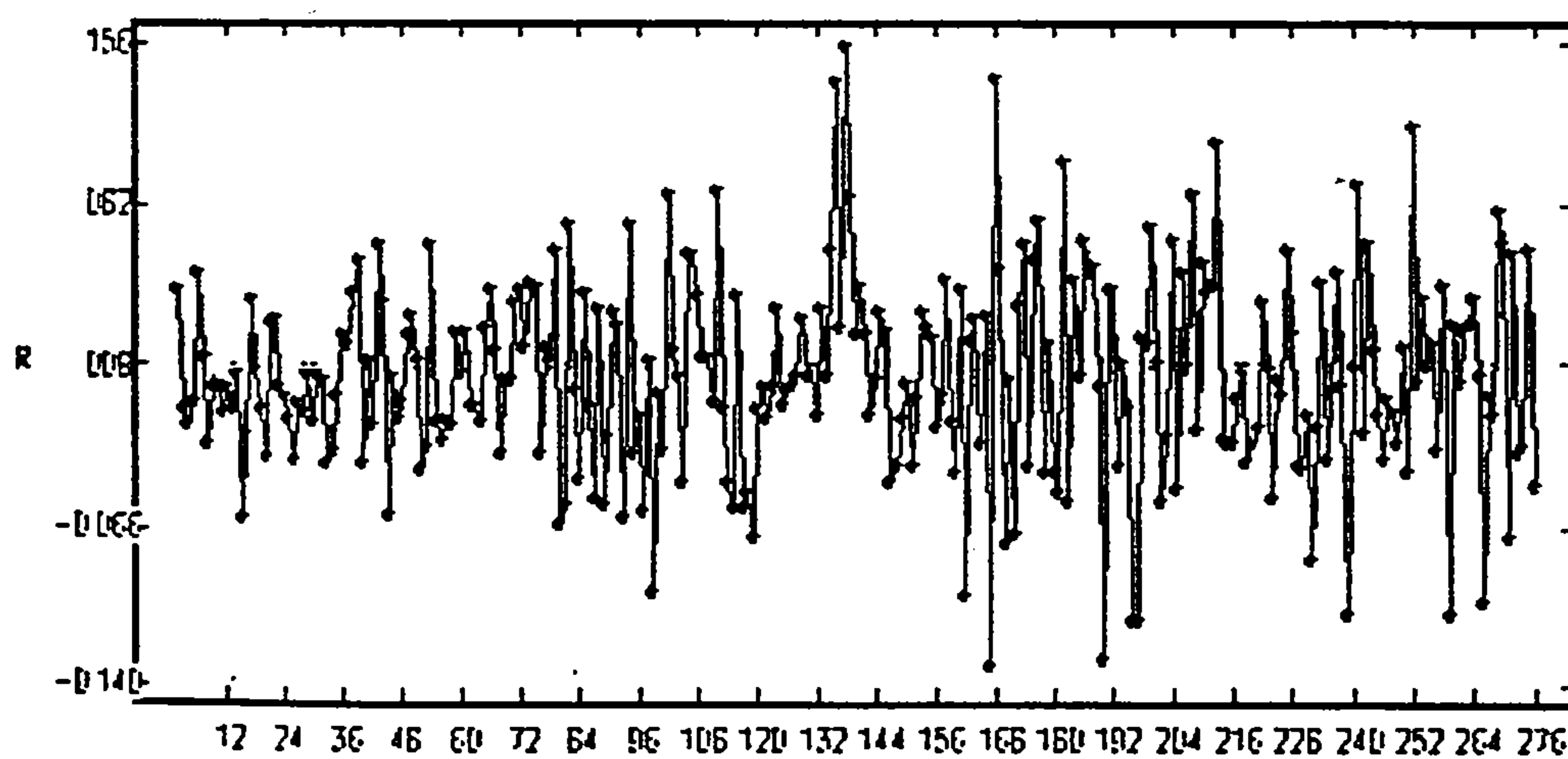
outlier mzz.types A0,I0,LS

INITIAL RESIDUAL STANDARD ERROR = .45416E-01

TIME	ESTIMATE	T-VALUE	TYPE
136	.15	4.36	LS
167	-.15	-3.76	A0
190	-.12	-3.01	A0

ADJUSTED RESIDUAL STANDARD ERROR = .42157E-01

graph r



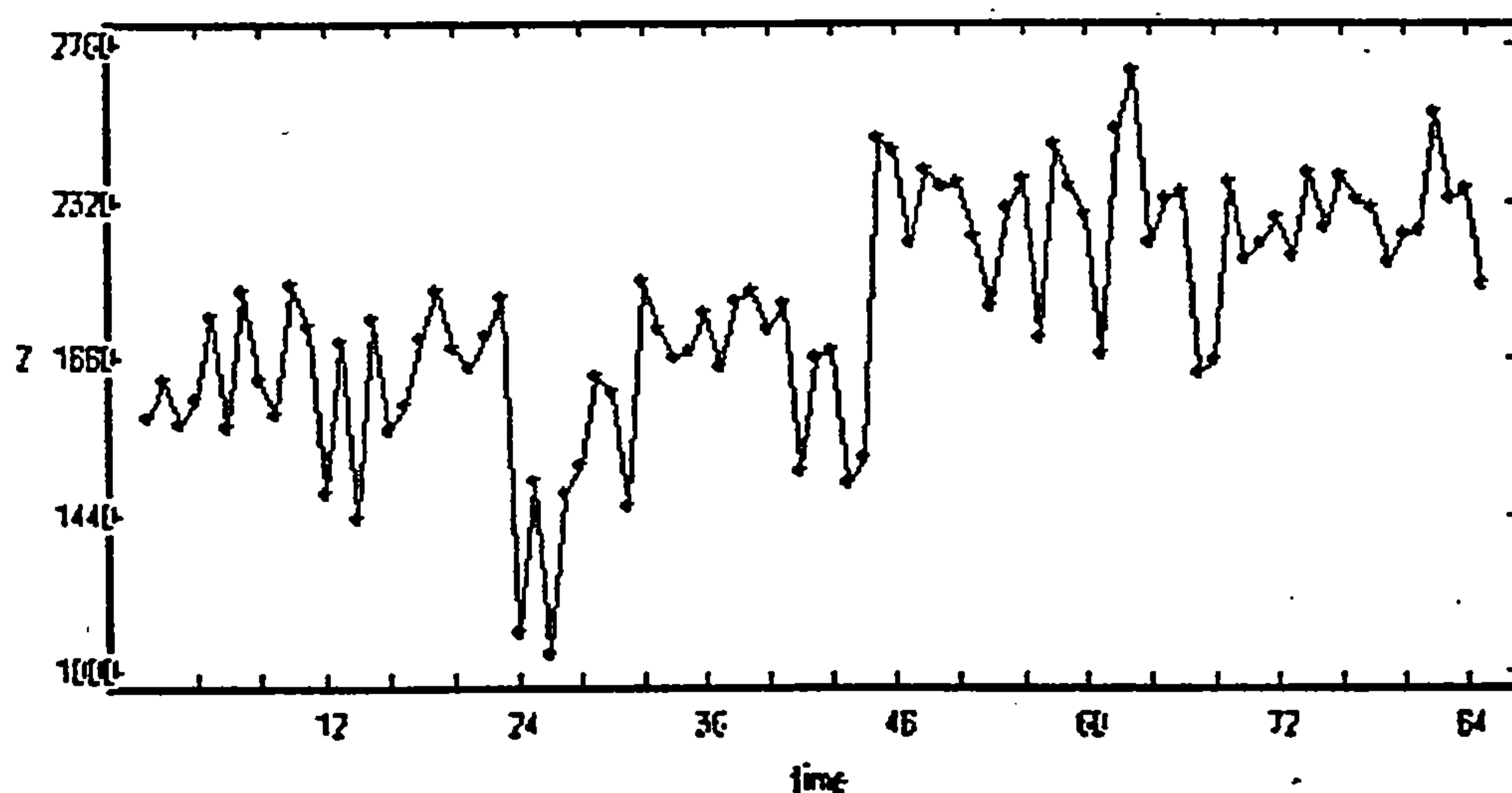
#It is interesting to compare this plot with the residuals plot on p. C-126 to see the effect of the intervention model.

C-127

#EXAMPLE XII. THE PRODUCTION DATA

The data consist of 85 consecutive daily output of a certain production process. We employ this data set to illustrate the intervention analysis and outlier detection methods.

```
--  
input z.file'c:\b325\product.dat'  
  
Z      , A  85 BY  1 VARIABLE, IS STORED IN THE WORKSPACE  
--  
graph z
```



#It is clear from the figure that there was a level shift at $t=47$, and indeed there was a change made to the production process at that time. In what follows we first model the data with a known level shift intervention at $t=47$.

```
--  
gene LSH.nrow 85.values 0 for 46,1 for 39  
  
THE SINGLE PRECISION VARIABLE    LSH    IS GENERATED  
--  
print LSH  
  
LSH      IS A  85 BY  1 VARIABLE
```

.000	.000	.000	.000	.000	.000	.000
.000	.000	.000	.000	.000	.000	.000
.000	.000	.000	.000	.000	.000	.000
.000	.000	.000	.000	.000	.000	.000
.000	.000	.000	.000	.000	.000	.000
.000	.000	.000	.000	.000	.000	.000
.000	.000	.000	.000	.000	.000	.000
.000	.000	.000	.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000
1.000	1.000	1.000	1.000	1.000	1.000	1.000

--
To build a model for the data, we employ the 46 observations before the intervention to tentatively specify a model.

eacf z.span 1,46

TIME PERIOD ANALYZED	1	TO	46
NAME OF THE SERIES			Z
EFFECTIVE NUMBER OF OBSERVATIONS			46
STANDARD DEVIATION OF THE SERIES			241.5422
MEAN OF THE (DIFFERENCED) SERIES			1795.5430
STANDARD DEVIATION OF THE MEAN			35.6135
T-VALUE OF MEAN (AGAINST ZERO)			50.4176

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.18	.31	.14	-.01	-.08	-.20	-.23	-.17	-.23	-.05	-.09	-.00	-.14
(P= 1)	-.47	.22	.17	-.05	.01	-.06	-.06	.05	-.19	.12	-.10	.04	-.15
(P= 2)	-.19	.37	-.04	-.05	.04	-.03	-.09	-.02	-.17	.02	.09	.06	-.11
(P= 3)	.30	.35	-.05	-.15	.08	-.02	-.03	.00	-.17	.04	.06	.09	-.12
(P= 4)	-.51	-.05	.12	.19	-.12	.06	-.01	-.02	-.14	.08	-.05	.13	-.12
(P= 5)	-.50	-.33	.03	.23	.01	-.00	-.00	-.02	-.13	.01	.01	.06	-.07
(P= 6)	-.50	.18	-.17	.23	.04	.02	.03	-.03	-.14	-.01	-.02	.01	-.03

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	0	0	0	0	0	0	0	0	0	0	0	0	0
(P= 1)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 2)	0	0	0	0	0	0	0	0	0	0	0	0	0
(P= 3)	0	0	0	0	0	0	0	0	0	0	0	0	0
(P= 4)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 5)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 6)	X	0	0	0	0	0	0	0	0	0	0	0	0

#The EACF indicates a white noise process. We shall assume an MA(1) model for the entire data span except for the intervention.

```
--
tsm ml.model z=c+(wo)lsh+(1)noise
estim ml.method exact
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
Z	RANDOM	ORIGINAL	NONE
LSH	RANDOM	ORIGINAL	NONE

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 C		CNST	1	0	NONE	1794.5036	34.7877	51.58
2 WO	LSH	NUM.	1	0	NONE	483.0816	51.2865	9.42
3	Z	MA	1	1	NONE	-.0974	.1078	-.90

TOTAL SUM OF SQUARES 888999E+07
TOTAL NUMBER OF OBSERVATIONS 85
RESIDUAL SUM OF SQUARES. 395193E+07
R-SQUARE 555
EFFECTIVE NUMBER OF OBSERVATIONS 85
RESIDUAL VARIANCE ESTIMATE 464933E+05
RESIDUAL STANDARD ERROR. 215623E+03

#The estimation results show that the MA(1) coefficient is not needed and the level shift is definitely real.

We now present an alternative analysis supposing that the level shift at t=47 was not known and was neglected by the analyst.

```
--
eacf z
```

TIME PERIOD ANALYZED 1 TO 85
NAME OF THE SERIES Z
EFFECTIVE NUMBER OF OBSERVATIONS 85
STANDARD DEVIATION OF THE SERIES 323.4009
MEAN OF THE (DIFFERENCED) SERIES 2016.4710
STANDARD DEVIATION OF THE MEAN 35.0778
T-VALUE OF MEAN (AGAINST ZERO) 57.4857

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.56	.52	.50	.43	.32	.31	.34	.34	.28	.34	.39	.38	.29
(P= 1)	-.44	-.05	.08	.09	-.13	-.03	.03	.07	-.11	.03	.04	.12	-.13
(P= 2)	-.50	-.35	.06	.10	-.12	-.15	.02	.03	-.12	.04	.01	.13	-.13
(P= 3)	-.30	.23	-.16	.02	-.11	.03	-.06	.00	-.12	.03	.03	.15	-.03
(P= 4)	.49	.19	-.21	-.14	-.12	-.08	-.11	-.06	-.04	.03	.03	.16	.06
(P= 5)	.26	.22	-.17	-.32	.11	-.04	-.06	-.08	-.08	.03	-.01	.14	.02
(P= 6)	-.19	.26	.08	-.38	-.00	-.08	.01	-.05	-.03	.05	.02	.14	.04

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	0	0	0	0	0	0	0	0	0
(P= 1)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 2)	X	X	0	0	0	0	0	0	0	0	0	0	0
(P= 3)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 4)	X	0	0	0	0	0	0	0	0	0	0	0	0
(P= 5)	X	0	0	X	0	0	0	0	0	0	0	0	0
(P= 6)	0	0	0	X	0	0	0	0	0	0	0	0	0

--
#This time an ARMA(1,1) model was suggested!

```
tsm m2.model (1)z=c1+(1)noise
estim m2.method exact
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M2

VARIABLE	TYPE OF	ORIGINAL	DIFFERENCING					
Z	VARIABLE	OR CENTERED	NONE					
	RANDOM	ORIGINAL						
PARAMETER	VARIABLE	NUM./	FACTOR	ORDER	CONS-	VALUE	STD	T
LABEL	NAME	DENOM.			TRAIINT		ERROR.	VALUE
1	C1	CNST	1	0	NONE	138.3807	109.6716	1.26
2	Z	MA	1	1	NONE	.6410	.1160	5.53
3	Z	AR	1	1	NONE	.9343	.0543	17.21
TOTAL SUM OF SQUARES				.888999E+07				
TOTAL NUMBER OF OBSERVATIONS				85				
RESIDUAL SUM OF SQUARES.				.512851E+07				
R-SQUARE				.416				
EFFECTIVE NUMBER OF OBSERVATIONS				84				
RESIDUAL VARIANCE ESTIMATE				.610537E+05				
RESIDUAL STANDARD ERROR.				.247090E+03				

```
--
tsm m3.model (1)z=(1)noise
estim m3.method exact. hold resi(r1)
```

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M3

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING					
Z	RANDOM	ORIGINAL	NONE					
PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	Z	MA	1	1	NONE	.7662	.0710	10.79
2	Z	AR	1	1	NONE	1.0026	.0033	301.03

```
TOTAL SUM OF SQUARES . . . . .888999E+07
TOTAL NUMBER OF OBSERVATIONS . . . . .85
RESIDUAL SUM OF SQUARES. . . . .523528E+07
R-SQUARE . . . . .404
EFFECTIVE NUMBER OF OBSERVATIONS . . . . .84
RESIDUAL VARIANCE ESTIMATE . . . . .623248E+05
RESIDUAL STANDARD ERROR. . . . .249649E+03
```

```
--
acf r1.maxlag 10
```

```
TIME PERIOD ANALYZED . . . . .2 TO 85
NAME OF THE SERIES . . . . .R1
EFFECTIVE NUMBER OF OBSERVATIONS . . . . .84
STANDARD DEVIATION OF THE SERIES . . . . .249.5157
MEAN OF THE (DIFFERENCED) SERIES . . . . .4.4024
STANDARD DEVIATION OF THE MEAN . . . . .27.2244
T-VALUE OF MEAN (AGAINST ZERO) . . . . .1617
```

AUTOCORRELATIONS

```
1- 10 .07 .03 .07 -.01 -.21 -.17 -.09 -.06 -.15 -.00
ST.E. .11 .11 .11 .11 .11 .12 .12 .12 .12 .12
Q .4 .5 .9 .9 5.1 7.9 8.5 8.9 10.9 10.9
```


outlier m3.types A0.I0.LS

INITIAL RESIDUAL STANDARD ERROR = 249.52

TIME	ESTIMATE	T-VALUE	TYPE
47	584.84	3.80	LS
24	-841.02	-3.99	I0
26	-651.09	-3.50	A0

ADJUSTED RESIDUAL STANDARD ERROR = 195.45

--
#Thus a level shift is again indicated at t=47. We now modify model m3 to become an (0.1.1) model with this level shift included, i.e.

$$(1-B)Z_t = \omega P_t^{(47)} + (1-\theta B)a_t$$

x=z-z

x(47)=1.

--

tsm m4.model z(1)=(w)x+(1)noise

estim m4.method exact

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M4

VARIABLE	TYPE OF VARIABLE	ORIGINAL OR CENTERED	DIFFERENCING
Z	RANDOM	ORIGINAL	$\frac{1}{(1-B)}$
X	RANDOM	ORIGINAL	NONE

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1 W	X	NUM.	1	0	NONE	481.5298	48.1096	10.01
2	Z	MA	1	1	NONE	1.0006	.0340	29.43

TOTAL SUM OF SQUARES 888999E+07
TOTAL NUMBER OF OBSERVATIONS 85
RESIDUAL SUM OF SQUARES. 399401E+07
R-SQUARE 545
EFFECTIVE NUMBER OF OBSERVATIONS 84
RESIDUAL VARIANCE ESTIMATE 475478E+05
RESIDUAL STANDARD ERROR. 218055E+03

--

#The estimated θ value is essentially 1 and the estimated ω value of 481.53 for the level shift is very close to the estimate for the model on p. C-130. Indeed with $\theta = 1$ the factors $(1-B)$ and $(1-\theta B)$ cancel each other and the model here becomes

$$Z_t = c + \omega S_t^{(4)} + a_t$$

which is the same model on p. C-130. This analysis shows the usefulness of plotting residuals and the outlier detection techniques.

outlier m4.types A0.I0.LS

INITIAL RESIDUAL STANDARD ERROR = 217.88

TIME	ESTIMATE	T-VALUE	TYPE
26	-745.40	-3.67	A0
24	-684.79	-3.64	A0

ADJUSTED RESIDUAL STANDARD ERROR = 186.94

--

stop

#EXAMPLE XIII. WEEKLY SALES OF A FOOD PRODUCT

We here consider weekly sales of a popular food product. This example serves to link time series analysis with regression techniques. The data consist of 156 weekly observations, beginning 9/14/91, of four variables:

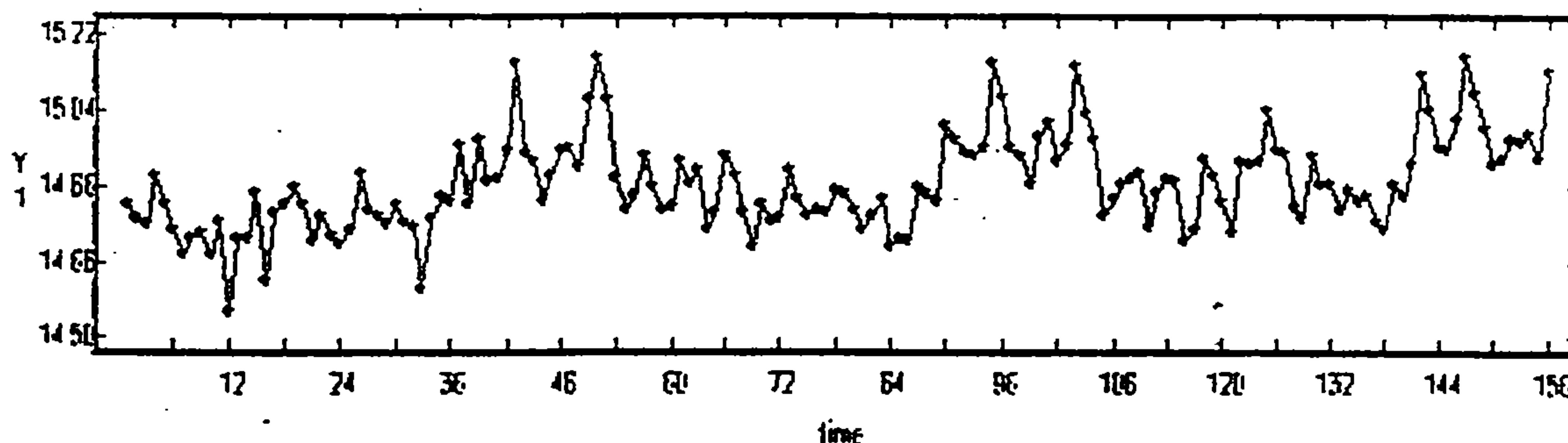
y1: ln(sales volume)
y2: average promoted price
y3: % feature ad } ?
y4: % display }

In what follows we first carry out an univariate time series analysis of y1, then do a regression of y1 on y2-y4 and finally link these two together.

```
input y1,y2,y3,y4.file'c:\b325\foodprod.dat'
```

```
Y1      , A 156 BY 1 VARIABLE, IS STORED IN THE WORKSPACE  
Y2      , A 156 BY 1 VARIABLE, IS STORED IN THE WORKSPACE  
Y3      , A 156 BY 1 VARIABLE, IS STORED IN THE WORKSPACE  
Y4      , A 156 BY 1 VARIABLE, IS STORED IN THE WORKSPACE
```

```
fsave y1,y2,y3,y4.file'foodp.fmt'
```



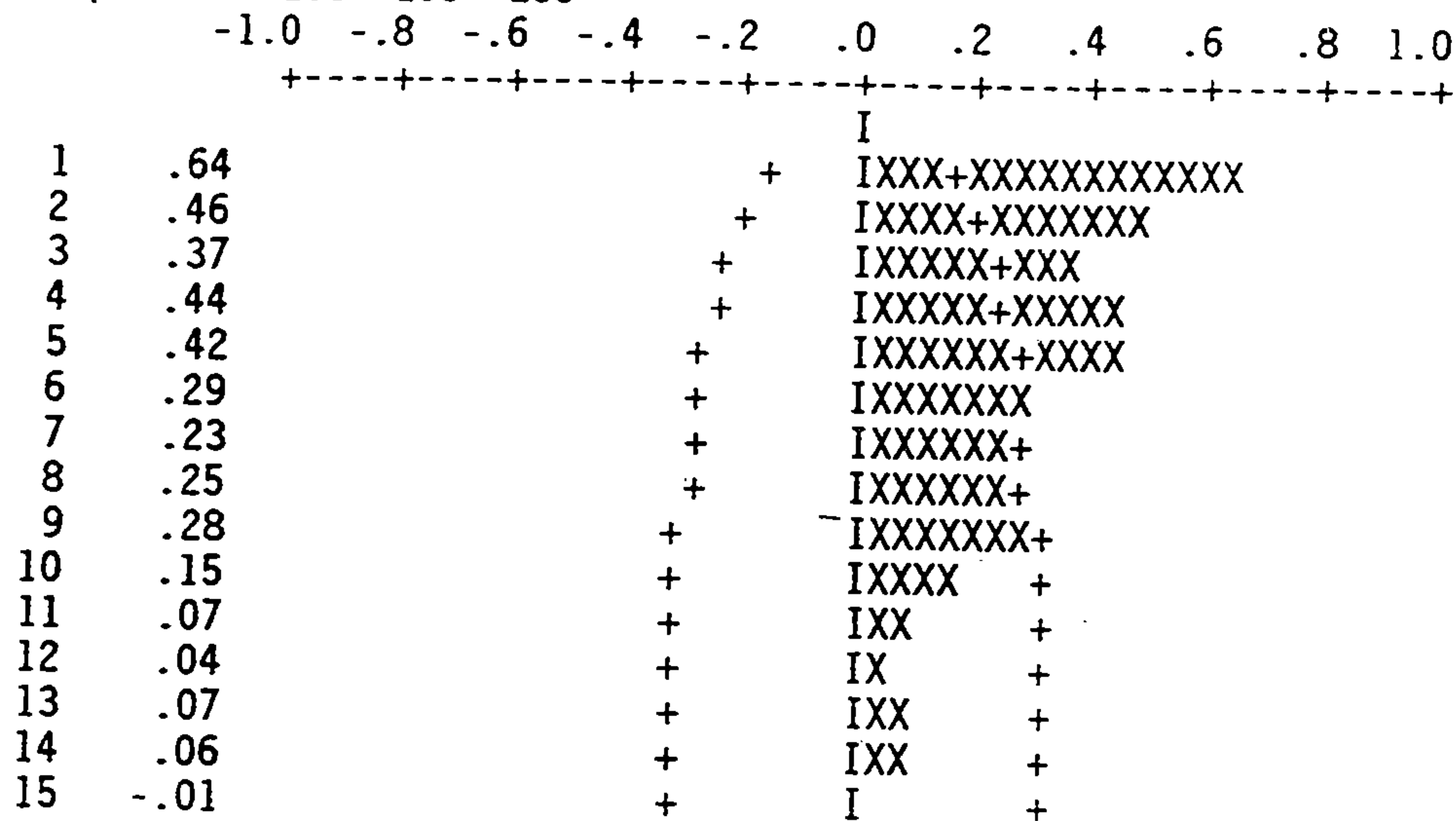
```
uiden y1.maxlag 15
```

```
TIME PERIOD ANALYZED . . . . . 1 TO 156  
NAME OF THE SERIES . . . . . Y1  
EFFECTIVE NUMBER OF OBSERVATIONS . . . 156  
STANDARD DEVIATION OF THE SERIES . . . .1108  
MEAN OF THE (DIFFERENCED) SERIES . . . 14.8700  
STANDARD DEVIATION OF THE MEAN . . . . .0089  
T-VALUE OF MEAN (AGAINST ZERO) . . . . 1676.7490
```

AUTOCORRELATIONS

1- 12	.64	.46	.37	.44	.42	.29	.23	.25	.28	.15	.07	.04
ST.E.	.08	.11	.12	.13	.14	.14	.15	.15	.15	.16	.16	.16
Q	64.9	99.0	121	153	183	196	205	215	229	233	234	234

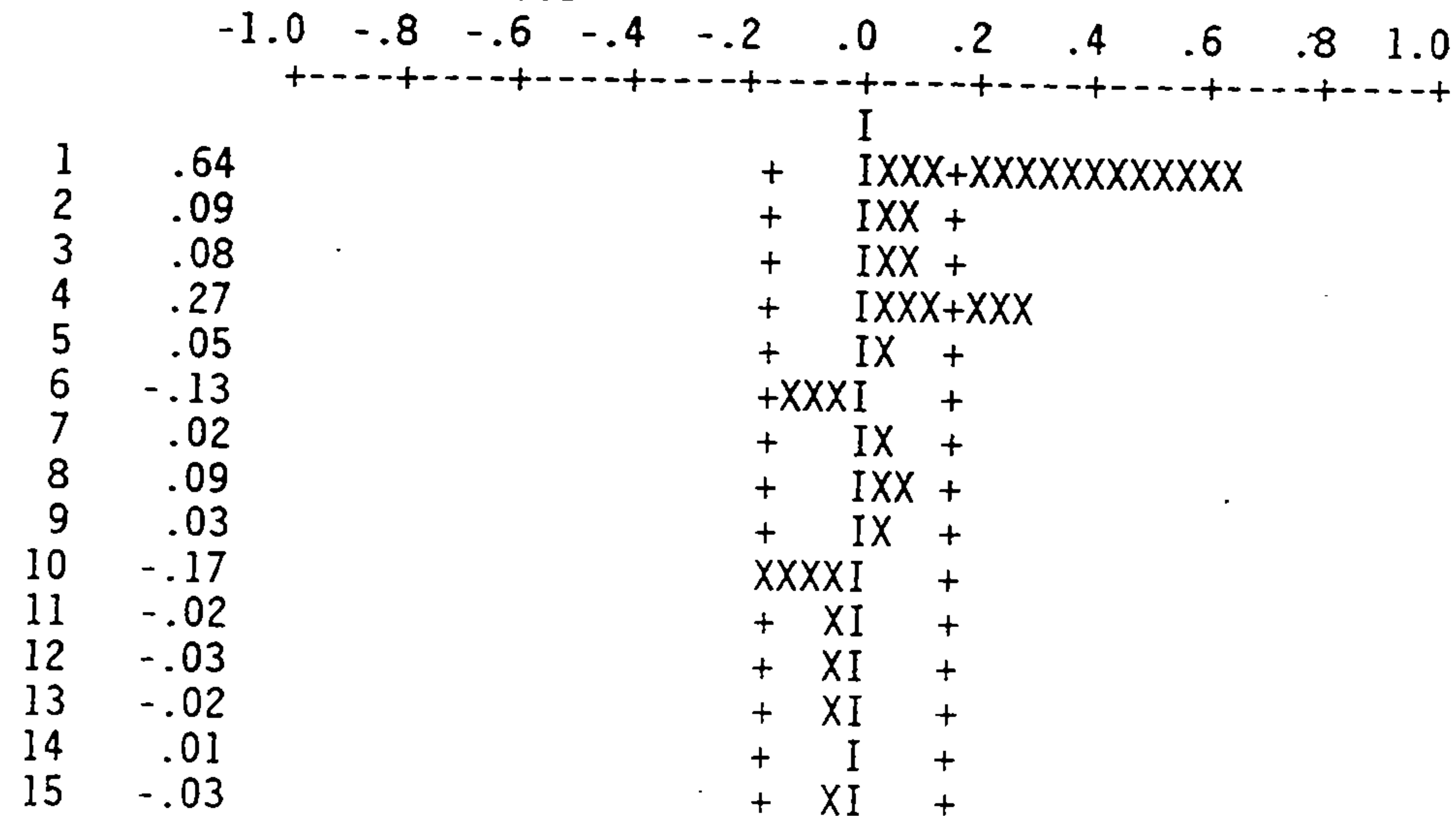
13- 15	.07	.06	-.01
ST.E.	.16	.16	.16
Q	235	235	235



PARTIAL AUTOCORRELATIONS

1- 12	.64	.09	.08	.27	.05	-.13	.02	.09	.03	-.17	-.02	-.03
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08

13- 15	-.02	.01	-.03
ST.E.	.08	.08	.08



eacf y1

TIME PERIOD ANALYZED 1 TO 156
NAME OF THE SERIES Y1
EFFECTIVE NUMBER OF OBSERVATIONS . . . 156
STANDARD DEVIATION OF THE SERIES 1108
MEAN OF THE (DIFFERENCED) SERIES . . . 14.8700
STANDARD DEVIATION OF THE MEAN 0089
T-VALUE OF MEAN (AGAINST ZERO) 1676.7490

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.64	.46	.37	.44	.42	.29	.23	.25	.28	.15	.07	.04	.07
(P= 1)	-.11	-.11	-.20	.13	.21	-.08	-.14	-.02	.24	.02	-.05	-.07	.04
(P= 2)	-.48	.01	-.29	.13	.24	-.06	-.14	-.00	.24	.06	-.02	-.11	.04
(P= 3)	-.30	-.16	-.29	.02	.00	-.07	.04	-.06	.21	-.07	.04	-.06	.03
(P= 4)	-.15	.38	-.21	.04	.01	-.02	.04	.04	.14	-.06	.07	.03	-.05
(P= 5)	.30	.30	-.20	-.02	-.11	.00	.02	-.01	.14	.02	.06	-.01	.00
(P= 6)	.24	.40	-.19	-.26	-.13	-.02	.01	.01	.13	-.11	.04	-.01	.00

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	X	0	0	0	0	0	0	0	0
(P= 1)	0	0	X	0	X	0	0	0	X	0	0	0	0
(P= 2)	X	0	X	0	X	0	0	0	X	0	0	0	0
(P= 3)	X	0	X	0	0	0	0	0	X	0	0	0	0
(P= 4)	0	X	X	0	0	0	0	0	0	0	0	0	0
(P= 5)	X	X	0	0	0	0	0	0	0	0	0	0	0
(P= 6)	X	X	0	X	0	0	0	0	0	0	0	0	0

--
tsm ml.model (1,2,3,4)y1=c+(1,2,3)noise.no show
estim ml.method exact.

ARMA(4, 2)

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M1

VARIABLE		TYPE OF	ORIGINAL	DIFFERENCING				
		VARIABLE	OR CENTERED					
Y1		RANDOM	ORIGINAL	NONE				
PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR T VALUE
1	C		CNST	1	0	NONE	3.7046	1.8936 1.96
2		Y1	MA	1	1	NONE	.2191	.1995 ~~1.10~~
3		Y1	MA	1	2	NONE	-.5842	.1435 -4.07
4		Y1	MA	1	3	NONE	.1291	.1793 .72
5		Y1	AR	1	1	NONE	.7882	.1892 4.17
6		Y1	AR	1	2	NONE	-.6156	.2118 -2.91
7		Y1	AR	1	3	NONE	.3260	.2143 1.52
8		Y1	AR	1	4	NONE	.2524	.1337 1.89
TOTAL SUM OF SQUARES					.191398E+01			
TOTAL NUMBER OF OBSERVATIONS					156			
RESIDUAL SUM OF SQUARES.					.947318E+00			
R-SQUARE					.492			
EFFECTIVE NUMBER OF OBSERVATIONS					152			
RESIDUAL VARIANCE ESTIMATE					.623236E-02			
RESIDUAL STANDARD ERROR.					.789453E-01			

tsm m2.model (1,2,3,4)y1=c+(2)noise.no show
estim m2.method exact.hold resi(rr)

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M2

VARIABLE		TYPE OF	ORIGINAL	DIFFERENCING				
		VARIABLE	OR CENTERED					
Y1		RANDOM	ORIGINAL	NONE				
PARAMETER LABEL		VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR T VALUE
1	C		CNST	1	0	NONE	4.9557	1.6896 2.93
2		Y1	MA	1	2	NONE	-.5227	.1539 -3.40
3		Y1	AR	1	1	NONE	.5930	.0771 7.69
4		Y1	AR	1	2	NONE	-.4217	.1572 -2.68
5		Y1	AR	1	3	NONE	.1550	.1271 1.22
6		Y1	AR	1	4	NONE	.3406	.0781 4.36
TOTAL SUM OF SQUARES					.191398E+01			
TOTAL NUMBER OF OBSERVATIONS					156			
RESIDUAL SUM OF SQUARES.					.953157E+00			
R-SQUARE					.489			
EFFECTIVE NUMBER OF OBSERVATIONS					152			
RESIDUAL VARIANCE ESTIMATE					.627077E-02			
RESIDUAL STANDARD ERROR.					.791882E-01			

why not remove AR - 3

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acf rr.maxlag 15

TIME PERIOD ANALYZED 5 TO 156
 NAME OF THE SERIES RR
 EFFECTIVE NUMBER OF OBSERVATIONS 152
 STANDARD DEVIATION OF THE SERIES 0792
 MEAN OF THE (DIFFERENCED) SERIES 0001
 STANDARD DEVIATION OF THE MEAN 0064
 T-VALUE OF MEAN (AGAINST ZERO) 0161

AUTOCORRELATIONS

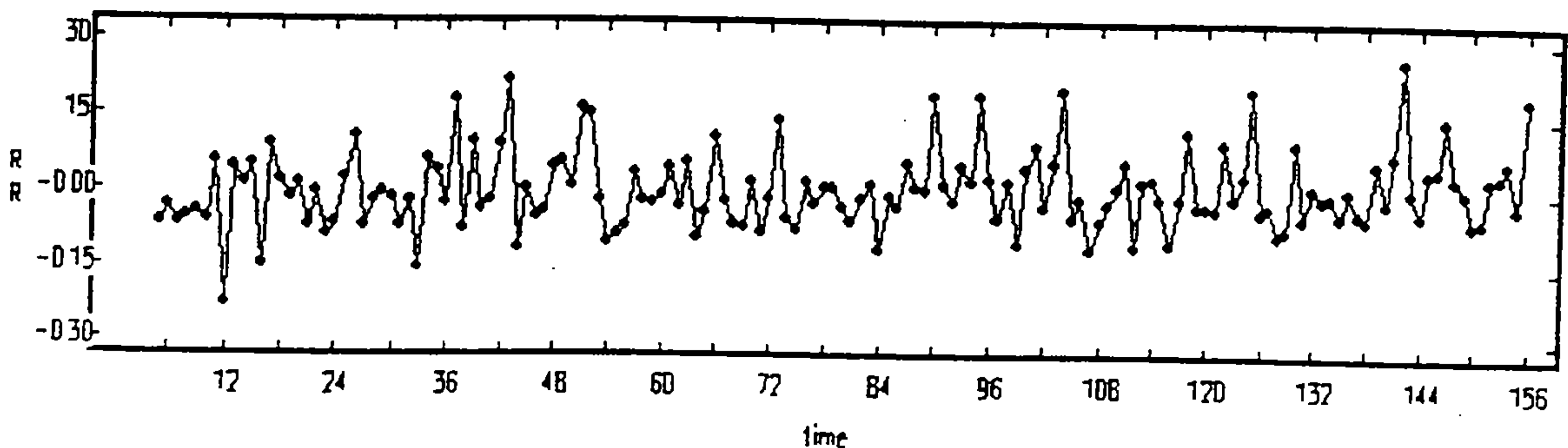
1- 12	-.02	-.02	.03	-.06	.01	.01	.01	.04	.10	-.04	.03	-.04
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08
Q	.1	.1	.2	.8	.8	.9	.9	1.1	2.6	2.9	3.0	3.3

13- 15	-.07	.11	.05
ST.E.	.08	.08	.08
Q	4.2	6.0	6.5

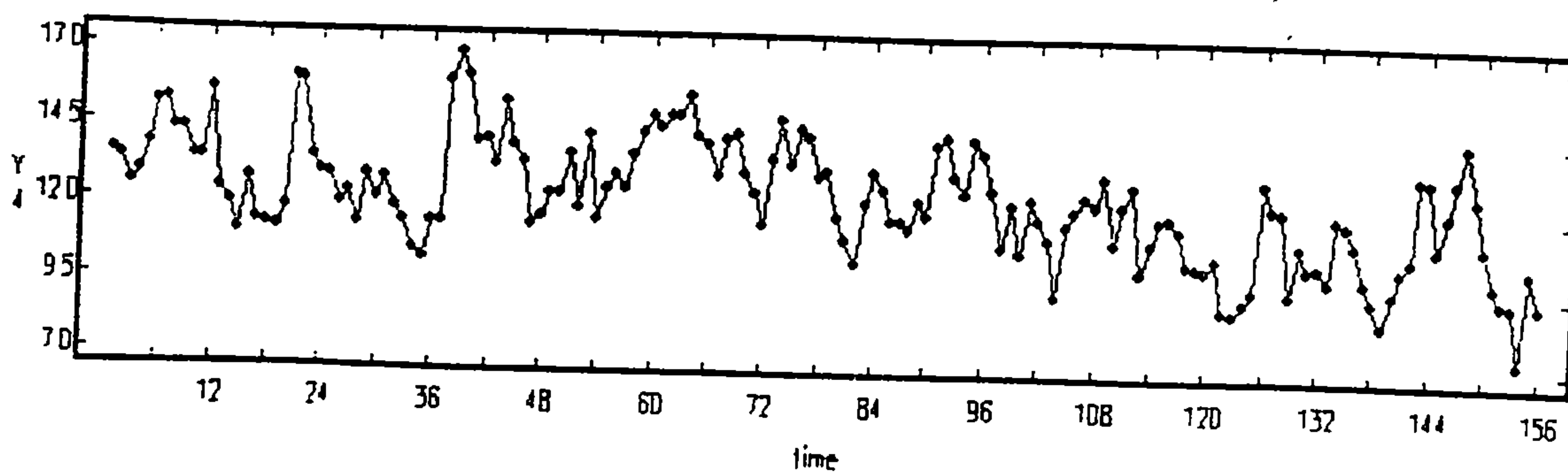
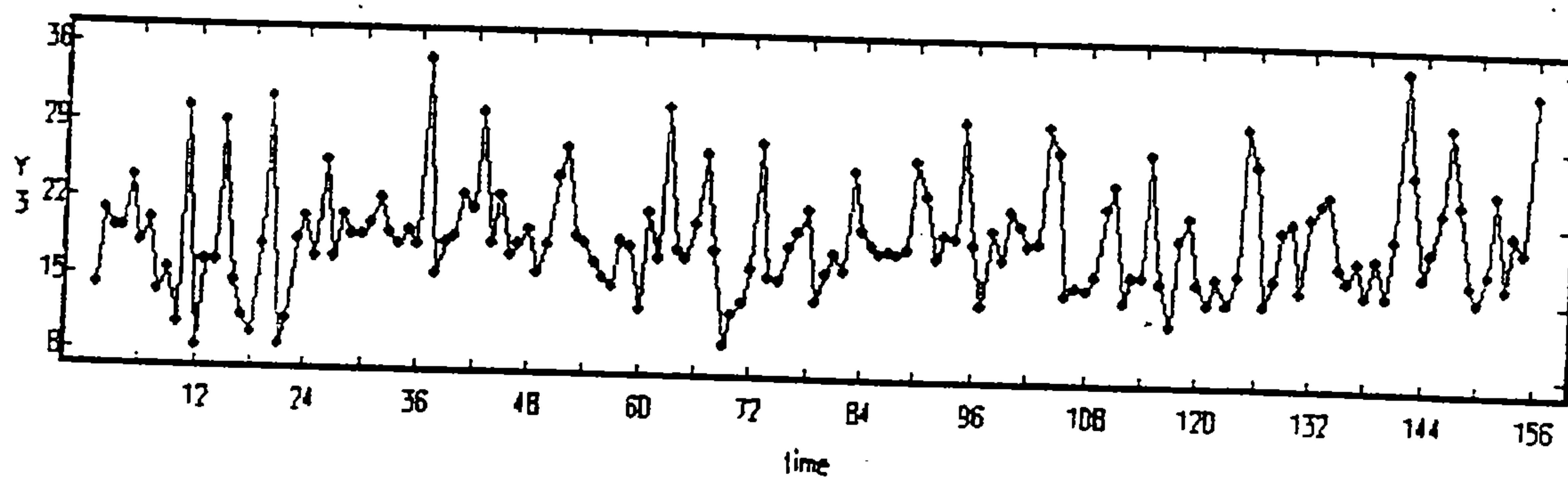
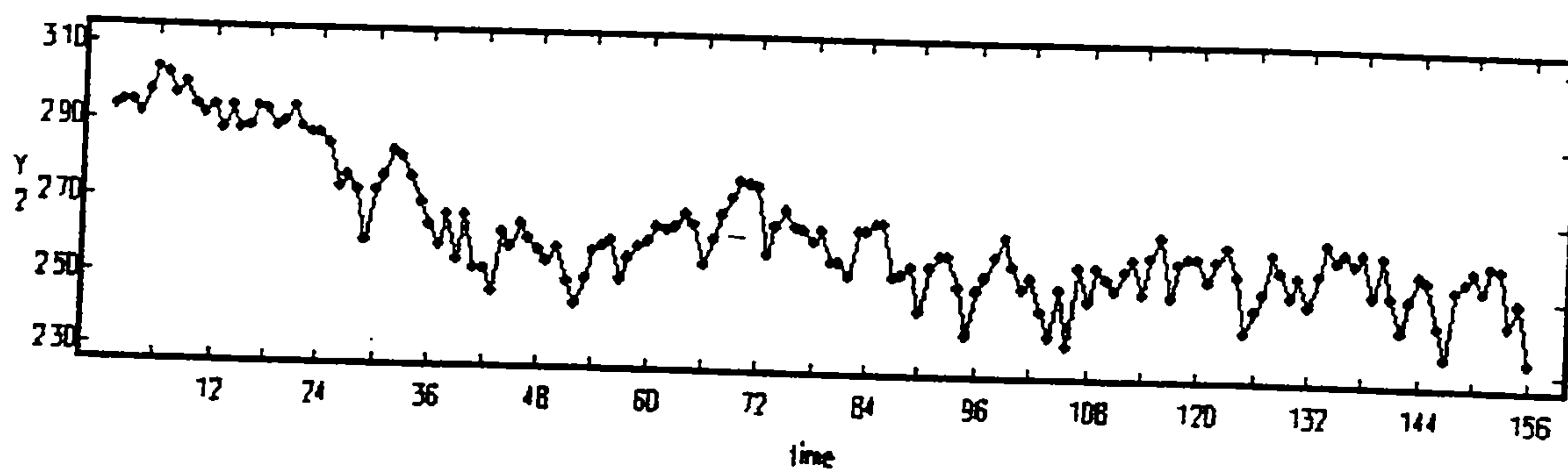
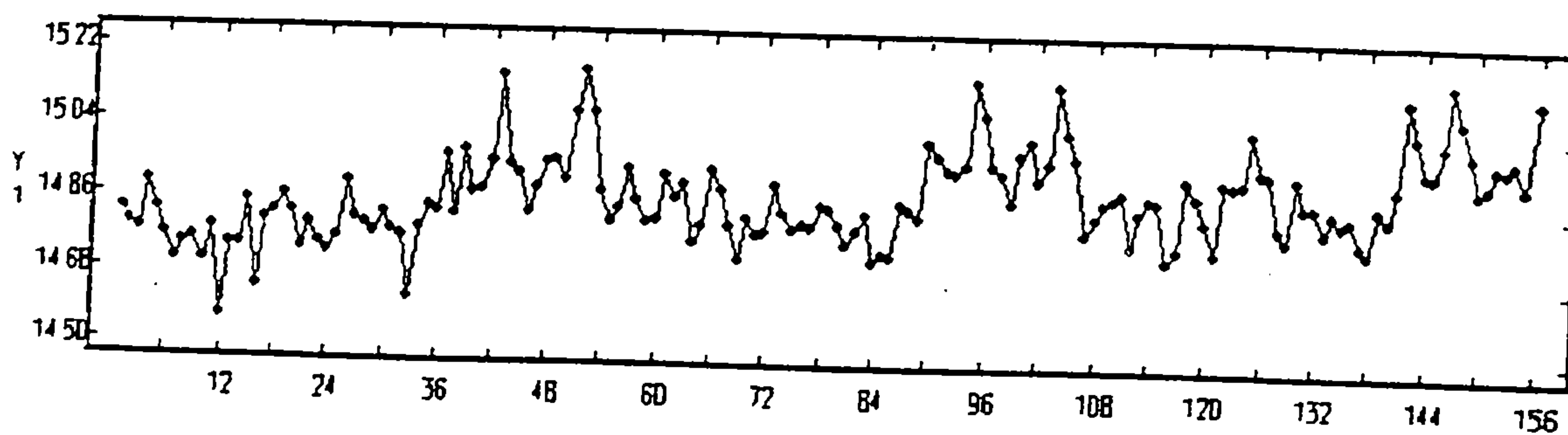
-1.0	-.8	-.6	-.4	-.2	.0	.2	.4	.6	.8	1.0
+	+	+	+	+	+	+	+	+	+	+

			I	
1	-.02		+ XI +	
2	-.02		+ XI +	
3	.03		+ IX +	
4	-.06		+ XXI +	
5	.01		+ I +	
6	.01		+ I +	
7	.01		+ I +	
8	.04		+ IX +	
9	.10		+ IXX +	
10	-.04		+ XI +	
11	.03		+ IX +	
12	-.04		+ XI +	
13	-.07		+ XXI +	
14	.11		+ IXXX+	
15	.05		+ IX +	

graph rr



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#Now we regress y1 on y2,y3,y4.

regr y1,y2,y3,y4.hold resi(rgr)

REGRESSION ANALYSIS FOR THE VARIABLE Y1

PREDICTOR	COEFFICIENT	STD. ERROR	T-VALUE
INTERCEPT	15.74663	.10783	146.03
Y2	-.41815	.03904	-10.71
Y3	.00801	.00113	7.10
Y4	.00541	.00324	1.67

CORRELATION MATRIX OF REGRESSION COEFFICIENTS

Y2	1.00		
Y3	.31	1.00	
Y4	-.30	.03	1.00
	Y2	Y3	Y4

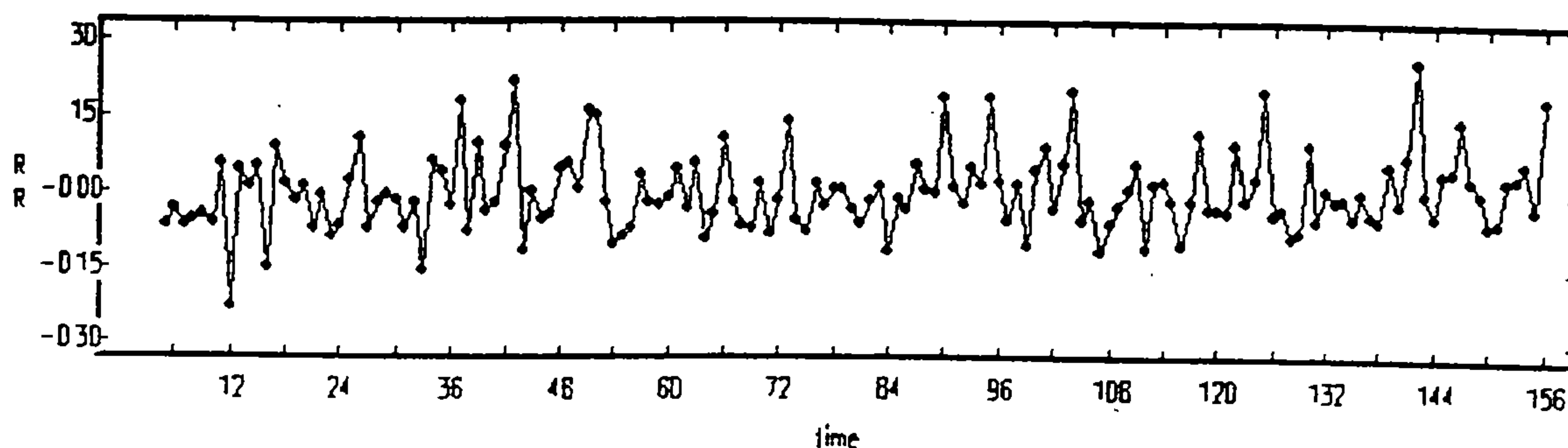
S = .0694 R**2 = 61.8%- R**2(ADJ) = 61.0%

ANALYSIS OF VARIANCE TABLE

SOURCE	SUM OF SQUARES	DF	MEAN SQUARE	F-RATIO
REGRESSION	1.182	3	.394	81.843
RESIDUAL	.732	152	.005	
ADJ. TOTAL	1.914	155		

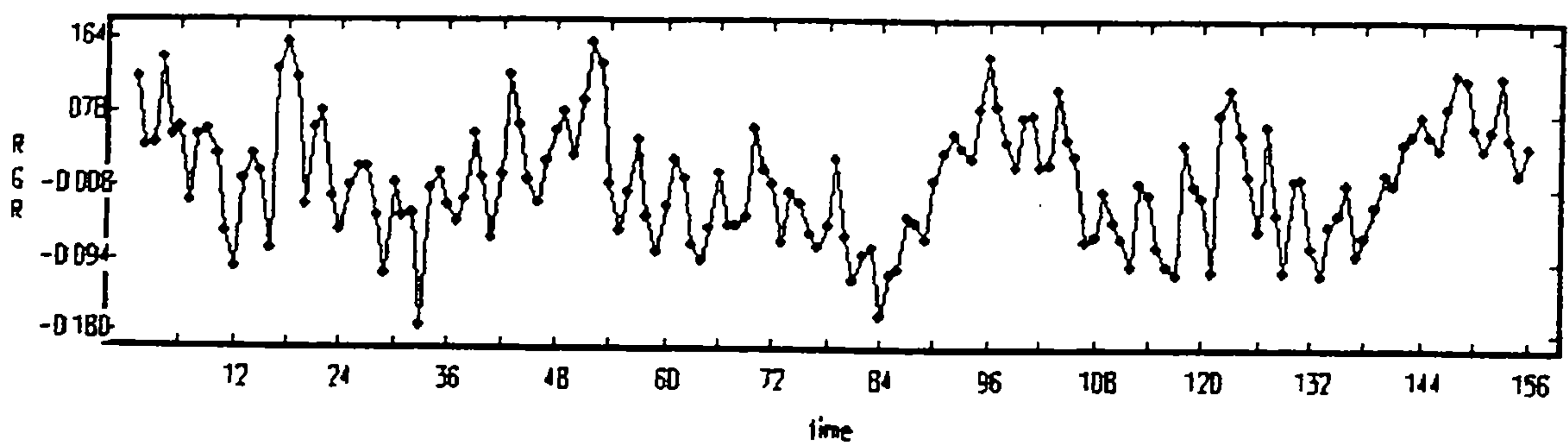
SOURCE	SEQUENTIAL SS	DF	MEAN SQUARE	F-RATIO
Y2	.929	1	.929	193.024
Y3	.239	1	.239	49.725
Y4	.013	1	.013	2.781

#It is interesting to compare the residuals rr with rgr.



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graph rgr



#The plot clearly shows that rgr is not a white noise !

eacf rgr

TIME PERIOD ANALYZED 1 TO 156
NAME OF THE SERIES RGR
EFFECTIVE NUMBER OF OBSERVATIONS . . . 156
STANDARD DEVIATION OF THE SERIES 0685
MEAN OF THE (DIFFERENCED) SERIES 0000
STANDARD DEVIATION OF THE MEAN 0055
T-VALUE OF MEAN (AGAINST ZERO)-.0001

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.56	.26	.29	.51	.35	-.03	-.08	.12	.17	-.17	-.33	-.20	-.00
(P= 1)	.16	-.38	-.08	.48	.35	-.18	-.03	.17	.09	-.19	-.32	-.20	-.00
(P= 2)	.30	-.18	-.15	.40	.33	.10	.04	.01	-.01	-.23	-.09	-.15	.12
(P= 3)	-.49	.37	-.28	.22	.03	-.03	.11	-.03	-.03	-.27	.04	-.28	.14
(P= 4)	.42	.54	.31	.23	.21	.00	.11	-.02	.01	-.19	-.15	-.29	-.19
(P= 5)	-.45	.45	-.23	.25	.02	-.02	.12	-.07	.12	-.11	.11	-.09	-.05
(P= 6)	.07	.18	.20	-.18	-.30	-.06	.05	.11	.08	.06	.11	-.08	-.05

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	X	X	X	X	0	0	0	0	0	X	0	0
(P= 1)	0	X	0	X	X	0	0	0	0	0	X	0	0
(P= 2)	X	X	0	X	X	0	0	0	0	X	0	0	0
(P= 3)	X	X	X	0	0	0	0	0	0	X	0	X	0
(P= 4)	X	X	X	0	0	0	0	0	0	0	0	X	0
(P= 5)	X	X	0	X	0	0	0	0	0	0	0	0	0
(P= 6)	0	X	X	0	X	0	0	0	0	0	0	0	0

--
tsm m3.model y1=c1+(c2)y2+(c3)y3+(c4)y4+(1,2,3)/(1,2,3,4)@
noise
estim m3.method exact

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M3

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	C1	CNST	1	0	NONE	15.9392	.1360	117.24
2	C2	Y2	NUM.	1	0	NONE	-.4700	.0492 -9.55
3	C3	Y3	NUM.	1	0	NONE	.0082	.0007 12.49
4	C4	Y4	NUM.	1	0	NONE	.0002	.0027 .08
5		Y1	MA	1	1	NONE	-.1526	.1940 <u>-.79</u>
6		Y1	MA	1	2	NONE	-.6270	.0940 <u>-6.67</u>
7		Y1	MA	1	3	NONE	-.3599	.1757 -2.05
8		Y1	D-AR	1	1	NONE	.3810	.1699 2.24
9		Y1	D-AR	1	2	NONE	-.4061	.1701 -2.39
10		Y1	D-AR	1	3	NONE	-.0314	.1962 <u>-.16</u>
11		Y1	D-AR	1	4	NONE	.5803	.1255 <u>4.62</u>
TOTAL SUM OF SQUARES						.191398E+01		
TOTAL NUMBER OF OBSERVATIONS						156		
RESIDUAL SUM OF SQUARES						.274778E+00		
R-SQUARE						.853		
EFFECTIVE NUMBER OF OBSERVATIONS						152		
RESIDUAL VARIANCE ESTIMATE						.180775E-02		
RESIDUAL STANDARD ERROR						.425176E-01		

tsm m4.model y1=c1+(c2)y2+(c3)y3+(2,3)/(1,2,4)noise.no show
estim m4.method exact.hold resi(rrr)

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M4

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRAINT	VALUE	STD ERROR	T VALUE
1	C1	CNST	1	0	NONE	15.9524	.1334	119.58
2	C2	Y2	NUM.	1	0	-.4746	.0488	-9.72
3	C3	Y3	NUM.	1	0	.0082	.0006	13.05
4		Y1	MA	1	2	-.6096	.0710	-8.58
5		Y1	MA	1	3	-.2724	.0663	-4.11
6		Y1	D-AR	1	1	.4322	.0348	12.42
7		Y1	D-AR	1	2	-.4254	.0573	-7.42
8		Y1	D-AR	1	4	.5614	.0527	10.65

TOTAL SUM OF SQUARES 191398E+01
TOTAL NUMBER OF OBSERVATIONS 156
RESIDUAL SUM OF SQUARES. 279019E+00
R-SQUARE 850
EFFECTIVE NUMBER OF OBSERVATIONS 152
RESIDUAL VARIANCE ESTIMATE 183565E-02
RESIDUAL STANDARD ERROR. 428445E-01

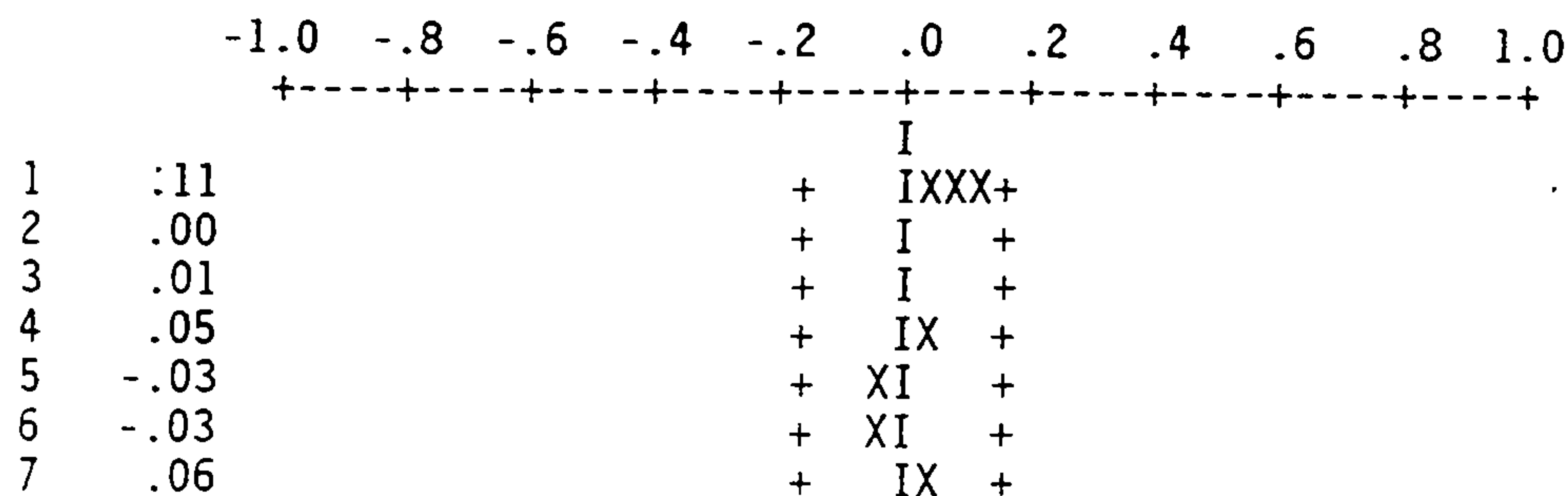
--
acf rrr.maxlag 15

TIME PERIOD ANALYZED 5 TO 156
NAME OF THE SERIES RRR
EFFECTIVE NUMBER OF OBSERVATIONS 152
STANDARD DEVIATION OF THE SERIES 0428
MEAN OF THE (DIFFERENCED) SERIES-0002
STANDARD DEVIATION OF THE MEAN 0035
T-VALUE OF MEAN (AGAINST ZERO)-0711

AUTOCORRELATIONS

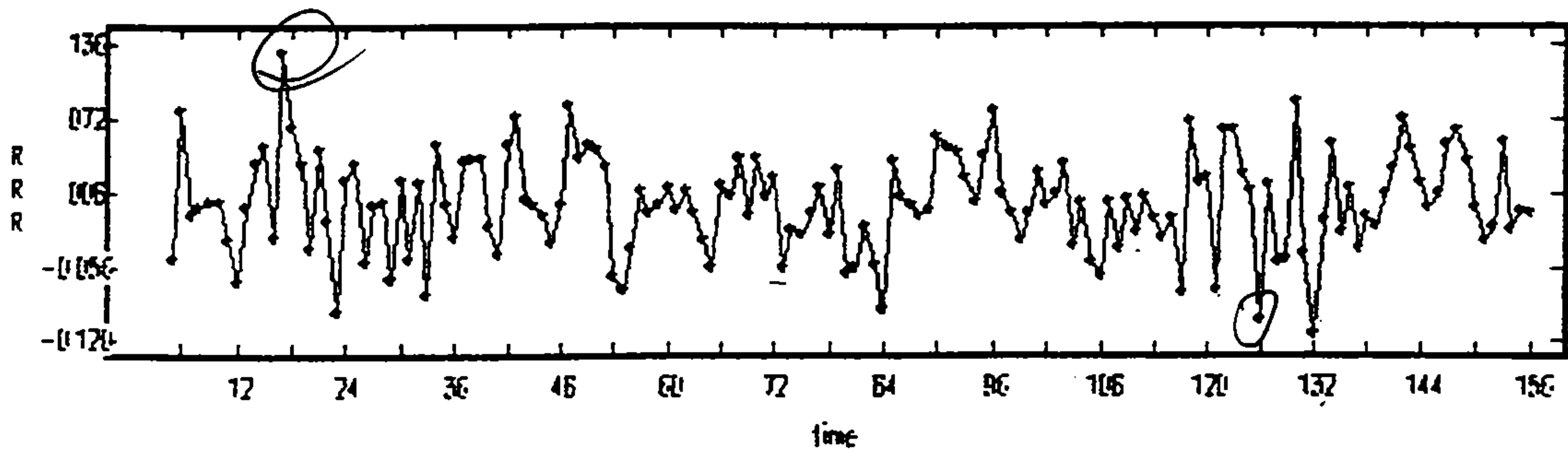
1- 12	.11	.00	.01	.05	-.03	-.03	.06	-.04	-.03	-.07	.06	-.17
ST.E.	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08	.08
Q	1.9	1.9	1.9	2.2	2.4	2.5	3.1	3.4	3.6	4.4	4.9	9.8

13- 15	-.13	-.14	-.08
ST.E.	.09	.09	.09
Q	12.6	16.2	17.3



8	-.04	+ XI	+
9	-.03	+ XI	+
10	-.07	+ XXI	+
11	.06	+ IX	+
12	-.17	XXXXI	+
13	-.13	+XXXI	+
14	-.14	XXXXI	+
15	-.08	+ XXI	+

graph rrr.



outlier m4.type A0,I0,LS

INITIAL RESIDUAL STANDARD ERROR = .42766E-01

TIME	ESTIMATE	T-VALUE	TYPE
126	-.11	-3.18	A0
17	.13	3.24	I0

ADJUSTED RESIDUAL STANDARD ERROR = .39990E-01

#We now conduct some commonly used regression diagnostics.

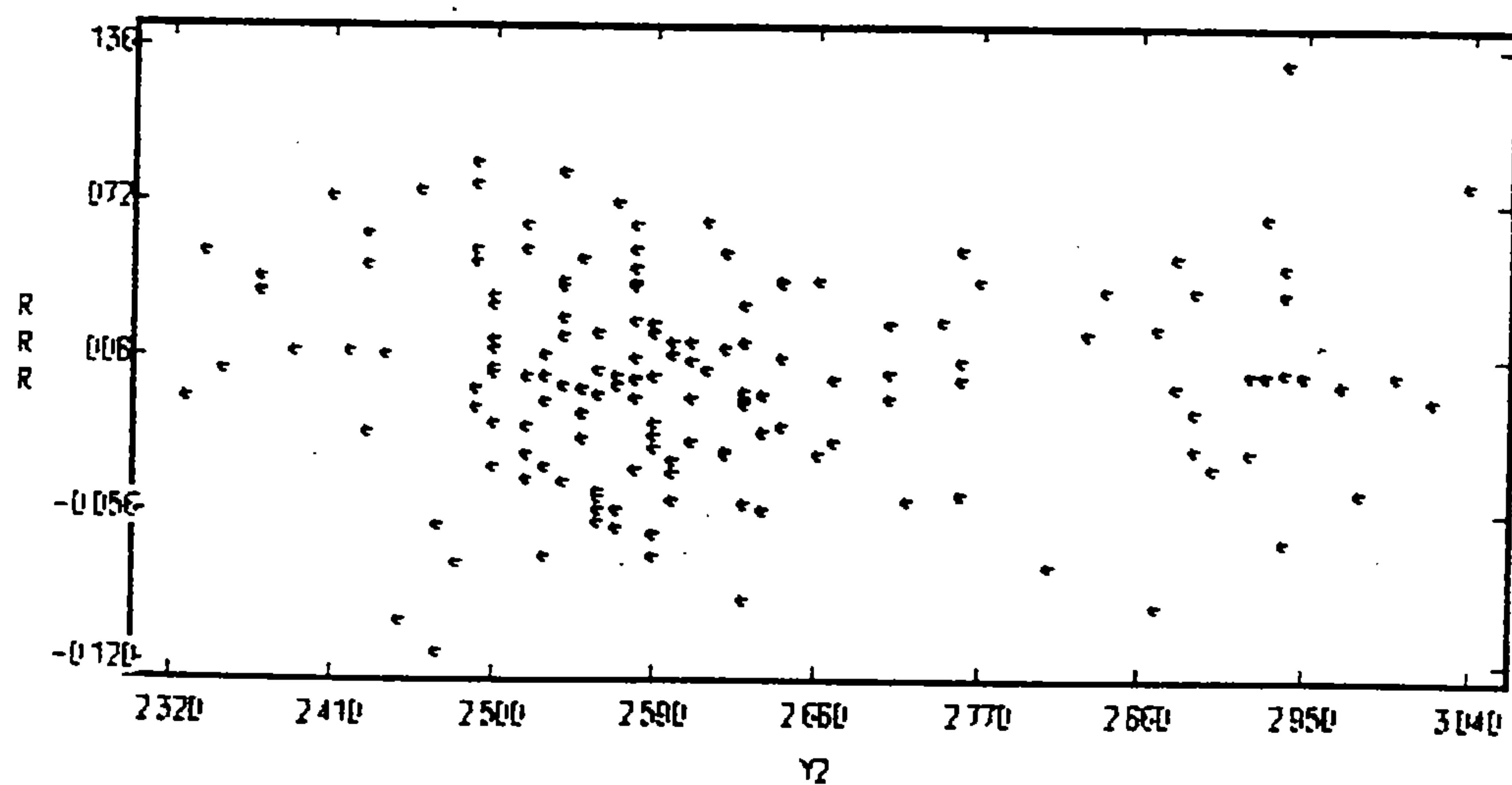
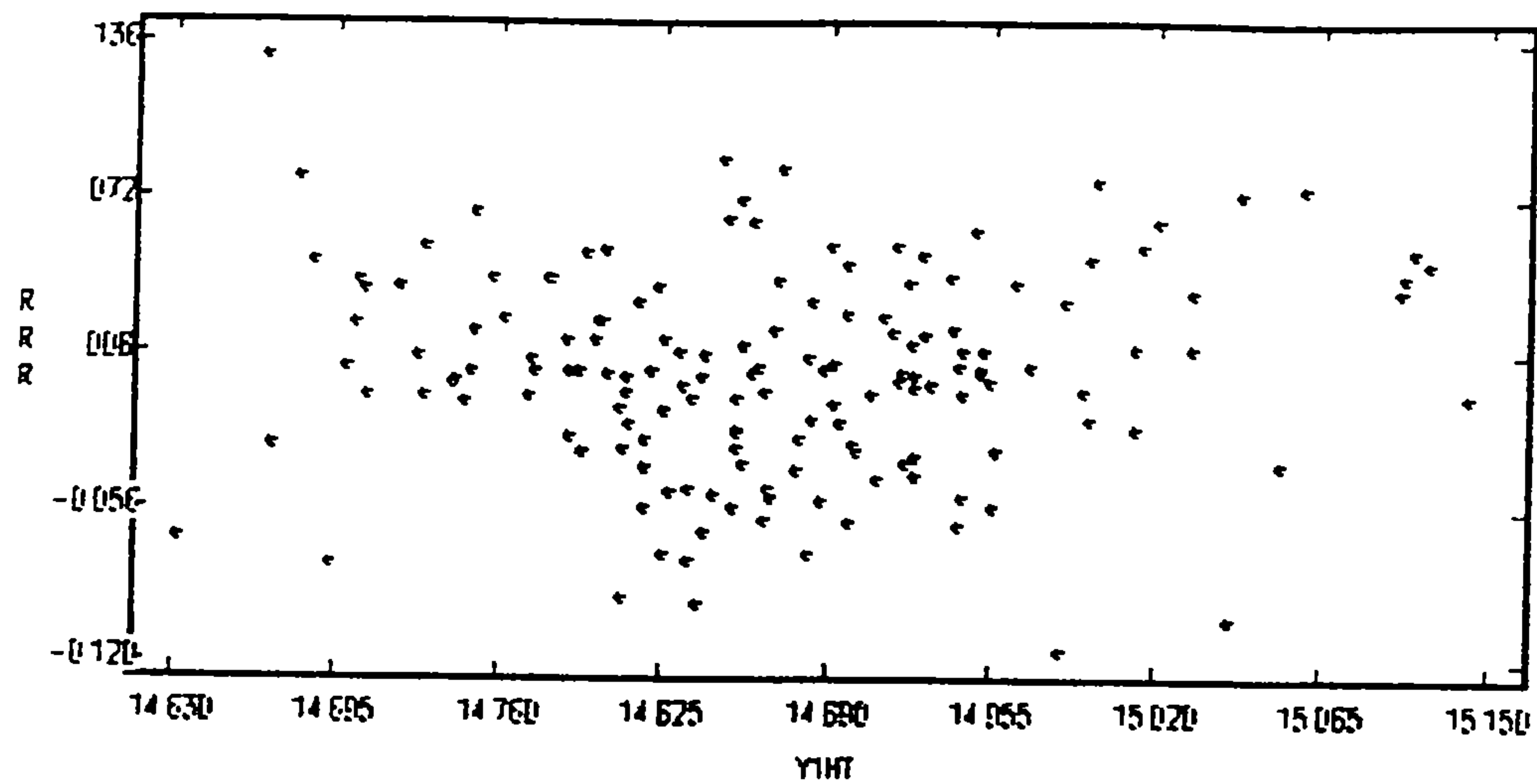
```

ylht=yl-rrr
**      4 MISSING DATA ARE FOUND IN THE VARIABLE   RRR   ; COMPUTATION ABORTED
--
rrr(1)=0
rrr(2)=0
rrr(3)=0
rrr(4)=0
ylht=yl-rrr

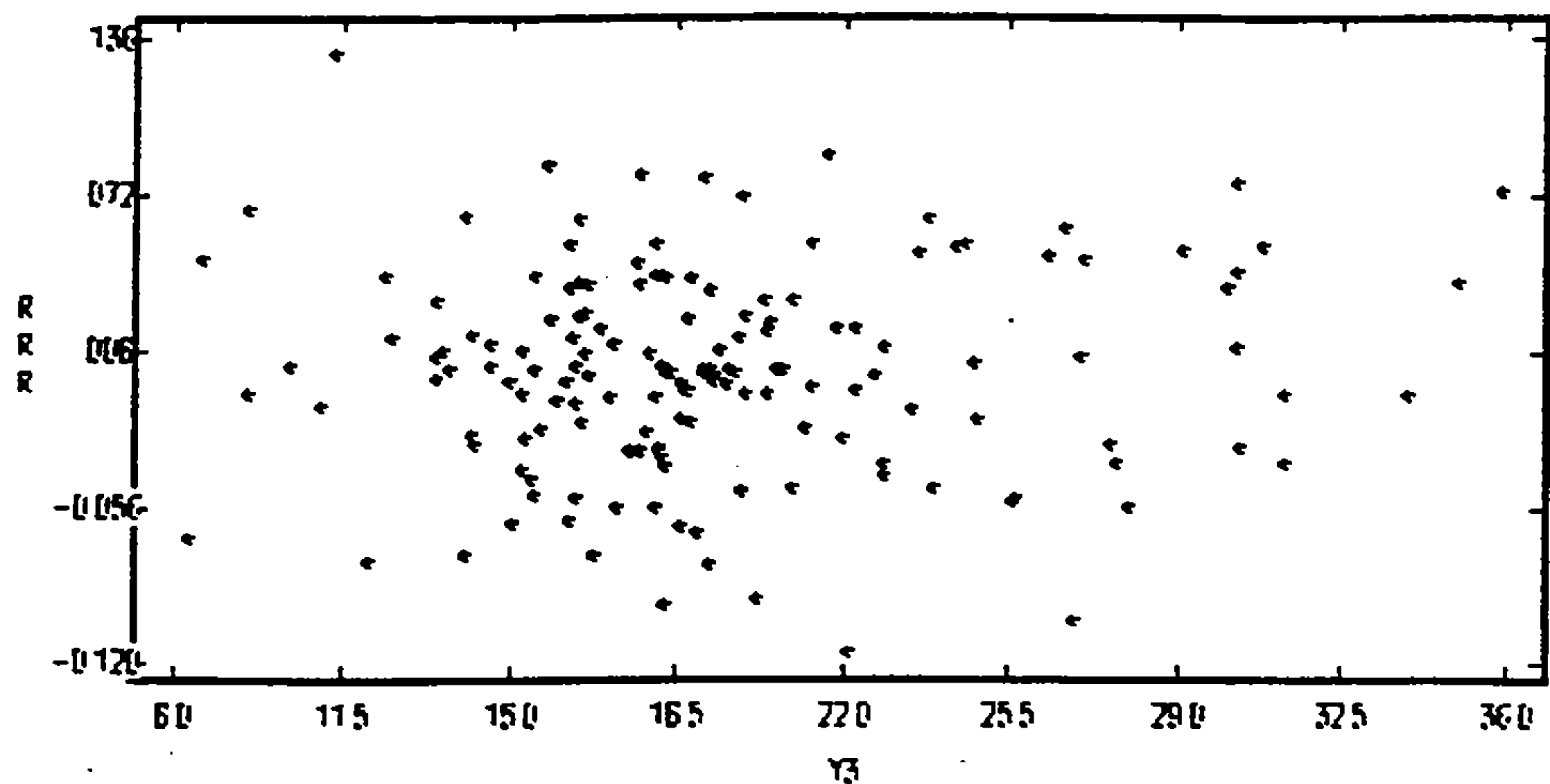
graph ylht

```

#In what follows we plot rrr against y1ht, y2, and y3, respectively.



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#No obvious inadequacy has been indicated by these plots. As a final step, we add lag 1 values of y2 and y3 to the regression time series model to see if they might lead to some further improvement.

lag y2.new y21

THE ORIGINAL SERIES IS Y2

THE LAG 1 SERIES IS STORED IN VARIABLE Y21 , WHICH HAS 157 ENTRIES

lag y3.new y31

THE ORIGINAL SERIES IS Y3

THE LAG 1 SERIES IS STORED IN VARIABLE Y31 , WHICH HAS 157 ENTRIES

tsm m5.model y1=c1+(c2)y2+(c3)y3+(c4)y21+(c5)y31+(2,3)/@
(1,2,4)noise.no show
estim m5

SUMMARY FOR UNIVARIATE TIME SERIES MODEL -- M5

PARAMETER LABEL	VARIABLE NAME	NUM./ DENOM.	FACTOR	ORDER	CONS- TRRAINT	VALUE	STD ERROR	T VALUE
1 C1		CNST	1	0	NONE	16.0511	.1836	87.42
2 C2	Y2	NUM.	1	0	NONE	-.4709	.0498	-9.45
3 C3	Y3	NUM.	1	0	NONE	.0082	.0007	12.51
4 C4	Y21	NUM.	1	0	NONE	-.0426	.0522	-.82
5 C5	Y31	NUM.	1	0	NONE	.0002	.0007	.34
6	Y1	MA	1	2	NONE	-.5421	.0782	-6.93
7	Y1	MA	1	3	NONE	-.3031	.0711	-4.26

8	Y1	D-AR	1	1	NONE	.4573	.0409	11.18
9	Y1	D-AR	1	2	NONE	-.4052	.0582	-6.96
10	Y1	D-AR	1	4	NONE	.5650	.0520	10.86

TOTAL SUM OF SQUARES	.191156E+01
TOTAL NUMBER OF OBSERVATIONS	155
RESIDUAL SUM OF SQUARES.	.280964E+00
R-SQUARE	.849
EFFECTIVE NUMBER OF OBSERVATIONS	151
RESIDUAL VARIANCE ESTIMATE	.186069E-02
RESIDUAL STANDARD ERROR.	.431357E-01

--

#In summary, by combining regression and time series techniques, we have obtained a much more satisfactory model for this data set than each one individually would have gotten.