Handouts Index

CHAPTER 1

Introduction

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1.1. EXAMPLES OF TIME SERIES PROBLEMS

Data in business, economics, engineering, environment, medicine, and other areas of scientific investigations are often collected in the form of time series, that is, a sequence of observations taken at regular intervals of time such as hourly temperature readings, daily stock prices, weekly traffic volume, monthly beer consumption, and annual growth rates. The main objectives of time series modeling and analysis are (1) understanding the dynamic or time-dependent structure of the observations of a single series—*univariate* time series analysis and (2) ascertaining the leading, lagging, and feedback relationships among several series—*multivariate* time series analysis.

Knowledge of the dynamic structure will help produce accurate forecasts of future observations and design optimal control schemes. This chapter presents first a number of univariate and multivariate time series data sets arisen from various scientific disciplines. These data examples are used to introduce and illustrate the following:

- Stationary versus nonstationary series
- Linear versus nonlinear dynamic relationship
- Homogeneity versus heterogeneity in variance
- Unidirectional versus feedback relation between series

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- Outlier, level shift, structural change, and intervention
- Comovement and cointegration

These concepts motivate many of the topics discussed in this book.

1.1.1. Stationary series

Figure 1.1a shows a series of the yield of 70 consecutive batches of a chemical process given in Box and Jenkins (1976). The observations fluctuate about a fixed mean level with constant variance over the observational period. In other words, the overall behavior of the series remains the same over time. Such a series is called a *stationary series*. A formal definition of stationarity will be given later.

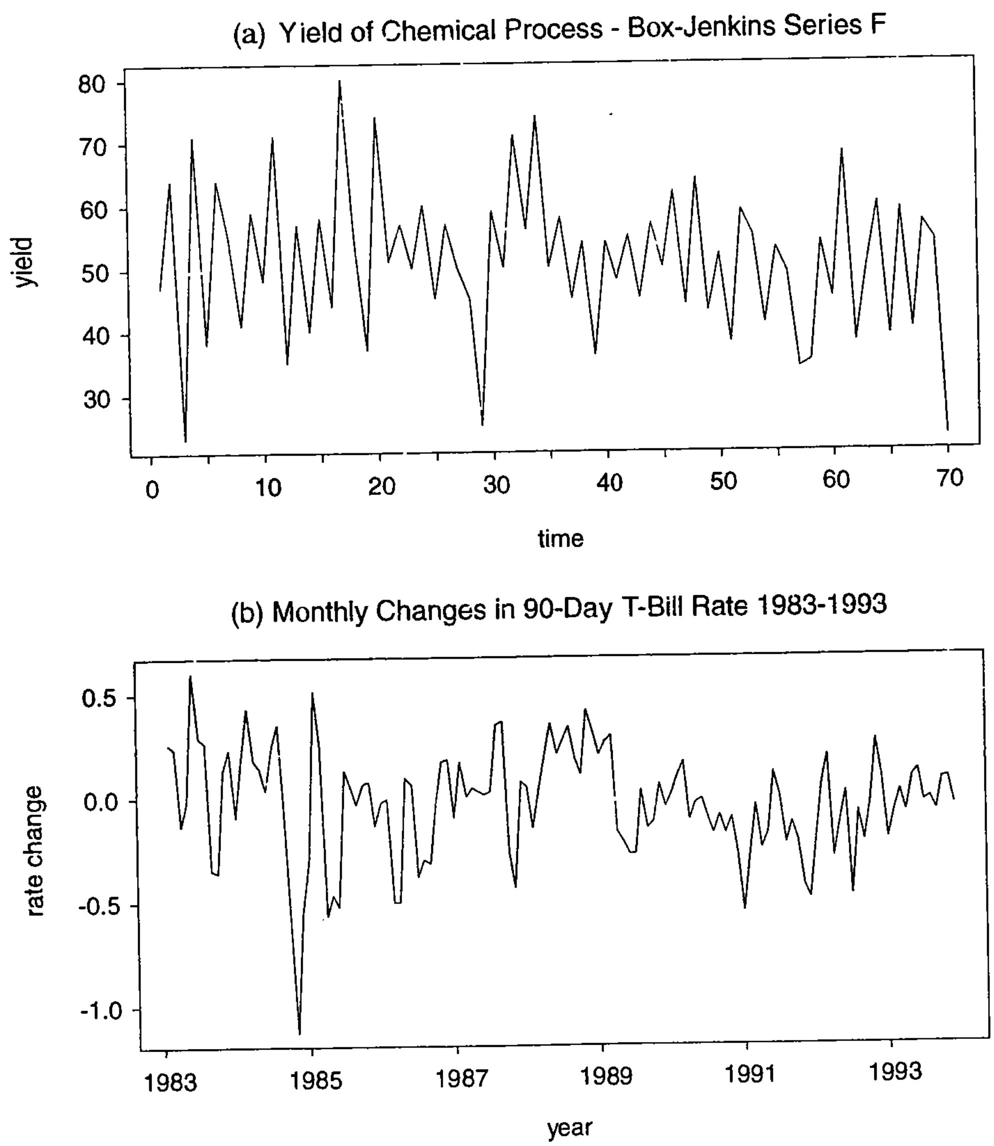

FIGURE 1.1 Two examples of stationary series.

As another example of stationary series, Figure 1.1b gives a series of the month to month changes in the interest rates of 90-day U.S. Treasury bills (T-bills) from 1983 to 1993. Except for the sharp dip near the end of 1984, this series appears to be quite stationary with a mean level close to zero over time.

In practice, temporal changes (week to week, month to month, or quarter to quarter) of many economic time series often exhibit this kind of stationary behavior. Good examples are stock returns and changes in exchange rates.

1.1.2. Nonstationary series

Instead of month to month changes, if we look at the series of monthly rates of the 90-day T-bills themselves, we see a vastly different behavior. This is shown in Figure 1.2a.

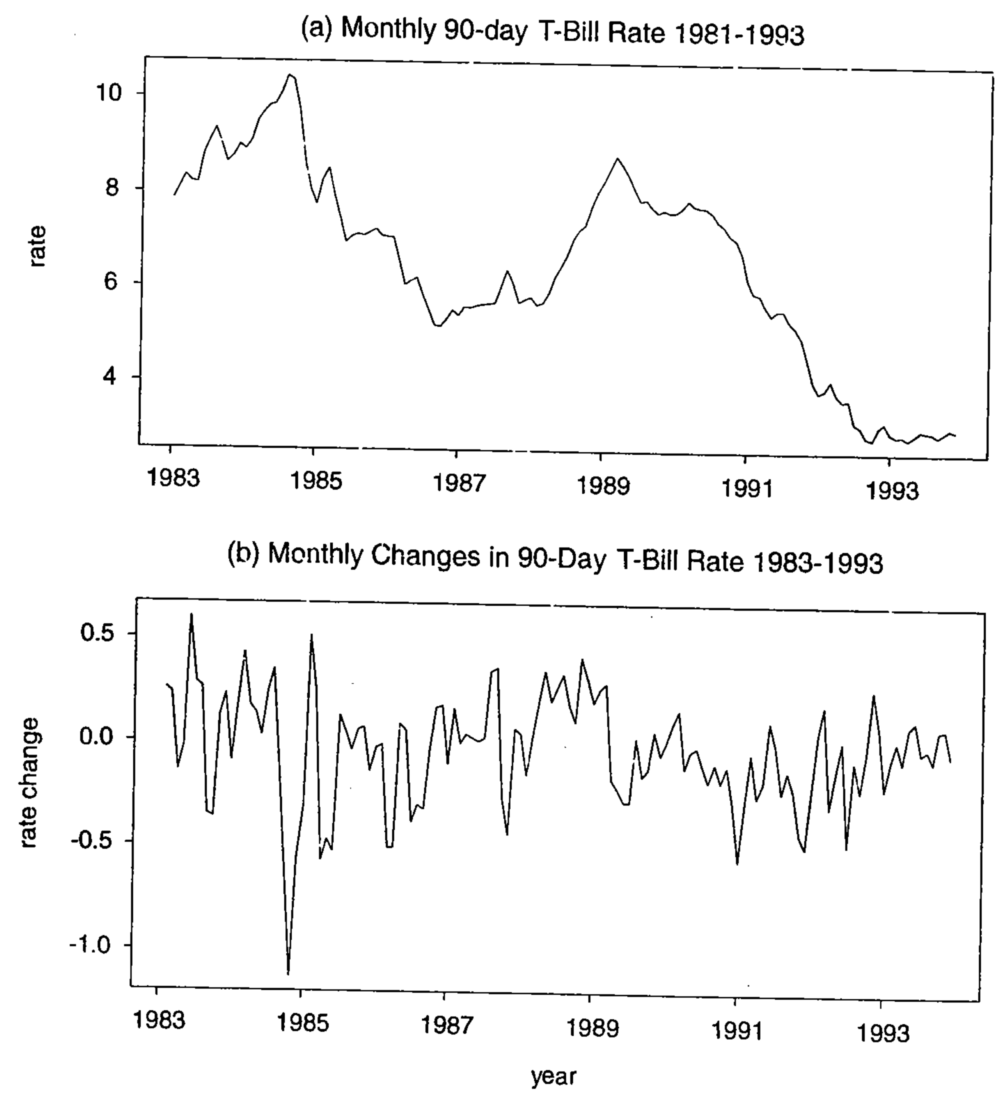

FIGURE 1.2 A nonstationary series and its first difference.

This series does not seem to have a mean level and exhibits a drifting or wandering behavior. It is clearly not a stationary series. Financial time series such as stock prices, prices of derivatives, and exchange rates often behave in this manner. However, by taking successive differences of the observations, we obtain the series of monthly changes in Figure 1.1b, which is reproduced in Figure 1.2b for easy comparison. This example shows that a drifting nonstationary series can be transformed into a stationary one by the differencing operation. The series in Figure 1.2b is called the first difference of the series in Figure 1.2a. In practice, sometimes the first difference series may not be stationary and it may be necessary to difference the series again to make it stationary.

Figure 1.3a shows quarterly data of U.S. real GNP (gross national product) over the period 1946–1991. The series shows an exponential growth. By taking a logarithmic

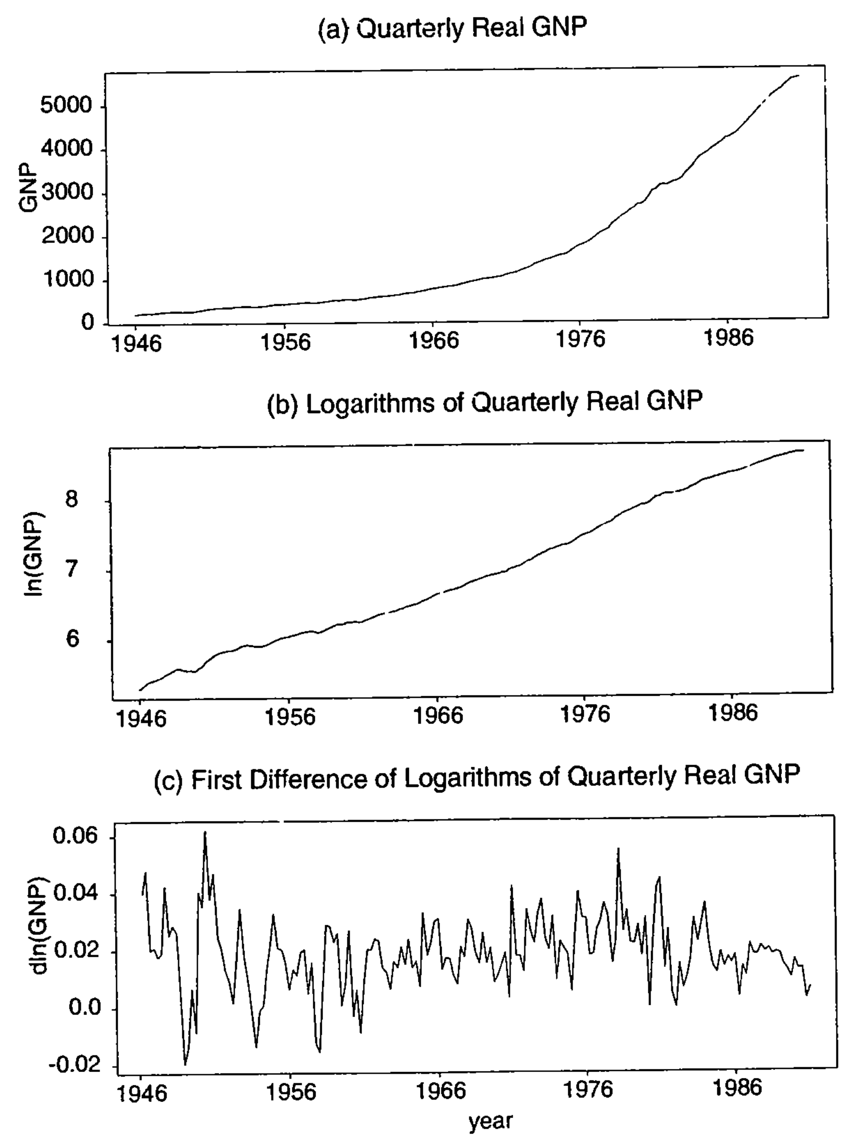

FIGURE 1.3 Quarterly U.S. real GNP 1946–1991.

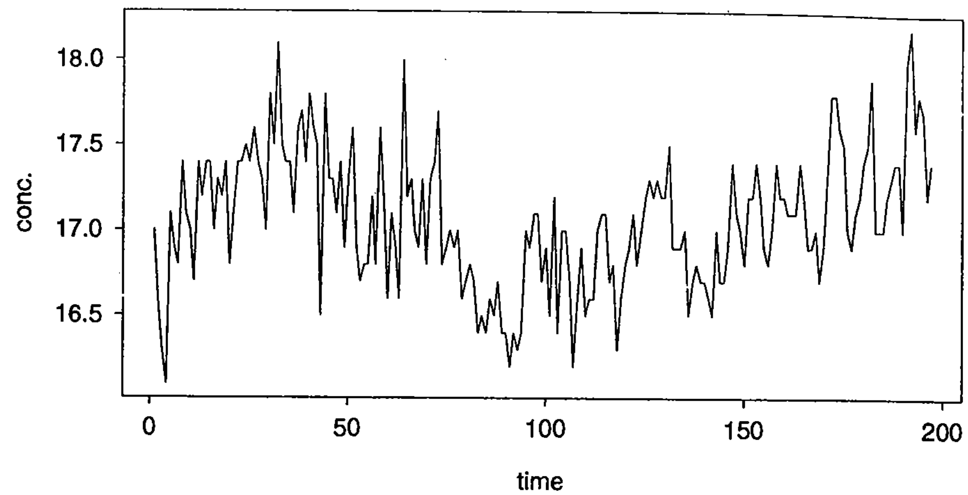

FIGURE 1.4 Concentration readings of a chemical process: Box–Jenkins series A.

transformation of the observations, we see a persistent linear growth in Figure 1.3b. The first difference series of the logged data is shown in Figure 1.3c, which is fairly stationary, although there appears to have some changes in the variability of the series over the data period.

Figure 1.4 presents series A and Figure 1.5 shows series C of Box and Jenkins (1976). The first appears to lie in the gray area of a stationary or a nonstationary series. For the second, differencing may be called for. Both have been used in the literature by other authors to illustrate novel methods for modeling time series.

1.1.3. Seasonal series

Time series data in business, economics, environment, and other disciplines often exhibit a strong cyclical or seasonal behavior. Modeling and analyzing such series

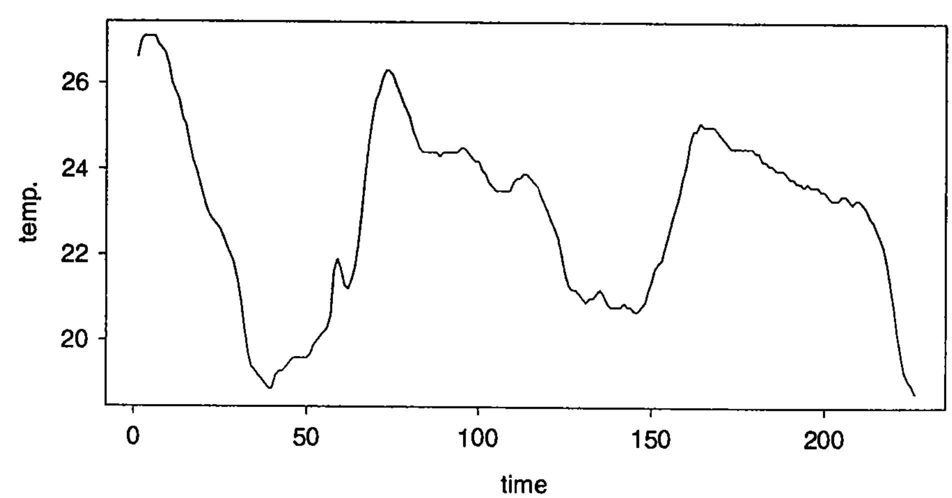

FIGURE 1.5 Temperature readings: Box–Jenkins series C.

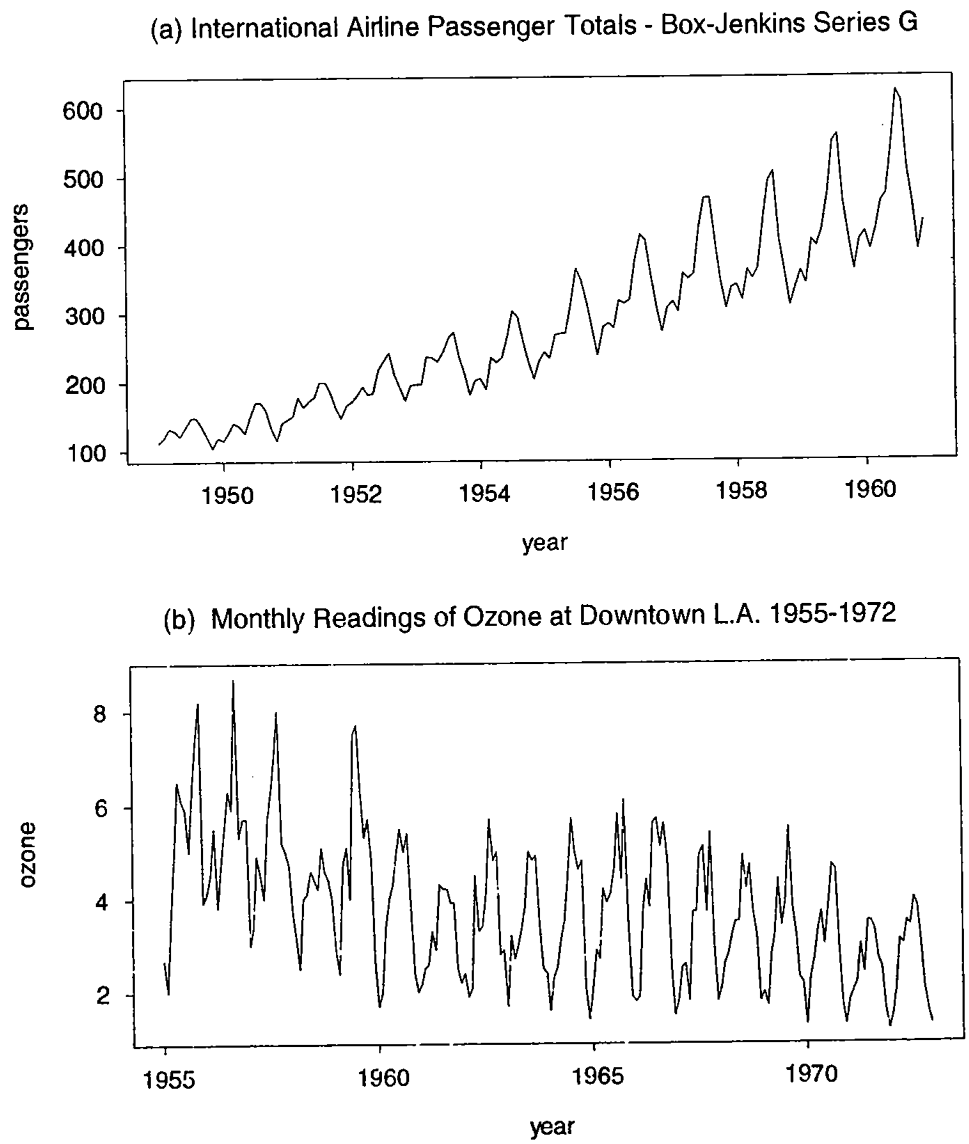

FIGURE 1.6 Two examples of seasonal series.

is an important topic in time series study. Figure 1.6a shows monthly international airline passenger totals in 1949–1959, which were used by Box and Jenkins (1976) to illustrate their innovative seasonal models. In practice, the user of the data may wish to remove the seasonality from the series in order to discern the "underlying trend," and this has led to the vast literature on seasonal decomposition and seasonal adjustment, which will be discussed later.

Figure 1.6b shows monthly averages of ozone in downtown Los Angeles during the period 1955–1972. Ambient ozone is an indicator of air pollution and is strongly seasonal: high in the summer months and low in the winter. In addition to seasonal cycles, there appears to be a level shift in the beginning of the sixth year and a down trend in the last 7 years of the data. The level shift may be associated with changes in

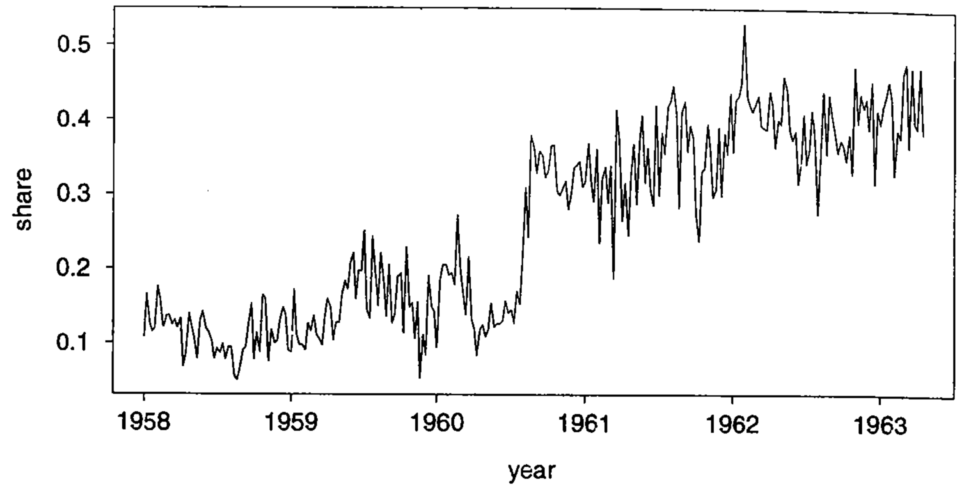

FIGURE 1.7 The Crest market share weekly data 1958–1963.

the traffic pattern and/or changes in the composition of gasoline sold in Los Angeles, and the down trend may be the result of progressively more stringent air quality standards at that time. This series was used by Box and Tiao (1975) to motivate intervention analysis in time series.

1.1.4. Level shifts and outliers in time series

Another example of a level shift is shown in Figure 1.7, the weekly market share data of Crest toothpaste from January 1958 to April 1963. In August 1960, the American Dental Association publicly endorsed Crest, and this led to a substantial jump in its market share as it is clearly seen in the figure. If the timing of this event is known, as in this case, the intervention analysis techniques can be applied to estimate its effect. In practice, such interventions are often unknown to the investigator, and detection of level shifts, outliers, and other types of structural changes becomes an important problem in time series analysis.

1.1.5. Variance changes

Figure 1.8 shows monthly returns of value-weighted S&P (Standard and Poor) 500 stocks from 1926 to 1991. While the mean level stayed close to zero over the entire period, it is clear that changes in the variance, called *volatility* in the finance literature, occurred. There has been an intense interest in modeling data of this kind in recent (at the time of writing) years, and some of the methods will be discussed later in the book.

1.1.6. Asymmetric time series

Two time series are shown in Figure 1.9. The first, in panel (a) is a series of annual sunspot numbers from 1700 to 1979; the other in panel (b), shows seasonally adjusted quarterly U.S. unemployment rates from 1948 to 1993. Both series share a common

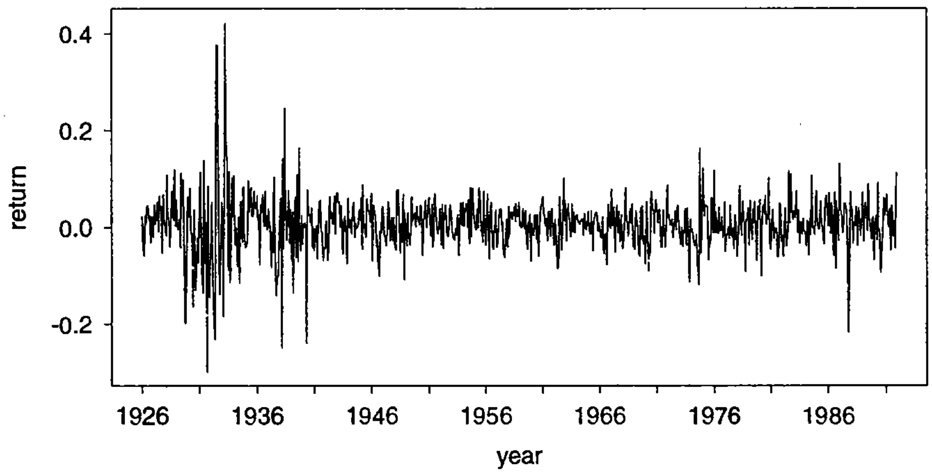

FIGURE 1.8 Value-weighted S&P 500 returns 1926–1991.

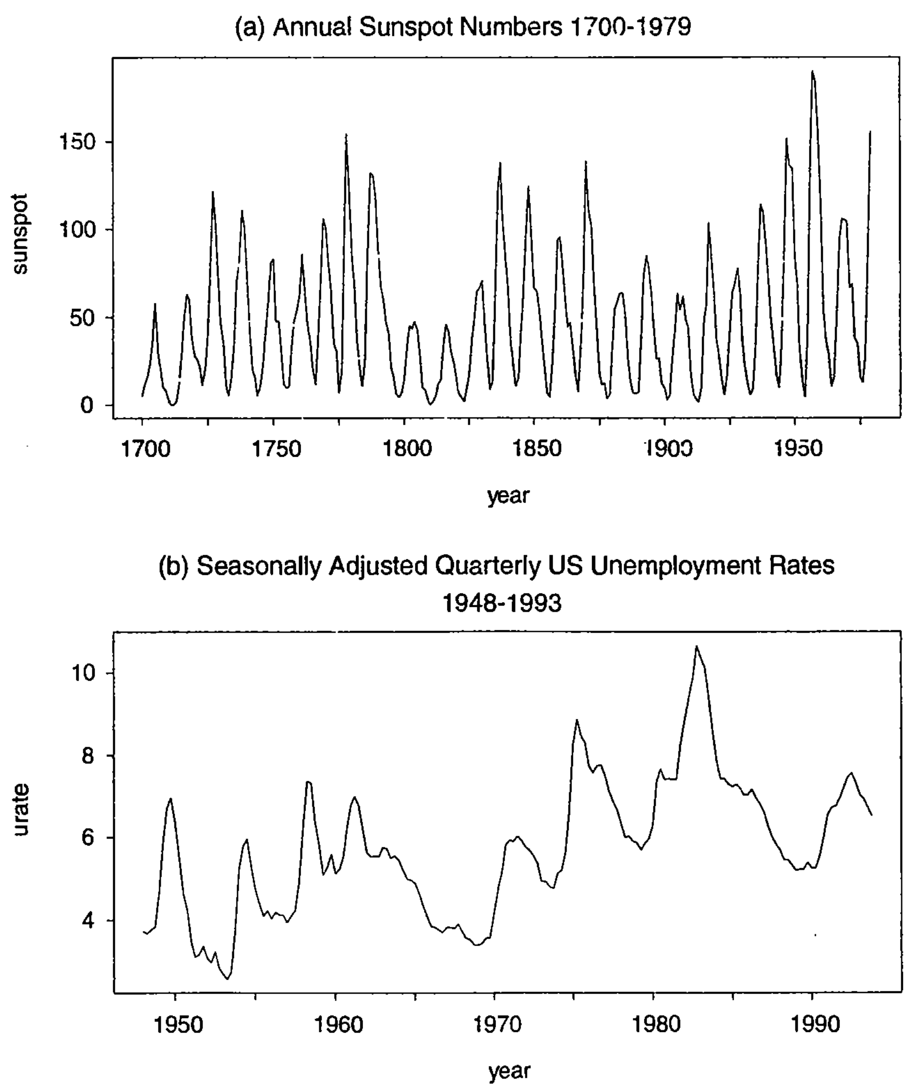

FIGURE 1.9 Two nonlinear time series.

feature; asymmetry in the rise and fall of the observations. Another way to express this asymmetric behavior is to say that these series are not time-reversible. The first will be used to illustrate nonlinear time series models, which is one of the most important research topics in time series analysis.

All the examples given above are univariate time series, and models will be introduced to relate the observations to their own past history. In practice, the principal purpose of modeling univariate series is on forecasting future observations of the series using its own past values. In the following examples, we now turn to consider several time series jointly.

1.1.7. Unidirectional–feedback relation between series

Figure 1.10 shows two series given in Box and Jenkins (1976), the input gas rate and the output CO_2 of a chemical reactor. The data were taken in 9-s intervals. These two

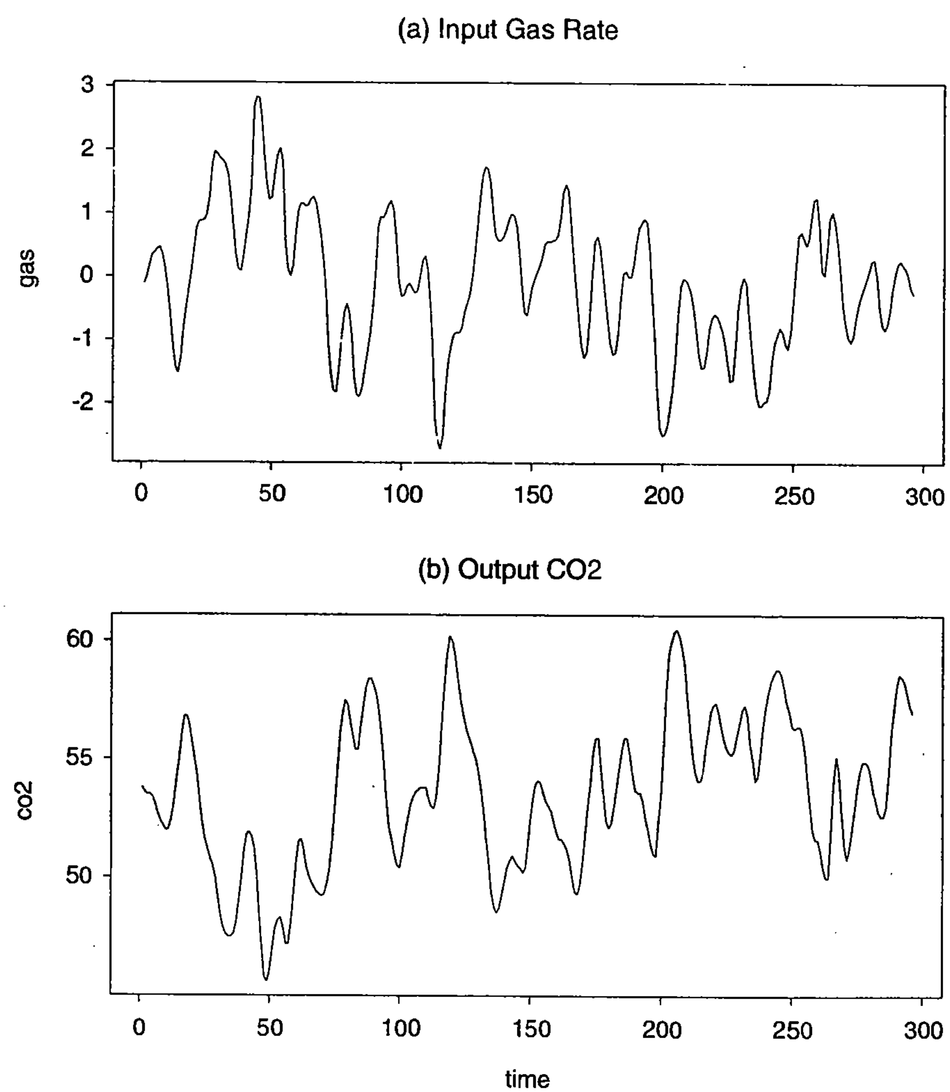

FIGURE 1.10 The gas furnace data: Box–Jenkins series J.

series are clearly related; when one goes up the other comes down. The figure also shows that, as expected, the input series leads the output series by several periods. Intuitively, the input series should therefore help to produce more efficient forecasts of the output series than just using the past values of the output. This example was used by Box and Jenkins to introduce their *transfer function* modeling techniques, but has also been used by others to illustrate modeling several series together.

The three series in Figure 1.11 are quarterly *Financial Times* stock, car production, and commodity indices over the period 1952–1967. They were first used by Coen et al. (1969) to establish an apparent regression relationship between stock index, car production index, which lagged six periods, as well as commodity index, which lagged seven periods. This result has led to many criticisms in the literature and has become a notorious example showing the importance of careful checking of the independence assumption of the residuals in regression of time series data.

1.1.8. Comovement and cointegration

Figure 1.12 gives five series, consisting of annual hog supply, hog prices, corn supply, corn prices and farm wages for the period 1867–1948. The data were given in Quenouille (1957), and all five series appear to be nonstationary. Box and Tiao (1977) used the data to illustrate their canonical analysis of multiple time series showing that linear combinations of nonstationary series can be stationary. Figure 1.13 shows five linearly transformed series, the first two of which are apparently stationary. This phenomenon has become known as "cointegration" (Engle and Granger 1986) and has been one of the most intensely studied topics in the econometrics literature in the last 10 years.

The three series in Figure 1.14 represent monthly logged flour price indices over the 9-year period 1972–1980 at the commodity exchanges of Buffalo, Minneapolis, and Kansas City, respectively. The example was used by Tiao and Tsay (1989) to illustrate their scalar component model technique in multiple time series model specification. The three series move in tandem as they should be, but they are not found to be cointegrated. This raises the interesting question as to how to characterize comovement in multiple time series.

As an further example of comovement of economic time series, Figure 1.15 shows three monthly series of 3-month, 6-month, and 9-month interest rates on bank deposits in Taiwan from 1961 to 1989. Again the three series move largely in tandem, but there is no cointegration. Tiao et al. (1993) employed this data set to illustrate the usefulness and limitations of various dimension reduction techniques including principal component, canonical correlation, and scalar component model methods.

1.2. OVERVIEW OF THE BOOK

The book is organized in three parts. Parts 1 and 2 concentrate on univariate time series models and Part 3, on multivariate models. A model for a univariate time series, z_t,

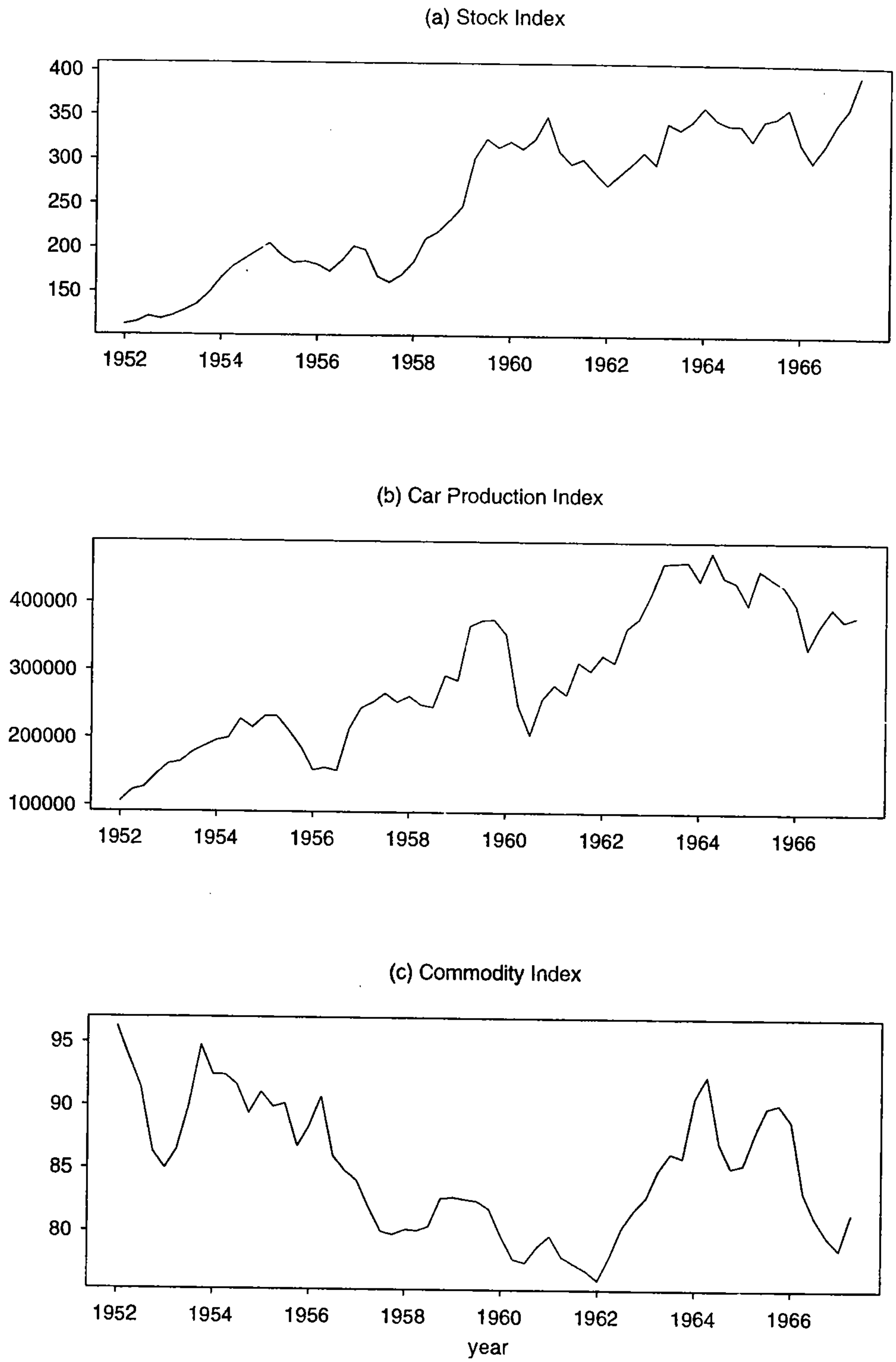

FIGURE 1.11 Quarterly *Financial Times* data 1952–1967.

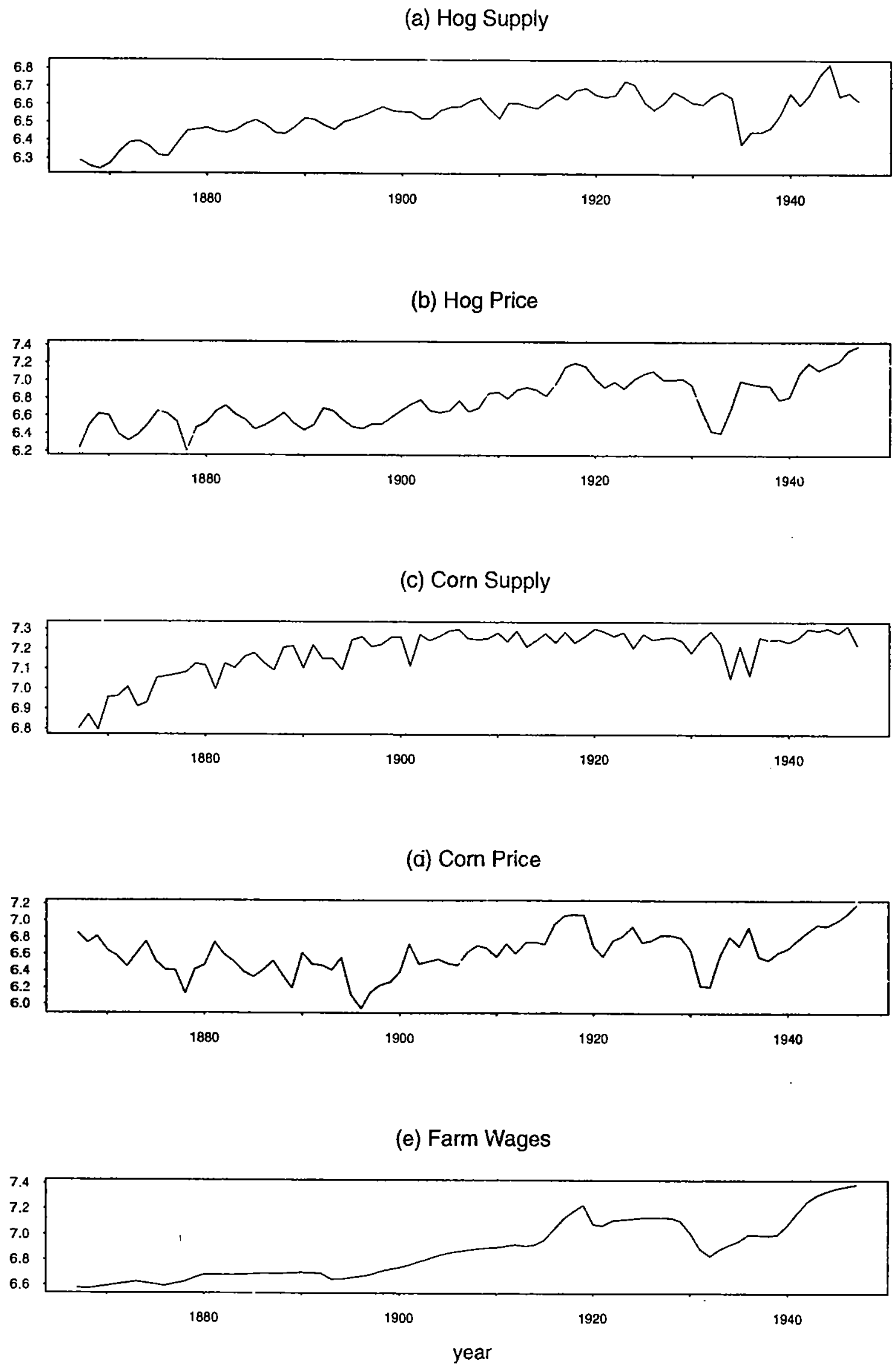

FIGURE 1.12 Quenouille's hog data 1867–1948.

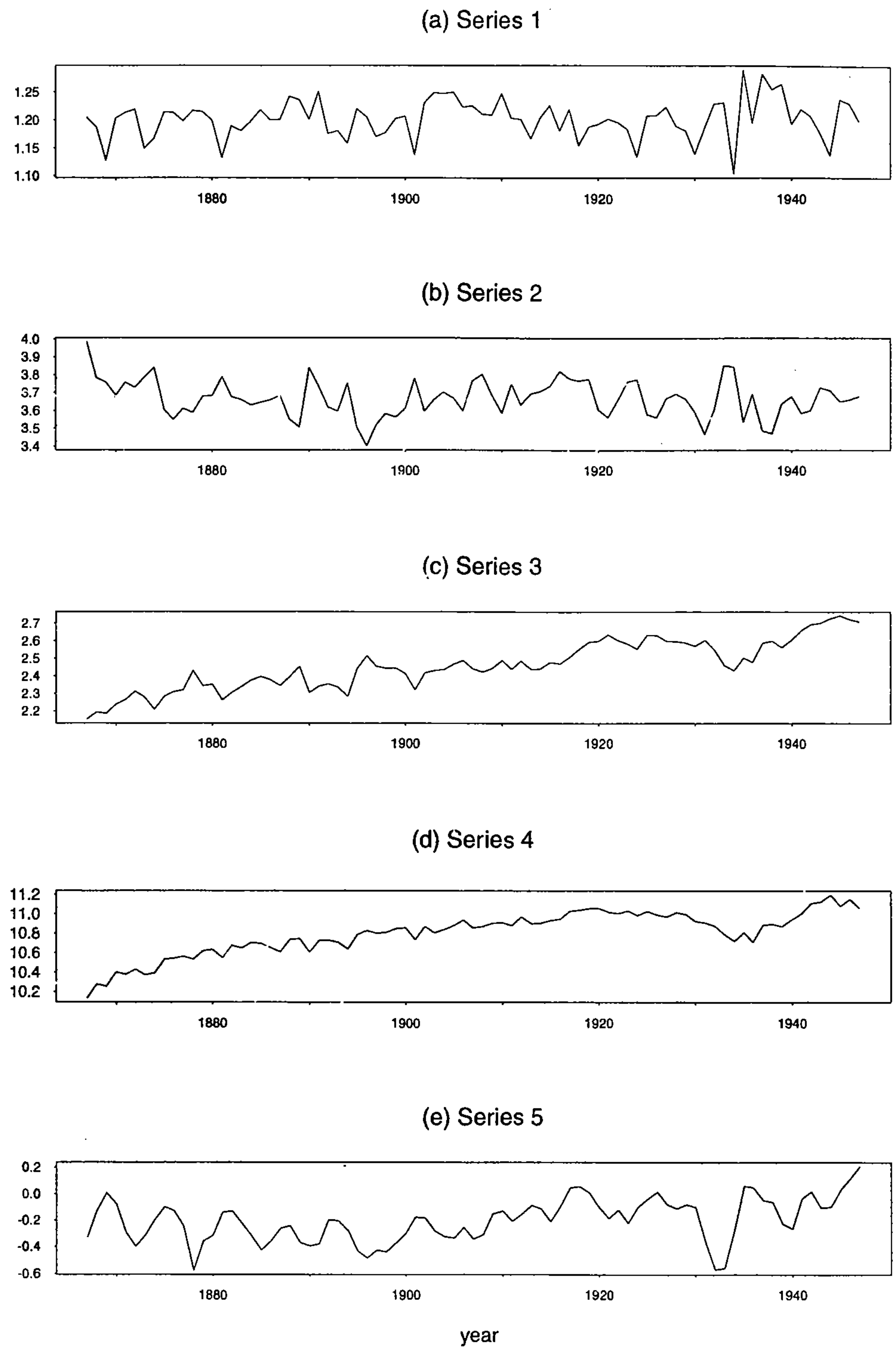

FIGURE 1.13 Linearly transformed hog data.

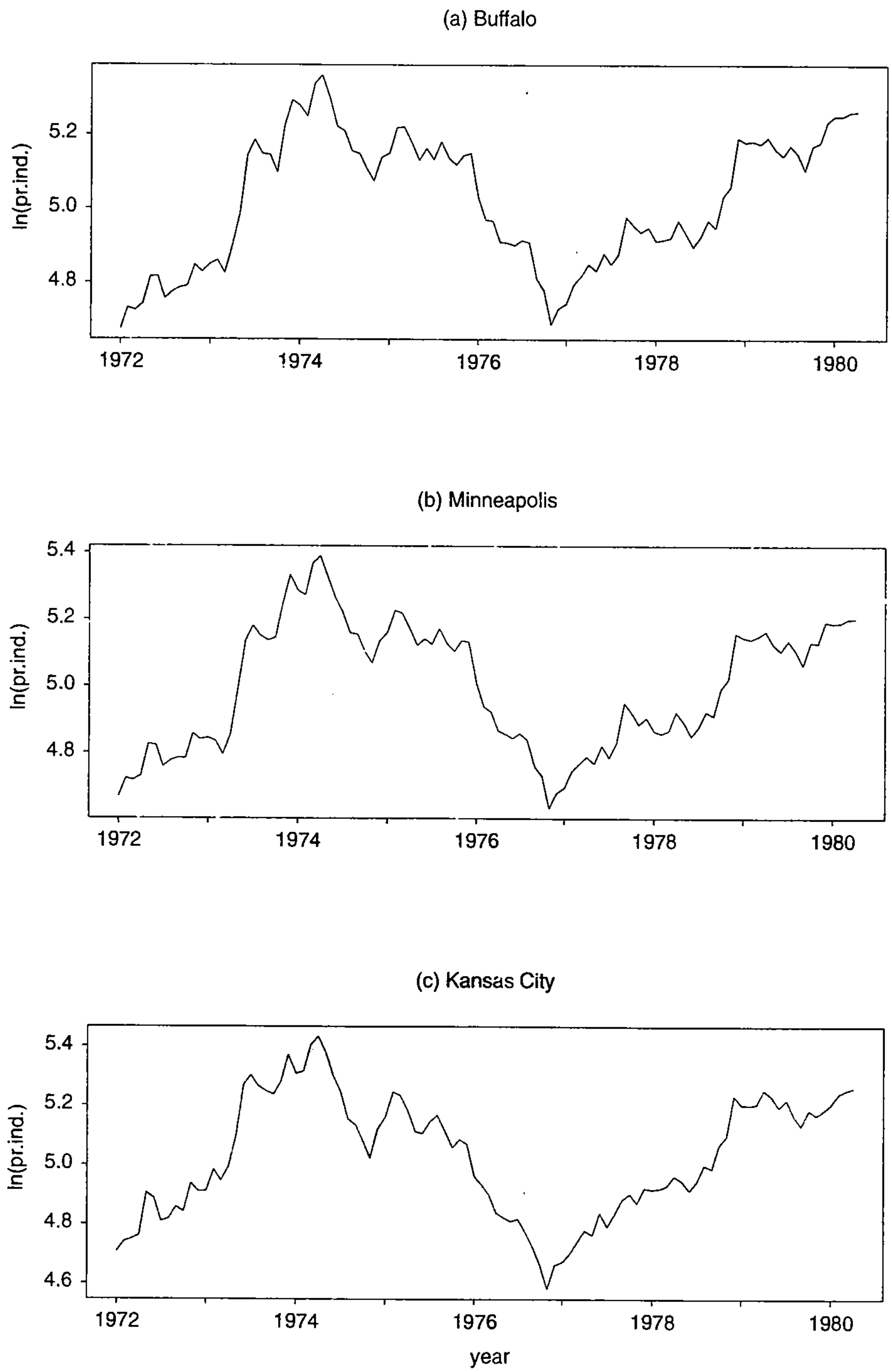

FIGURE 1.14 Logarithms of monthly flour price indices 1972–1980.

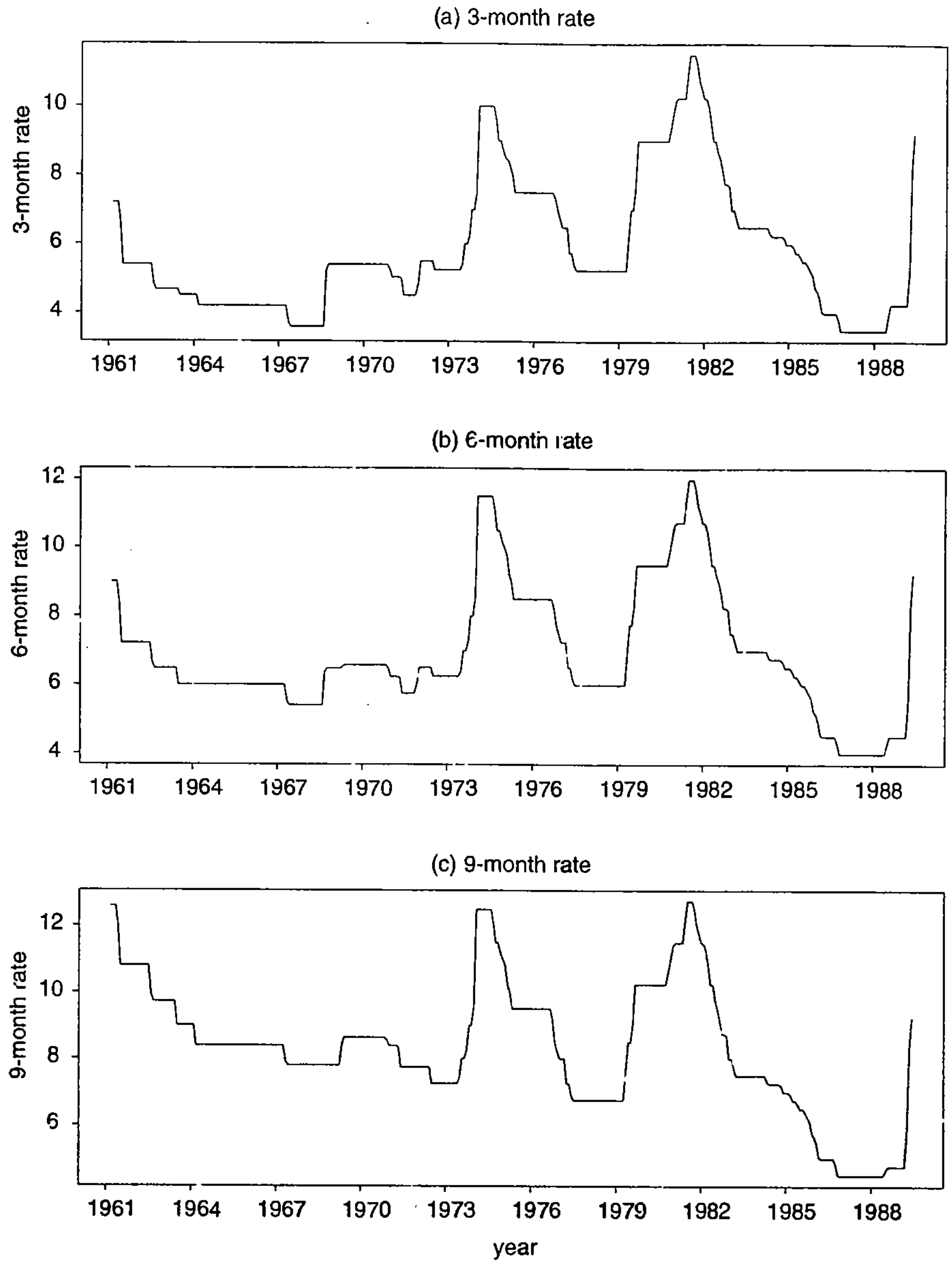

FIGURE 1.15 Taiwan's interest rate data 1961–1989.

takes usually the additive decomposition form

$$z_t = f(z_{t-1}, \dots, z_1) + a_t \tag{1.1}$$

where $f(z_{t-1}, \dots, z_1)$ is a function of the past values of the series, to be determined from the data, and a_t is a sequence of independent and identically distributed (iid) variables. The time series model can be seen as a way to decompose the data into a systematic part (the signal part), which depends on past values and therefore can be forecasted, $f(z_{t-1}, \ldots, z_1)$, and a noise part, a_t, which is independent from previous values and therefore it is unpredictable from its past. In some cases obtaining the structure of the function f is the main objective of the analysis whereas in other cases our interest is mostly in obtaining forecasts.

Part 1, on basic concepts for univariate time series modeling, presents the main ideas and tools for building univariate time series models. The presentation emphasizes linear autoregressive integrated moving-average (ARIMA) models, in which it is assumed that the function f has a linear form, that is, it can be written as

$$f(z_{t-1}, \ldots, z_1) = \pi_1 z_{t-1} + \cdots + \pi_{t-1} z_1 \qquad (1.2)$$

and a key problem is how to approximate the sequences of weights $(\pi_1, \pi_2, \ldots.)$ by using a small number of parameters. This is the idea of ARIMA models studied by Box and Jenkins (1976) in a landmark book on time series analysis. In addition to ARIMA models, this part also presents a brief analysis of two alternative time series approaches. The first one is the spectral approach in which it is assumed that the function f can be represented as a sum of sine and cosine waves. The second is the state space approach in which the evolution of the series is assumed to be a linear function of some unobserved factors or states as

$$z_t = \mu_t + a_t,$$

where μ_t is the mean, and a_t has the same interpretation as before. Then we have to assume some equation for the evolution of the mean as, for instance

$$\mu_t = \mu_{t-1} + u_t,$$

where u_t is another sequence of iid variables. State-space models and ARIMA models are closely related, but the latter provide more flexibility as we do not need to determine the evolution of the state variables; it is identified from the data.

Part 2, on advanced topics for univariate time series, covers Chapters 9–13, and includes more sophisticated time series models. The first generalization is to assume that the variability of the noise process is not constant but depends on past values of the process. This allows for particular forms of heteroscedasticity in a_t that have many applications in financial data. A second generalization is to assume some parametric nonlinear structures for the function f, and then we have nonlinear time series models. A third generalization is to try to estimate the function f without assuming a priori any parametric structure. This can be carried out by the nonparametric approach, if we have a large sample size, or we can try to approximate f by a general method as neural network models.

The third part of the book deals with multivariate models. In the simplest case of a bivariate system the two components series can be split into a dependent response time series, y_t, and an independent input series x_t. The first model equation will describe the dynamic evolution of the response as a function of the input variable in what is called a *dynamic regression model*

$$y_t = g(x_t, \ldots, x_1) + f(y_{t-1}, \ldots, y_1) + a_t \tag{1.3}$$

where, as before, a_t is a sequence of independent and identically distributed (iid) variables and the functions f and g are to be determined from the data. In addition to this equation, the time series model has to specify the model for the evolution of the univariate independent series x_t, that will be a univariate time series model. In the general case, we cannot say that the variable x_t causes y_t or vice versa because there is feedback between the two variables. The time series model will describe this situation by two dynamic regression equations. In general, letting $\mathbf{z}_t$ be a vector of k related time series, a multivariate time series model takes the form

$$\mathbf{z}_t = \mathbf{f}(\mathbf{z}_{t-1}, \ldots, \mathbf{z}_1) + \mathbf{a}_t \tag{1.4}$$

where now $\mathbf{f}$ is a vector of functions of the past values of all the components of the vector time series to be determined from the data and $\mathbf{a}_t$ is a sequence of vector variables without lag dependency. This model will be equivalent to k dynamic regression equations, in which each series is explained as a function of the past of all the other series and its own past. Note, however, that the noises from different equations are usually correlated. If we assume that f has a linear form, we obtain the *multivariate* ARIMA models and the *linear system models*. A key problem in multivariate modeling is finding simplifying structures in order to reduce the number of parameters in the model and facilitate its interpretation. An interesting structure with clear economic meaning is that linear combinations of the components of the vector time series are more stable over time than the series themselves. In particular, if the vector of series has some stochastic trend, it is interesting to find linear combinations that are stable around some fixed mean. This is the idea of cointegration. Finally, multivariate model can be analyzed form the state-space approach and a review of the field is also included in this third part of the book.

The subject matter of Part 1 is distributed into chapters as follows. Chapter 2 introduces stationary time series models and presents the three most important classical approaches to analyze time series data: ARIMA models, periodicity analysis in the frequency domain, and state-space models. The chapter introduces the basic tools for each of these analysis: the autocorrelation function and the partial autocorrelation function for linear stationary process, the periodogram, and the spectrum for periodicity analysis and the dynamic linear system for the state-space representation. Chapter 3 considers the model specification strategy for univariate ARIMA models. Assuming that we have a linear process, the key problem is how to parametrize it in order to represent its structure with a small number of parameter values that can be estimated from the data. The chapter explains three statistical tools that can be used for this objective: the autocorrelation function, partial autocorrelation function, and the extended autocorrelation function. It is shown how an ARIMA model can be identified by these tools, and examples are given of the use of this methodology. Chapter 4 presents an introduction to the maximum likelihood estimation of ARMA models and discusses the diagnostic checking of the fitted model. It also includes procedures for parameter estimation using the sample spectrum and the estimation of state-space models by the Kalman filter. Chapter 5 presents the prediction problem and concentrates in the computation of ARIMA forecasts with emphasis on understanding the structure of the forecasts generated by these models. It is shown that the forecast function can be easily understood in terms of the main parts of the ARIMA model. This chapter also includes an introduction to the combination of forecast from different sources and to the problem of model selection in time series. Time series, as any kind of statistical data, are often subject to outliers, and Chapter 6 discusses outliers and influential observations in univariate time series. Different types of outliers are introduced, and a methodology is presented to identify them and estimate their effects. It is shown that outlier analysis is related to the estimation of missing values in a time series, and a brief introduction to this important practical problem is presented. Chapter 7 presents a procedure for automatic ARIMA modeling of univariate time series that is implemented in the program TRAMO. This program allows a powerful and fast application of the methodology presented in the previous chapters. Finally, Chapter 8 discusses the use of ARIMA models in the important problem of seasonal adjustment in economic time series. The chapter shows how ARIMA models provide a powerful tool for decomposing the observed time series and carrying out seasonal adjustment and discusses this topic within the broader context of signal extraction.

Part 2, on advanced topics, includes Chapters 9–13. Chapter 9 considers a particular class of nonlinear time series models with many application in finance: heteroscedastic models. The chapter concentrates in the most often used ARCH and GARCH models and presents examples of their applications. Chapter 10 considers more general nonlinear time series models. This chapter presents a general test to detect three kinds of non linearity (bilinear, exponential autorregressive, and threshold autorregressive) often found in time series and discusses in more detail the fitting of threshold models. Chapter 11 analyzes in a common framework linear and nonlinear model by using the Bayesian approach. It is shown how Markov chain Monte Carlo methods (MCMC) provides a powerful tool for the analysis of complex models within the Bayesian framework. Chapter 12 presents an alternative way to analyze time series: the nonparametric approach. In particular, it is shown how this approach can be applied to the decomposition and seasonal adjustment of economic time series. Finally, Chapter 13 includes an introduction of neural network models in time series. This method provides a simple way to generate forecast for a time series with a minimum set of assumptions about the underlying structure.

Part 3, on multiple time series analyses, includes Chapter 14–16. Chapter 14 presents a methodology for building multivariate time series ARIMA models. The three stages of identification, estimation, and diagnostic are presented, and illustrated with examples. The chapter also discusses the important problem of model simplification by different types of eigenvalue analysis. A key idea in multivariate modeling is finding simplifying structure in the vector time series, and, in particular, this includes finding linear combination of the vector time series that are more stable than the observed series. This leads to the idea of cointegration, developed in Chapter 15, in which a general methodology is presented for testing and estimating cointegration relationships. Finally, Chapter 16 presents an introduction to the multivariate analyses of linear system from the linear system approach. This methodology offers an alternative to the vector ARIMA methodology for multivariate analysis of time series.

1.3. FURTHER READING

The reader interested in a deeper analysis of the basic concepts in time series should consult the books by Abraham and Ledolter (1983), Anderson (1971), Box and Jenkins (1976), Box et al. (1994), Brockwell and Davis (1987, 1996), Gourieroux and Monfort (1997), Granger and Newbold (1977), Fuller (1976), Pandit and Wu (1983), Shumway (1988), Shumway and Stoffer (2000), and Wei (1990). The spectral approach is presented in Brillinger (1975), Granger and Hatanaka (1964), Jenkins and Watts (1968), and Priestley (1981). Harvey (1989) discusses with detail the structural approach in time series based on the state-space representation (see Anderson and Moore 1979) for economic time series. Hamilton (1994) and Enders (1995) also emphasize economic time series and econometrics. Hendry and Clements (1998) concentrates on economic forecasting.

Moving to the advanced topic section, ARCH and GARCH models are discussed in recent econometric texts, and the reader can find a deeper study in the books by Engle (1995) and Gourieroux (1997). Nonlinear models are discussed by Granger and Andersen (1980), Priestley (1988), and Tong (1990). Bayesian models are discussed by West and Harrison (1997), nonparametric regression by Hardle (1990), and neural networks by Ripley (1996).

Multivariate ARMA time series models are considered by Hannan (1970), Lutkepohl (1993), Reinsel (1993), and Reinsel and Velu (1998). Aoki (1990) and Hannan and Deistler (1988) are important references for state-space modeling of multivariate time series.

The limitation of space and time has made that many interesting development in time series have not been introduced in this text. Among them are long memory processes (Beran 1994), wavelets (Hardle et al. 1998, Morettin 1999), and discrimination and clustering in time series (Karizawa et al. 1998).

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#EXAMPLE I. A GENERATED MA(1) EXAMPLE

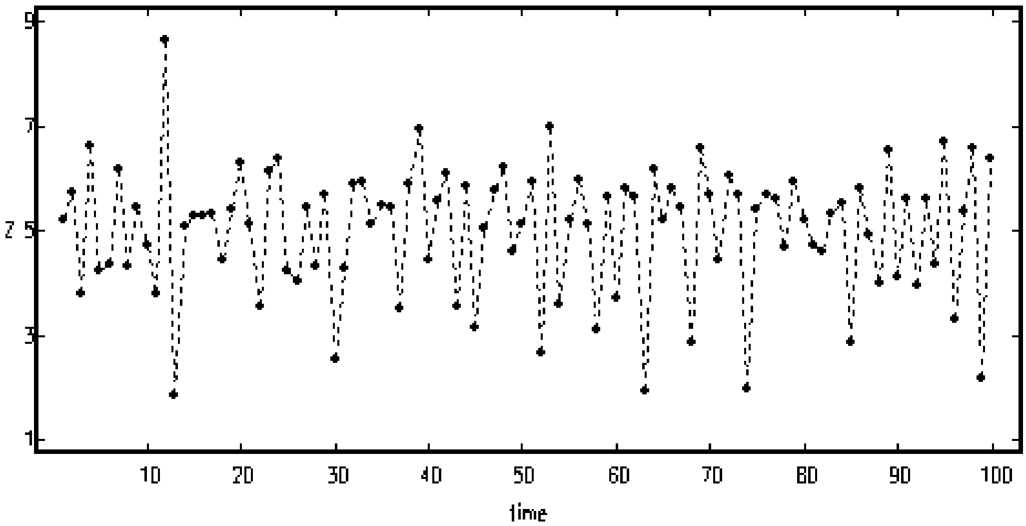
The following data are generated from the MA(1) model

$$Z_t = 5 + a_t - .7a_{t-1}$$

where the a_t's are iid N(0,1.).

```
tsmodel m1.model z=c+(1-th*b)noise.simulate
c=5
      #Here we set the values of  c and th
th=.7
simulate model m1.noise n(0,1.).nobs 100

THE UNIVARIATE TIME SERIES   Z    IS SIMULATED USING MODEL M1

print z

Z       IS  A   100  BY    1  VARIABLE
   5.212    5.734    3.822    6.633    4.258    4.355    6.173
   4.337    5.473    4.736    3.827    8.644    1.854    5.109
   5.318    5.293    5.331    4.462    5.437    6.309    5.149
   3.566    6.139    6.391    4.263    4.031    5.474    4.315
   5.701    2.572    4.280    5.903    5.964    5.126    5.512
   5.475    3.507    5.914    6.951    4.435    5.604    6.113
   3.568    5.885    3.148    5.054    5.783    6.228    4.621
   5.139    5.947    2.685    7.020    3.625    5.206    6.000
   5.149    3.134    5.666    3.711    5.812    5.673    1.948
   6.174    5.230    5.815    5.465    2.898    6.592    5.704
   4.472    6.062    5.690    1.988    5.424    5.713    5.610
   4.681    5.938    5.208    4.737    4.615    5.337    5.550
   2.866    5.839    4.945    4.019    6.561    4.140    5.615
   3.983    5.618    4.356    6.710    3.335    5.379    6.604
   2.209    6.389

desc z

VARIABLE   NAME   IS       Z
NUMBER OF OBSERVATIONS      100
NUMBER OF MISSING VALUES      0

                STATISTIC     STD. ERROR    STATISTIC/S.E.

MEAN              5.0317         .1236        40.7257
VARIANCE           1.5265
STD DEVIATION       1.2355
C.  V.             .2455
SKEWNESS           -.5064          .2414
KURTOSIS            .3302          .4783

                QUARTILE

MINIMUM            1.8542
1ST QUARTILE        4.2797
MEDIAN             5.3245
3RD QUARTILE        5.8149
```

```
MAXIMUM              8.6437
```




```
                    RANGE

MAX - MIN            6.7895
Q3  - Q1            1.5353
--
```
graph z. Type tsplot

#Here we go to the SCA GRAPHICS and obtain the following figure.

#In what follows we give three methods for tentative specification of a model for this data set.

acf z.maxlag 12

```
TIME PERIOD ANALYZED . . . . . . . . . .  1  TO   100
NAME OF THE SERIES . . . . . . . . . .             Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .           100
STANDARD DEVIATION OF THE SERIES . . .        1.2293
MEAN OF THE (DIFFERENCED) SERIES . . .        5.0317
STANDARD DEVIATION OF THE MEAN . . . .         .1229
T-VALUE OF MEAN (AGAINST ZERO) . . . .       40.9309

AUTOCORRELATIONS
  1- 12    -.47 -.05  .16 -.17  .11 -.04 -.08  .08  .06 -.23  .17 -.04
  ST.E.     .10  .12  .12  .12  .12  .13  .13  .13  .13  .13  .13  .13
   Q      22.5 22.7 25.6 28.6 29.9 30.1 30.7 31.5 31.9 37.8 41.0 41.2
```

```
       -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
          +----+----+----+----+----+----+----+----+----+----+
                                  I
  1   -.47            XXXXXXX+XXXXI    +
  2   -.05                 +   XI    +
  3    .16                 +    IXXXX +
  4   -.17                 + XXXXI    +
  5    .11                 +    IXXX  +
  6   -.04                 +   XI    +
  7   -.08                 +  XXI    +
  8    .08                 +    IXX   +
  9    .06                 +    IX   +
 10   -.23                 XXXXXXI    +
 11    .17                 +    IXXXX +
 12   -.04                +    XI     +
```

pacf z.maxlag 12

```
TIME PERIOD ANALYZED . . . . . . . . . .  1  TO   100
NAME OF THE SERIES . . . . . . . . . .             Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .           100
STANDARD DEVIATION OF THE SERIES . . .        1.2293
MEAN OF THE (DIFFERENCED) SERIES . . .        5.0317
STANDARD DEVIATION OF THE MEAN . . . .         .1229
T-VALUE OF MEAN (AGAINST ZERO) . . . .       40.9309
PARTIAL AUTOCORRELATIONS
1- 12    -.47 -.35 -.04 -.15 -.00 -.03 -.10 -.05  .10 -.19 -.06 -.07
ST.E.     .10  .10  .10  .10  .10  .10  .10  .10  .10  .10  .10  .10
```

```
        -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
          +----+----+----+----+----+----+----+----+----+----+
                                   I
  1   -.47             XXXXXXX+XXXXI    +
  2   -.35               XXXX+XXXXI    +
  3   -.04                   +   XI    +
  4   -.15                   +XXXXI    +
  5    .00                   +    I    +
  6   -.03                   +   XI    +
  7   -.10                   + XXXI    +
  8   -.05                   +   XI    +
  9    .10                   +    IXX  +
 10   -.19                   XXXXXI    +
 11   -.06                   +  XXI    +
 12   -.07                   +  XXI    +
```

eacf z

```
 TIME PERIOD ANALYZED . . . . . . . . . .  1  TO   100
 NAME OF THE SERIES . . . . . . . . . . .            Z
 EFFECTIVE NUMBER OF OBSERVATIONS . . .           100
 STANDARD DEVIATION OF THE SERIES . . .        1.2293
 MEAN OF THE (DIFFERENCED) SERIES . . .        5.0317
 STANDARD DEVIATION OF THE MEAN . . . .         .1229
 T-VALUE OF MEAN (AGAINST ZERO) . . . .       40.9309


 THE EXTENDED ACF TABLE

 (Q-->)   0    1    2    3    4    5    6    7    8    9   10   11   12

------------------------------------------------------------------------

 (P= 0) -.47 -.05  .16 -.17  .11 -.04 -.08  .08  .06 -.23  .17 -.04 -.10
 (P= 1) -.52 -.20  .08 -.06  .08 -.03 -.10  .07  .03 -.22  .13 -.04 -.12
 (P= 2) -.10 -.34 -.18  .01 -.01 -.09 -.01 -.02 -.02 -.21 -.10 -.10  .02
 (P= 3) -.20 -.38 -.21 -.03  .00 -.07 -.05 -.02 -.03 -.18  .08 -.12  .04
 (P= 4) -.00 -.18 -.14  .07 -.08 -.06 -.01  .02 -.01 -.15 -.09 -.15  .04
 (P= 5) -.00 -.01 -.40 -.16  .09 -.05 -.01  .02 -.02 -.14  .06 -.01 -.06
 (P= 6) -.24  .08 -.15 -.27  .04  .04  .15 -.00 -.05 -.13  .06 -.02 -.04

  SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

 (Q-->)  0  1  2  3  4  5  6  7  8  9 10 11 12
 ----------------------------------------------
 (P= 0)  X  O  O  O  O  O  O  O  O  O  O  O  O
 (P= 1)  X  O  O  O  O  O  O  O  O  O  O  O  O
 (P= 2)  O  X  O  O  O  O  O  O  O  O  O  O  O
 (P= 3)  O  X  O  O  O  O  O  O  O  O  O  O  O
 (P= 4)  O  O  O  O  O  O  O  O  O  O  O  O  O
 (P= 5)  O  O  X  O  O  O  O  O  O  O  O  O  O
 (P= 6)  X  O  O  O  O  O  O  O  O  O  O  O  O
```

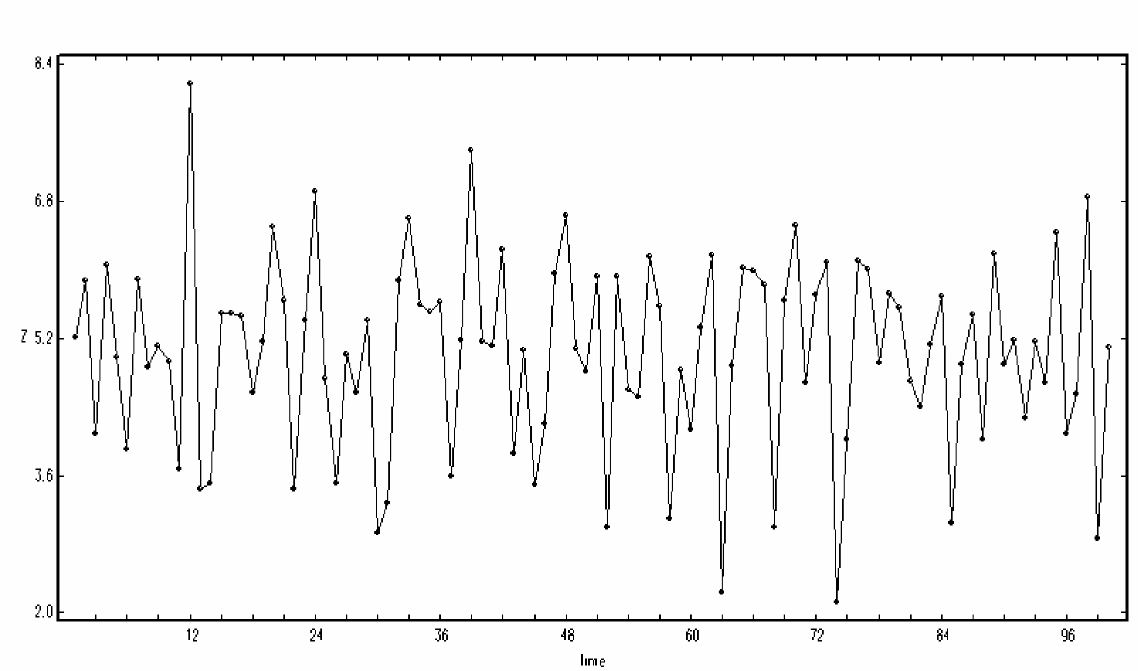
save z.file'c:\b325\gma1.dat'.format 'f12.4'

```
** THE SPECIFIED FILE DOES NOT EXIST. A NEW FILE IS OPENED.
   THE FILE NAME IS C:\B325\GMA1.DAT
THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE  7
```

```
stop
```

A GENERATED *MA(2)* EXAMPLE

```
tsm m1.model z=c+(1-t1*b-t2*b**2)noise.simulate

c=5

t1=.3

t2=.4

simu model m1.nobs 100. noise n(0,1)

THE UNIVARIATE TIME SERIES    Z     IS SIMULATED USING MODEL M1

graph z. type tsplot
```




```
acf z. maxl 12

NAME OF THE SERIES . . . . . . . . . .            Z
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
MEAN OF THE (DIFFERENCED) SERIES . . .       5.0254
STANDARD DEVIATION OF THE SERIES . . .       1.1246
T-VALUE OF MEAN (AGAINST ZERO) . . . .      44.6848

AUTOCORRELATIONS
```

```
  1- 12    -.18 -.29  .11 -.09  .06 -.04 -.09  .13  .01 -.22  .11 -.01
  ST.E.     .10  .10  .11  .11  .11  .11  .11  .11  .12  .12  .12  .12
   Q        3.3 11.9 13.2 14.1 14.5 14.7 15.6 17.5 17.5 23.0 24.4 24.5


          -1.0 -0.8 -0.6 -0.4 -0.2  0.0  0.2  0.4  0.6  0.8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1  -0.18                      +XXXXI    +
  2  -0.29                    XX+XXXXI    +
  3   0.11                      +    IXXX +
  4  -0.09                      +  XXI    +
  5   0.06                     +     IXX   +
  6  -0.04                     +    XI     +
  7  -0.09                     +   XXI     +
  8   0.13                     +     IXXX  +
  9   0.01                     +     I     +
 10  -0.22                     XXXXXXI     +
 11   0.11                     +     IXXX  +
 12  -0.01                     +     I     +
```

eacf z

```
NAME OF THE SERIES . . . . . . . . . .            Z
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
MEAN OF THE (DIFFERENCED) SERIES . . .       5.0254
STANDARD DEVIATION OF THE SERIES . . .       1.1246
T-VALUE OF MEAN (AGAINST ZERO) . . . .      44.6848


THE EXTENDED ACF TABLE

(Q-->)   0    1    2    3    4    5    6    7    8    9   10   11   12
-------------------------------------------------------------------------

(P= 0) -.18 -.29  .11 -.09  .06 -.04 -.09  .13  .01 -.22  .11 -.01 -.12

(P= 1) -.42 -.35 -.09 -.02  .00 -.03 -.06  .13  .01 -.20  .08 -.01 -.09

(P= 2) -.02 -.44  .05 -.02  .01 -.05 -.04 -.01 -.03 -.21 -.08 -.11  .00

(P= 3) -.03 -.45 -.20  .01  .02 -.06 -.02  .01 -.01 -.15 -.04 -.11  .01

(P= 4)  .20 -.42 -.04  .02  .11 -.06 -.02  .01 -.02 -.15  .02 -.05  .01

(P= 5)  .29 -.30 -.39 -.25  .03  .06  .00  .02 -.01 -.14  .06 -.03 -.04

(P= 6) -.47  .36 -.26 -.24  .09  .01  .05  .06 -.04 -.12  .04 -.02 -.05


SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)  0  1  2  3  4  5  6  7  8  9 10 11 12
--------------------------------------------------
(P= 0)  O  X  O  O  O  O  O  O  O  O  O  O  O
(P= 1)  X  X  O  O  O  O  O  O  O  O  O  O  O
(P= 2)  O  X  O  O  O  O  O  O  O  O  O  O  O
(P= 3)  O  X  O  O  O  O  O  O  O  O  O  O  O
(P= 4)  O  X  O  O  O  O  O  O  O  O  O  O  O
(P= 5)  X  X  X  O  O  O  O  O  O  O  O  O  O
(P= 6)  X  X  X  O  O  O  O  O  O  O  O  O  O
```

```
--

print z

Z         IS  A   100  BY     1  VARIABLE
     5.212       5.883       4.091       6.056       4.979       3.910       5.900
     4.873       5.113       4.939       3.674       8.173       3.452       3.511
     5.501       5.506       5.470       4.574       5.168       6.510       5.639
     3.445       5.421       6.918       4.733       3.508       5.015       4.571
     5.417       2.928       3.277       5.884       6.600       5.596       5.506
     5.625       3.597       5.190       7.407       5.168       5.112       6.236
     3.858       5.070       3.504       4.208       5.953       6.639       5.086
     4.822       5.933       2.998       5.934       4.599       4.530       6.159
     5.578       3.094       4.832       4.140       5.331       6.176       2.244
     4.892       6.023       5.992       5.823       3.009       5.643       6.517
     4.686       5.719       6.086       2.116       4.033       6.113       6.006
     4.917       5.732       5.566       4.696       4.398       5.136       5.698
     3.055       4.898       5.475       4.032       6.200       4.904       5.181
     4.270       5.166       4.694       6.441       4.089       4.556       6.846
     2.869       5.093
--
```

#EXAMPLE III. A GENERATED AR(1) EXAMPLE

Here we generate 100 observations from the following AR(1) model,

$$Z_t - 5 = .7(Z_{t-1} - 5) + a_t$$

or

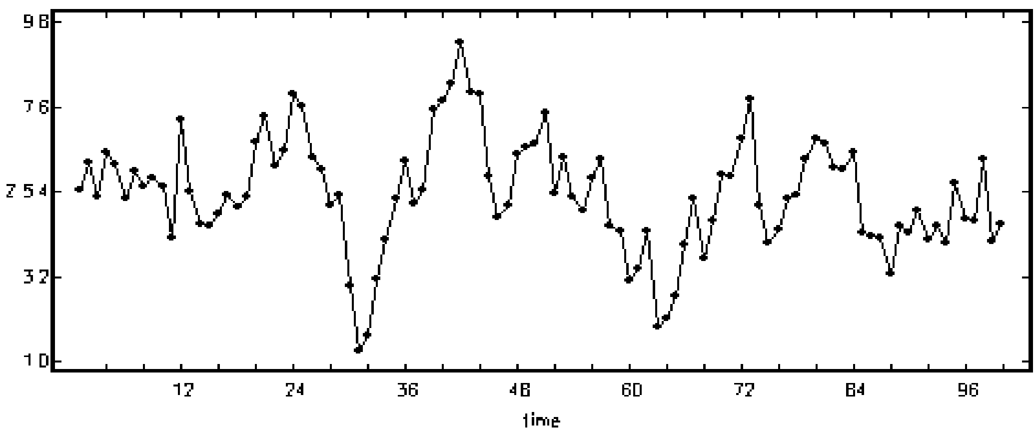
$$Z_t = 1.5 + .7Z_{t-1} + a_t$$

where a_t's are iid N(0,1).

```
tsmodel m1.model (1-ph*b)z=c+noise.simulate
c=1.5
ph=.7
simulate model m1.noise n(0,1.).nobs 100
THE UNIVARIATE TIME SERIES    Z     IS SIMULATED USING MODEL M1
print z

Z        IS  A   100  BY     1  VARIABLE
    5.449     6.198     5.279     6.436     6.132     5.237     5.950
    5.550     5.778     5.555     4.223     7.284     5.433     4.595
    4.540     4.847     5.343     5.017     5.292     6.710     7.400
    6.088     6.485     7.938     7.648     6.299     5.994     5.071
    5.313     2.976     1.292     1.704     3.166     4.174     5.255
    6.236     5.112     5.466     7.548     7.773     8.238     9.288
    7.985     7.962     5.833     4.768     5.050     6.412     6.573
    6.649     7.485     5.356     6.301     5.272     4.949     5.796
    6.288     4.547     4.401     3.094     3.438     4.419     1.901
    2.120     2.716     4.029     5.225     3.689     4.646     5.850
    5.836     6.815     7.822     5.049     4.109     4.442     5.265
    5.326     6.264     6.818     6.663     6.053     5.996     6.428
    4.377     4.267     4.224     3.292     4.549     4.346     4.920
    4.191     4.525     4.087     5.665     4.713     4.652     6.257
    4.140     4.569

desc z

VARIABLE    NAME    IS         Z
NUMBER OF OBSERVATIONS       100
NUMBER OF MISSING VALUES       0


                 STATISTIC      STD. ERROR     STATISTIC/S.E.

MEAN                 5.3572           .1470          36.4516
VARIANCE             2.1600
STD DEVIATION        1.4697
C.  V.                .2743
SKEWNESS             -.1521           .2414
KURTOSIS              .3249           .4783

                   QUARTILE

MINIMUM              1.2922
1ST QUARTILE         4.4418
MEDIAN               5.3027
3RD QUARTILE         6.2644
```

```
 MAXIMUM               9.2883

                        RANGE

 MAX - MIN             7.9961
 Q3  - Q1              1.8226
 --
```
#Again we go to SCA GRAPHICS to get the Z series plotted.

graph z. type tsplot



#Compared with the generated MA(1)series,we see that this series exhibits more momentum over time.
#The following are the three method to help specify a model for this series: acf,pacf and eacf.

```
acf z        #Here without specifying the maximun number of lags we get
              36 autocorrelations.
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
NAME OF THE SERIES . . . . . . . . . .            Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .          100
STANDARD DEVIATION OF THE SERIES . . .       1.4623
MEAN OF THE (DIFFERENCED) SERIES . . .       5.3572
STANDARD DEVIATION OF THE MEAN . . . .        .1462
T-VALUE OF MEAN (AGAINST ZERO) . . . .      36.6352

AUTOCORRELATIONS
  1- 12      .72  .51  .37  .17  .06 -.06 -.16 -.19 -.26 -.31 -.24 -.21
  ST.E.      .10  .14  .16  .17  .17  .17  .17  .17  .17  .18  .18  .19
   Q        53.1 80.5 94.6 97.8 98.2 98.6  101  105  113  124  131  136

 13- 24     -.16 -.06 -.02 -.02 -.03 -.10 -.15 -.16 -.15 -.07 -.06 -.06
  ST.E.      .19  .19  .19  .19  .19  .19  .19  .19  .19  .19  .19  .19
   Q         139  139  139  139  139  140  143  146  149  150  151  151

 25- 36     -.05 -.03  .07  .18  .24  .29  .32  .28  .22  .12  .05  .05
  ST.E.      .20  .20  .20  .20  .20  .20  .20  .21  .21  .22  .22  .22
   Q         151  152  152  157  165  178  193  205  212  214  214  215

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .72                      +    IXXXX+XXXXXXXXXXXXX
  2    .51                    +      IXXXXXX+XXXXXX
  3    .37                   +       IXXXXXXX+X
  4    .17                   +       IXXXX   +
  5    .06                   +       IXX     +
  6   -.06                   +     XXI       +
  7   -.16                   +   XXXXI       +
  8   -.19                   +  XXXXXI       +
  9   -.26                  +  XXXXXXI        +
 10   -.31                  +XXXXXXXXI        +
 11   -.24                  +  XXXXXXI        +
 12   -.21                  +   XXXXXI        +
 13   -.16                  +    XXXXI        +
 14   -.06                  +       XI        +
 15   -.02                  +        I        +
 16   -.02                  +       XI        +
 17   -.03                  +       XI        +
 18   -.10                  +      XXI        +
 19   -.15                  +    XXXXI        +
 20   -.16                  +    XXXXI        +
 21   -.15                  +    XXXXI        +
 22   -.07                 +       XXI         +
 23   -.06                 +       XXI         +
 24   -.06                 +       XXI         +
 25   -.05                 +        XI         +
 26   -.03                 +        XI         +
 27    .07                 +         IXX       +
 28    .18                 +         IXXXX     +
 29    .24                 +         IXXXXXX   +
 30    .29                 +         IXXXXXXX  +
 31    .32                 +         IXXXXXXXX +
 32    .28                 +         IXXXXXXX  +
```

```
 33    .22                   +         IXXXXX    +
 34    .12                  +          IXXXX      +
 35    .05                  +          IX         +
 36    .05                  +          IX         +
```

pacf z.maxlag 12

```
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
NAME OF THE SERIES . . . . . . . . . .            Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .          100
STANDARD DEVIATION OF THE SERIES . . .       1.4623
MEAN OF THE (DIFFERENCED) SERIES . . .       5.3572
STANDARD DEVIATION OF THE MEAN . . . .        .1462
T-VALUE OF MEAN (AGAINST ZERO) . . . .      36.6352
```

```
PARTIAL AUTOCORRELATIONS

  1- 12     .72 -.00 -.00 -.18  .01 -.14 -.04 -.03 -.11 -.12  .14 -.06
  ST.E.     .10  .10  .10  .10  .10  .10  .10  .10  .10  .10  .10  .10
```

```
          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .72                      +    IXXXX+XXXXXXXXXXXXX
  2    .00                      +    I    +
  3    .00                      +    I    +
  4   -.18                      XXXXXI    +
  5    .01                      +    I    +
  6   -.14                      + XXXI    +
  7   -.04                      +   XI    +
  8   -.03                      +   XI    +
  9   -.11                      + XXXI    +
 10   -.12                      + XXXI    +
 11    .14                      +    IXXXX+
 12   -.06                      +   XI    +
```

eacf z

```
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
NAME OF THE SERIES . . . . . . . . . .            Z
EFFECTIVE NUMBER OF OBSERVATIONS . . .          100
STANDARD DEVIATION OF THE SERIES . . .       1.4623
MEAN OF THE (DIFFERENCED) SERIES . . .       5.3572
STANDARD DEVIATION OF THE MEAN . . . .        .1462
T-VALUE OF MEAN (AGAINST ZERO) . . . .      36.6352
```

```
THE EXTENDED ACF TABLE

(Q-->)   0    1    2    3    4    5    6    7    8    9   10   11   12
-----------------------------------------------------------------------

(P= 0)  .72  .51  .37  .17  .06 -.06 -.16 -.19 -.26 -.31 -.24 -.21 -.16
(P= 1)  .00  .01  .22  .06  .07 -.07 -.10  .01 -.03 -.24  .06 -.06 -.13
(P= 2) -.03 -.00  .23 -.04  .15 -.02 -.12 -.05 -.00 -.25 -.10 -.03 -.13
(P= 3) -.06 -.03  .12 -.06  .14 -.01 -.12  .01 -.02 -.25  .05 -.13 -.10
(P= 4)  .08 -.48 -.20  .06  .03 -.08  .02  .00  .04 -.15 -.02 -.16 -.14
(P= 5)  .10 -.45 -.25  .19  .03 -.02  .01  .05  .06 -.16 -.00 -.11  .04
(P= 6) -.24 -.05 -.37 -.13 -.05  .00 -.01  .07  .08 -.13  .02 -.04 -.08
```

```
SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)  0  1  2  3  4  5  6  7  8  9 10 11 12
----------------------------------------------
(P= 0)  X  X  X  O  O  O  O  O  O  O  O  O  O
(P= 1)  O  O  O  O  O  O  O  O  O  X  O  O  O
(P= 2)  O  O  X  O  O  O  O  O  O  X  O  O  O
(P= 3)  O  O  O  O  O  O  O  O  O  X  O  O  O
(P= 4)  O  X  O  O  O  O  O  O  O  O  O  O  O
(P= 5)  O  X  O  O  O  O  O  O  O  O  O  O  O
(P= 6)  X  O  X  O  O  O  O  O  O  O  O  O  O
```

```
save z.file'c:\b325\gar1.dat'.format 'f12.4'
```

A GENERATED AR(2) EXAMPLE

```
tsm m1.model (1-p1*b-p2*b**2)y=c+noise.simulate
c=5
p1=.75
p2=-.5
simu model m1.noise n(0.,1.0).nobs 150
```

```
THE UNIVARIATE TIME SERIES    Y     IS SIMULATED USING MODEL M1
```

```
select y.new zz.span(21,150)
```

```
VARIABLE    Y     IS EDITED, THE RESULT IS STORED IN VARIABLE    ZZ
VARIABLE    ZZ    IS  A  130  BY   1  MATRIX
```

```
graph zz.type tsplot
acf zz.maxl 12
```

```
NAME OF THE SERIES . . . . . . . . . .           ZZ
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   130
MEAN OF THE (DIFFERENCED) SERIES . . .       6.8785
STANDARD DEVIATION OF THE SERIES . . .       1.3078
T-VALUE OF MEAN (AGAINST ZERO) . . . .      59.9664


AUTOCORRELATIONS

  1- 12     .48 -.08 -.22 -.16 -.04  .08  .04 -.07 -.09 -.07 -.06 -.09
  ST.E.     .09  .11  .11  .11  .11  .11  .11  .11  .11  .11  .11  .11
   Q       30.8 31.6 38.4 42.0 42.2 43.1 43.3 43.9 45.0 45.7 46.2 47.5

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .48                       +   IXXX+XXXXXXXX
  2   -.08                      +  XXI    +
  3   -.22                     X+XXXXI    +
  4   -.16                      +XXXXI    +
  5   -.04                      +   XI    +
  6    .08                      +    IXX  +
  7    .04                     +     IX    +
  8   -.07                     +   XXI     +
  9   -.09                     +   XXI     +
 10   -.07                     +   XXI     +
 11   -.06                     +    XI     +
 12   -.09                     +   XXI     +
```

```
pacf zz.maxl 12
```

```
NAME OF THE SERIES . . . . . . . . . .           ZZ
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   130
MEAN OF THE (DIFFERENCED) SERIES . . .       6.8785
STANDARD DEVIATION OF THE SERIES . . .       1.3078
T-VALUE OF MEAN (AGAINST ZERO) . . . .      59.9664
```

PARTIAL AUTOCORRELATIONS

```
  1- 12      .48 -.40  .03 -.09  .02  .06 -.11 -.03 -.02 -.05 -.04 -.13
  ST.E.      .09  .09  .09  .09  .09  .09  .09  .09  .09  .09  .09  .09

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .48                       +   IXXX+XXXXXXXX
  2   -.40                  XXXXXX+XXXI   +
  3    .03                       +   IX  +
  4   -.09                       + XXI   +
  5    .02                       +   I   +
  6    .06                       +   IXX +
  7   -.11                       +XXXI   +
  8   -.03                       +  XI   +
  9   -.02                       +   I   +
 10   -.05                       +  XI   +
 11   -.04                       +  XI   +
 12   -.13                       +XXXI   +
```

eacf zz

```
NAME OF THE SERIES . . . . . . . . . .           ZZ
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   130
MEAN OF THE (DIFFERENCED) SERIES . . .       6.8785
STANDARD DEVIATION OF THE SERIES . . .       1.3078
T-VALUE OF MEAN (AGAINST ZERO) . . . .      59.9664
```

THE EXTENDED ACF TABLE

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	.48	-.08	-.22	-.16	-.04	.08	.04	-.07	-.09	-.07	-.06	-.09	-.07
(P= 1)	.55	-.23	-.22	-.15	-.06	.14	.03	-.08	-.02	.02	.01	-.08	-.08
(P= 2)	.09	-.18	.06	.08	-.14	-.07	-.07	-.12	-.06	.02	.01	-.07	-.03
(P= 3)	.42	-.17	.06	.05	-.15	.08	.01	-.08	.04	-.07	-.00	-.07	-.07
(P= 4)	.27	.42	-.03	-.03	-.23	.04	.05	-.05	-.01	-.09	.02	-.06	-.08
(P= 5)	-.33	.24	-.42	.01	-.14	-.00	.06	-.07	-.01	-.09	.04	-.06	-.08
(P= 6)	.37	.34	-.18	-.09	-.15	-.01	.01	-.07	-.00	-.09	.02	-.10	-.06

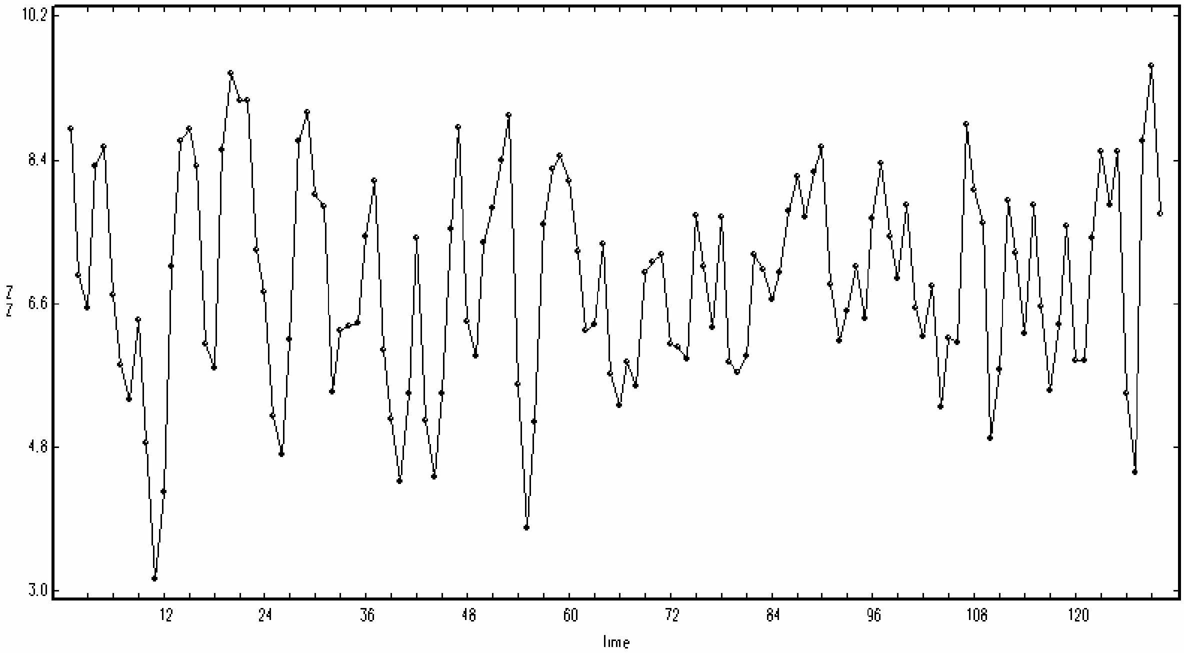
SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)	0	1	2	3	4	5	6	7	8	9	10	11	12
(P= 0)	X	O	X	O	O	O	O	O	O	O	O	O	O
(P= 1)	X	X	X	O	O	O	O	O	O	O	O	O	O
(P= 2)	O	O	O	O	O	O	O	O	O	O	O	O	O
(P= 3)	X	O	O	O	O	O	O	O	O	O	O	O	O
(P= 4)	X	X	O	O	X	O	O	O	O	O	O	O	O
(P= 5)	X	O	X	O	O	O	O	O	O	O	O	O	O
(P= 6)	X	X	O	O	O	O	O	O	O	O	O	O	O

print zz

```
ZZ       IS  A   130  BY     1  VARIABLE
    8.794     6.958     6.545     8.328     8.565     6.705     5.831
    5.395     6.394     4.855     3.153     4.237     7.074     8.645
    8.780     8.320     6.097     5.800     8.523     9.482     9.147
    9.141     7.265     6.751     5.190     4.702     6.144     8.634
    8.988     7.972     7.816     5.493     6.263     6.312     6.362
    7.446     8.134     6.023     5.168     4.378     5.471     7.427
    5.142     4.432     5.486     7.526     8.806     6.373     5.940
    7.367     7.796     8.393     8.949     5.588     3.792     5.115
    7.596     8.280     8.448     8.129     7.263     6.271     6.331
    7.343     5.720     5.322     5.868     5.575     6.992     7.118
    7.220     6.103     6.059     5.912     7.708     7.073     6.303
    7.692     5.877     5.734     5.955     7.224     7.021     6.658
    6.989     7.767     8.187     7.689     8.248     8.564     6.843
    6.127     6.509     7.062     6.410     7.658     8.365     7.442
    6.910     7.837     6.544     6.181     6.817     5.310     6.161
    6.106     8.839     8.015     7.619     4.919     5.769     7.886
    7.234     6.218     7.826     6.572     5.520     6.336     7.576
    5.897     5.887     7.418     8.499     7.831     8.504     5.477
    4.483     8.644     9.566     7.728
```

save zz.file'\b325\gar2.dat'.format 'f12.4'



3

A GENERATED RANDOM WALK

tsmodel m1.model z(1)=noise.simulate
simu model m1.noise n(0.,1.).nobs 500
uiden z.maxl 15
diff z.new dz
print z,dz. span 1,10
uiden dz.maxl 15
graph z,dz
stop

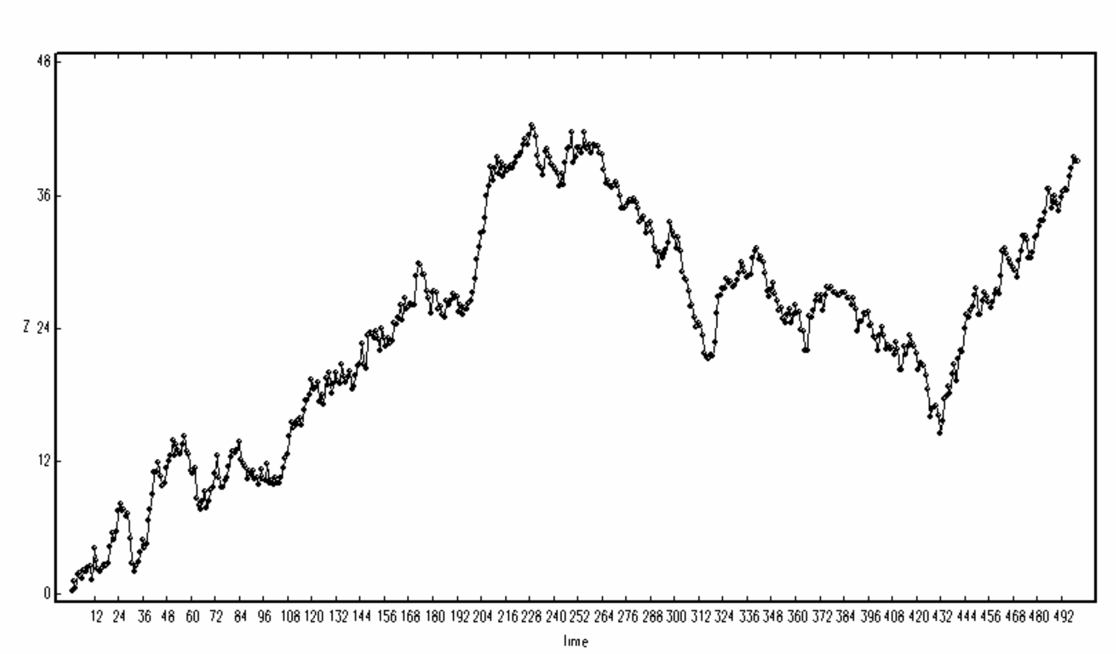
tsmodel m1.model z(1)=noise.simulate
simu model m1.noise n(0.,1.).nobs 500

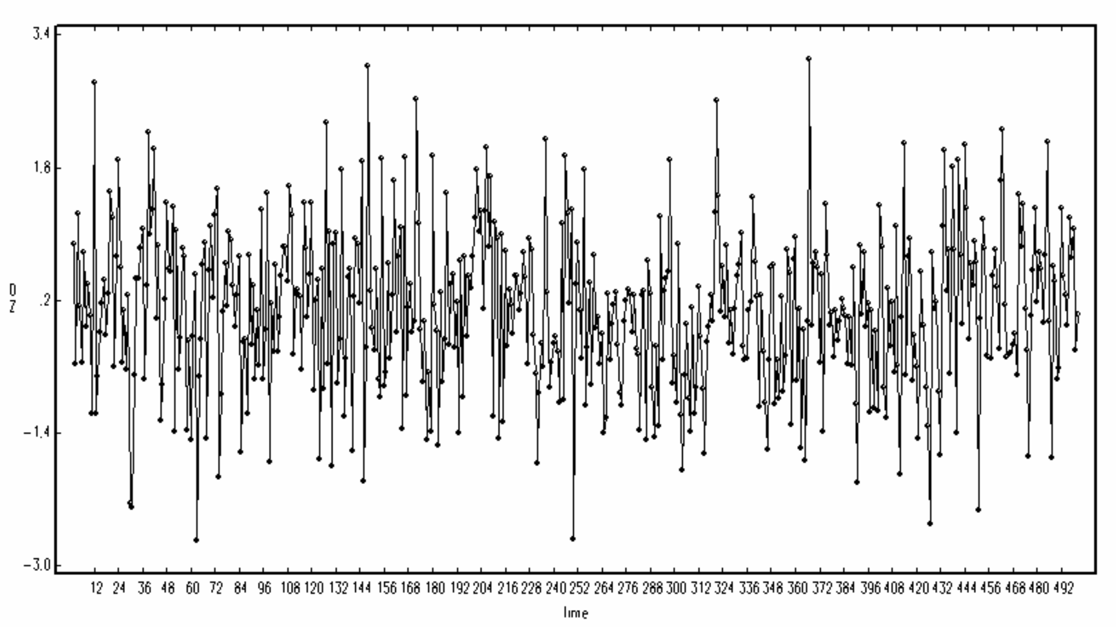
```
THE UNIVARIATE TIME SERIES    Z     IS SIMULATED USING MODEL M1

uiden z.maxl 15

NAME OF THE SERIES . . . . . . . . . .            Z
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   500
MEAN OF THE (DIFFERENCED) SERIES . . .      23.9836
STANDARD DEVIATION OF THE SERIES . . .      10.3808
T-VALUE OF MEAN (AGAINST ZERO) . . . .      51.6618

AUTOCORRELATIONS
  1- 12     .99  .98  .96  .95  .94  .93  .92  .91  .89  .88  .87  .86
  ST.E.     .04  .08  .10  .12  .13  .14  .16  .17  .18  .18  .19  .20
   Q        491  972 1442 1901 2350 2789 3217 3636 4044 4443 4832 5211

 13- 15     .85  .84  .82
  ST.E.     .21  .21  .22
   Q       5581 5942 6293

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .99                         + IX+XXXXXXXXXXXXXXXXXXXXXXX
  2    .98                       +   IXXX+XXXXXXXXXXXXXXXXXXXX
  3    .96                      +    IXXXX+XXXXXXXXXXXXXXXXXXX
  4    .95                     +     IXXXXX+XXXXXXXXXXXXXXXXXX
  5    .94                     +     IXXXXX+XXXXXXXXXXXXXXXXXX
  6    .93                    +      IXXXXXX+XXXXXXXXXXXXXXXX
  7    .92                   +       IXXXXXXX+XXXXXXXXXXXXXXX
  8    .91                   +       IXXXXXXX+XXXXXXXXXXXXXXX
  9    .89                  +        IXXXXXXXX+XXXXXXXXXXXXX
 10    .88                  +        IXXXXXXXX+XXXXXXXXXXXXX
 11    .87                  +        IXXXXXXXX+XXXXXXXXXXXXX
 12    .86                 +         IXXXXXXXXX+XXXXXXXXXXX
 13    .85                 +         IXXXXXXXXX+XXXXXXXXXXX
 14    .84                 +         IXXXXXXXXX+XXXXXXXXXXX
 15    .82                +          IXXXXXXXXXX+XXXXXXXXXX
```

```
PARTIAL AUTOCORRELATIONS

  1- 12      .99  .00 -.03 -.00 -.00 -.00  .00 -.01 -.02  .00 -.03  .02
  ST.E.      .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04

 13- 15     -.00  .01 -.04
  ST.E.      .04  .04  .04


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .99                         + IX+XXXXXXXXXXXXXXXXXXXXXXX
  2    .00                         + I +
  3   -.03                         +XI +
  4    .00                         + I +
  5    .00                         + I +
  6    .00                         + I +
  7    .00                         + I +
  8   -.01                         + I +
  9   -.02                         + I +
 10    .00                         + I +
 11   -.03                         +XI +
 12    .02                         + I +
 13    .00                         + I +
 14    .01                         + I +
 15   -.04                         +XI +

diff z.new dz
                            1
DIFFERENCE ORDERS ARE  (1-B  )
SERIES    Z     IS DIFFERENCED, THE RESULT IS STORED IN VARIABLE    DZ
SERIES    DZ    HAS  500  ENTRIES

print z,dz. span 1,10

Z        IS  A   500  BY     1  VARIABLE
DZ       IS  A   500  BY     1  VARIABLE


VARIABLE        Z        DZ
COLUMN-->       1         1
  ROW
    1         .290       ***
    2        1.173      .883
    3         .613     -.560
    4        1.855     1.241
    5        1.982      .127
    6        1.426     -.556
    7        2.210      .784
    8        2.095     -.115
    9        2.488      .392
   10        2.499      .011
--
```

```
uiden dz.maxl 15

NAME OF THE SERIES . . . . . . . . . .           DZ
TIME PERIOD ANALYZED . . . . . . . . .  2  TO   500
MEAN OF THE (DIFFERENCED) SERIES . . .        .0779
STANDARD DEVIATION OF THE SERIES . . .        .9752
T-VALUE OF MEAN (AGAINST ZERO) . . . .       1.7835


AUTOCORRELATIONS

  1- 12      .01  .01  .08  .01 -.00  .01  .00  .06  .01 -.02  .05 -.05
  ST.E.      .04  .04  .04  .05  .05  .05  .05  .05  .05  .05  .05  .05
   Q          .1   .1  3.4  3.5  3.5  3.5  3.5  5.1  5.1  5.3  6.3  7.6

 13- 15     -.06  .02 -.02
  ST.E.      .05  .05  .05
   Q         9.7  9.9 10.2


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .01                         + I +
  2    .01                         + I +
  3    .08                         + IXX
  4    .01                         + I +
  5    .00                         + I +
  6    .01                         + I +
  7    .00                         + I +
  8    .06                         + IX+
  9    .01                         + I +
 10   -.02                         + I +
 11    .05                         + IX+
 12   -.05                         +XI +
 13   -.06                         XXI +
 14    .02                         + IX+
 15   -.02                         +XI +


PARTIAL AUTOCORRELATIONS

  1- 12      .01  .01  .08  .01 -.01 -.00 -.00  .06  .01 -.02  .04 -.05
  ST.E.      .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04

 13- 15     -.06  .02 -.02
  ST.E.      .04  .04  .04
```

```
         -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .01                         + I +
  2    .01                         + I +
  3    .08                         + IXX
  4    .01                         + I +
  5   -.01                         + I +
  6    .00                         + I +
  7    .00                         + I +
  8    .06                         + IX+
  9    .01                         + I +
 10   -.02                         + I +
 11    .04                         + IX+
 12   -.05                         +XI +
 13   -.06                         XXI +
 14    .02                         + I +
 15   -.02                         + I +
--


graph z,dz
THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE  7
--


stop


THE CURRENT SCA SESSION IS TERMINATED.  WORKSPACE USED IS    14417
WORDS
```


48
36
Z 24
12
0
12 24 36 48 60 72 84 96 108 120 132 144 156 168 180 192 204 216 228 240 252 264 276 288 300 312 324 336 348 360 372 384 396 408 420 432 444 456 468 480 492
time


3.4
1.8
D Z .2
-1.4
-3.0
12 24 36 48 60 72 84 96 108 120 132 144 156 168 180 192 204 216 228 240 252 264 276 288 300 312 324 336 348 360 372 384 396 408 420 432 444 456 468 480 492
time

5

GENERATED (0,2,0) DATA

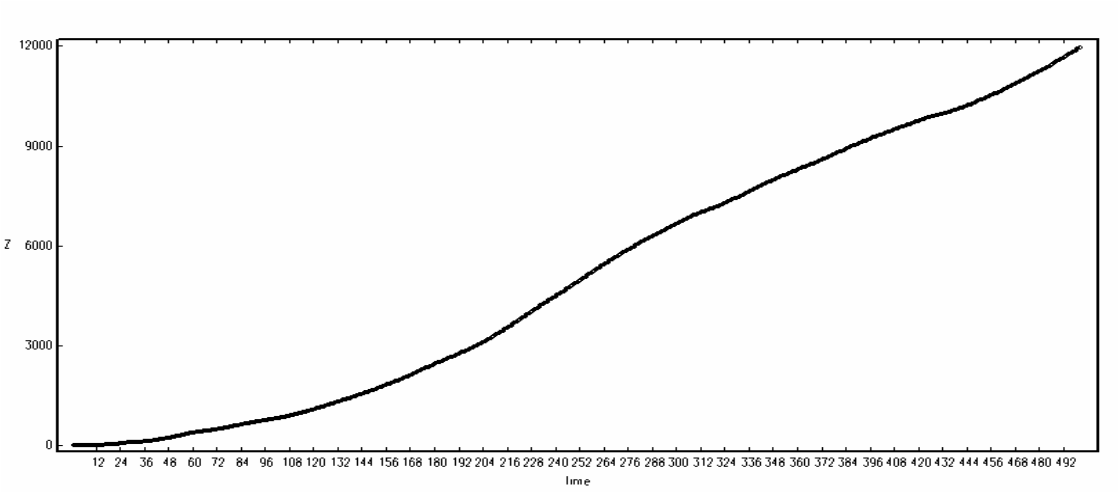
tsm m1.model z(1,1)=noise.simu
simu model m1.noise n(0.,1.).nobs 500
diff z.new dz
diff dz.new d2z
uiden z.maxl 12
uiden dz.maxl 12
uiden d2z.maxl 12
graph z,dz,d2z
stop

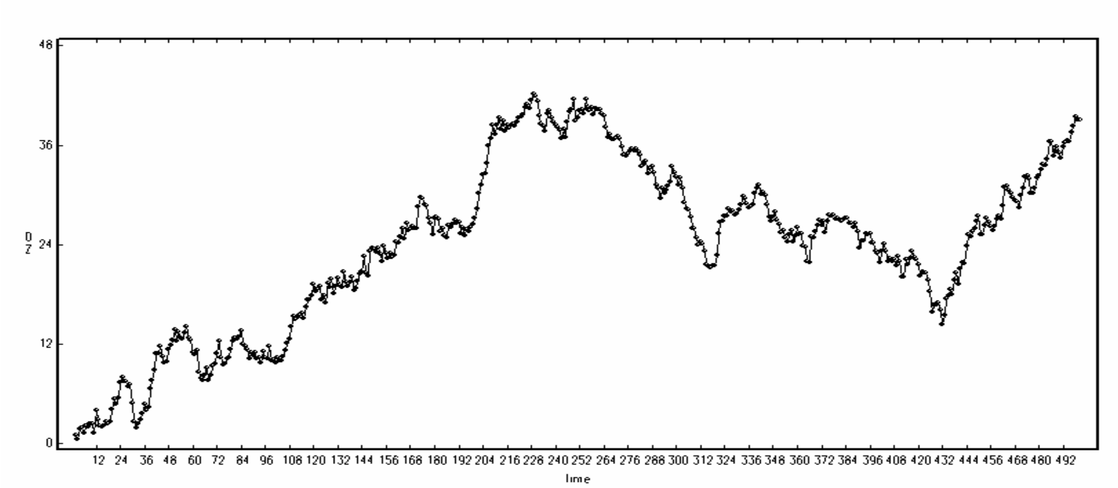
```
tsm m1.model z(1,1)=noise.simu
simu model m1.noise n(0.,1.).nobs 500

THE UNIVARIATE TIME SERIES    Z     IS SIMULATED USING MODEL M1

diff z.new dz

DIFFERENCE ORDERS ARE  (1-B  )
SERIES    Z     IS DIFFERENCED, THE RESULT IS STORED IN VARIABLE    DZ
SERIES    DZ    HAS  500  ENTRIES

diff dz.new d2z

                            1
DIFFERENCE ORDERS ARE  (1-B  )
SERIES    DZ    IS DIFFERENCED, THE RESULT IS STORED IN VARIABLE   D2Z
SERIES   D2Z    HAS  500  ENTRIES

uiden z.maxl 12

NAME OF THE SERIES . . . . . . . . . .             Z
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   500
MEAN OF THE (DIFFERENCED) SERIES . . .    5089.3320
STANDARD DEVIATION OF THE SERIES . . .    3854.9390
T-VALUE OF MEAN (AGAINST ZERO) . . . .      29.5208
```

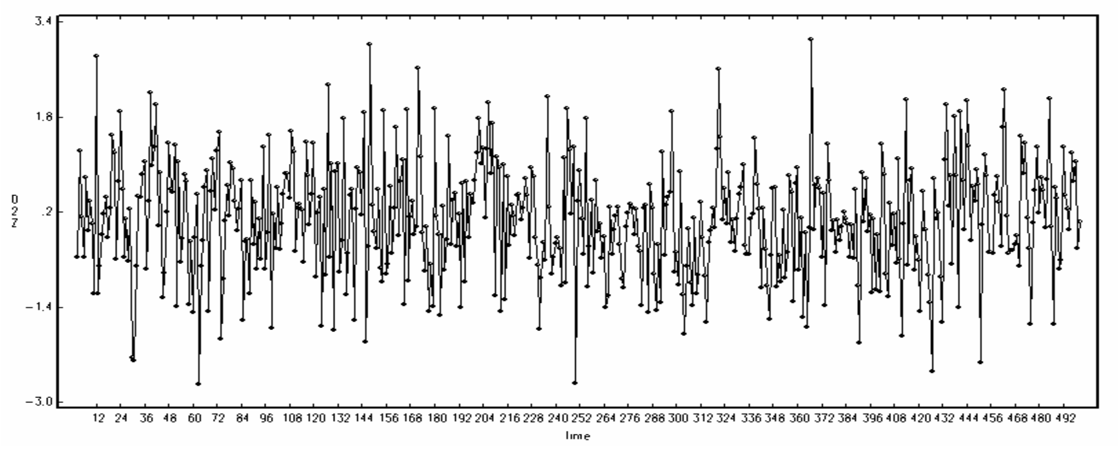
```
AUTOCORRELATIONS

   1- 12    1.00  .99  .99  .98  .98  .97  .97  .96  .96  .95  .95  .94
   ST.E.     .04  .08  .10  .12  .13  .15  .16  .17  .18  .19  .20  .21
    Q        498  992 1482 1969 2451 2929 3403 3874 4340 4803 5262 5716


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1   1.00                         + IX+XXXXXXXXXXXXXXXXXXXXXXX
   2    .99                       +   IXXX+XXXXXXXXXXXXXXXXXXXXX
   3    .99                      +    IXXXX+XXXXXXXXXXXXXXXXXXXX
   4    .98                     +     IXXXXX+XXXXXXXXXXXXXXXXXXX
   5    .98                    +      IXXXXXX+XXXXXXXXXXXXXXXXX
   6    .97                    +      IXXXXXX+XXXXXXXXXXXXXXXXX
   7    .97                   +       IXXXXXXX+XXXXXXXXXXXXXXXX
   8    .96                   +       IXXXXXXX+XXXXXXXXXXXXXXXX
   9    .96                  +        IXXXXXXXX+XXXXXXXXXXXXXXX
  10    .95                  +        IXXXXXXXX+XXXXXXXXXXXXXXX
  11    .95                 +         IXXXXXXXXX+XXXXXXXXXXXXXX
  12    .94                 +         IXXXXXXXXX+XXXXXXXXXXXXXX

 PARTIAL AUTOCORRELATIONS

   1- 12    1.00 -.00 -.00 -.00 -.00 -.00 -.00 -.00 -.00 -.00 -.00 -.00
   ST.E.     .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1   1.00                         + IX+XXXXXXXXXXXXXXXXXXXXXXX
   2    .00                         + I +
   3    .00                         + I +
   4    .00                         + I +
   5    .00                         + I +
   6    .00                         + I +
   7    .00                         + I +
   8    .00                         + I +
   9    .00                         + I +
  10    .00                         + I +
  11    .00                         + I +
  12    .00                         + I +

 uiden dz.maxl 12

 NAME OF THE SERIES . . . . . . . . . .           DZ
 TIME PERIOD ANALYZED . . . . . . . . .  2  TO   500
 MEAN OF THE (DIFFERENCED) SERIES . . .      23.9529
 STANDARD DEVIATION OF THE SERIES . . .      10.3368
 T-VALUE OF MEAN (AGAINST ZERO) . . . .      51.7636
```

```
AUTOCORRELATIONS

   1- 12      .99  .98  .96  .95  .94  .93  .92  .91  .89  .88  .87  .86
   ST.E.      .04  .08  .10  .12  .13  .14  .16  .17  .18  .18  .19  .20
    Q         491  970 1440 1898 2346 2784 3211 3629 4036 4434 4822 5200


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1    .99                         + IX+XXXXXXXXXXXXXXXXXXXXXXX
   2    .98                       +   IXXX+XXXXXXXXXXXXXXXXXXXX
   3    .96                      +    IXXXX+XXXXXXXXXXXXXXXXXXX
   4    .95                     +     IXXXXX+XXXXXXXXXXXXXXXXXX
   5    .94                     +     IXXXXX+XXXXXXXXXXXXXXXXXX
   6    .93                    +      IXXXXXX+XXXXXXXXXXXXXXXX
   7    .92                   +       IXXXXXXX+XXXXXXXXXXXXXXX
   8    .91                   +       IXXXXXXX+XXXXXXXXXXXXXXX
   9    .89                  +        IXXXXXXXX+XXXXXXXXXXXXX
  10    .88                  +        IXXXXXXXX+XXXXXXXXXXXXX
  11    .87                  +        IXXXXXXXX+XXXXXXXXXXXXX
  12    .86                 +         IXXXXXXXXX+XXXXXXXXXXX


 PARTIAL AUTOCORRELATIONS

   1- 12      .99 -.02  .01 -.02 -.01  .02 -.01  .00 -.03 -.02  .05 -.06
   ST.E.      .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1    .99                         + IX+XXXXXXXXXXXXXXXXXXXXXXX
   2   -.02                         +XI +
   3    .01                         + I +
   4   -.02                         +XI +
   5   -.01                         + I +
   6    .02                         + IX+
   7   -.01                         + I +
   8    .00                         + I +
   9   -.03                         +XI +
  10   -.02                         + I +
  11    .05                         + IX+
  12   -.06                         XXI +

 uiden d2z.maxl 12

 NAME OF THE SERIES . . . . . . . . . .          D2Z
 TIME PERIOD ANALYZED . . . . . . . . .  3  TO   500
 MEAN OF THE (DIFFERENCED) SERIES . . .        .0762
 STANDARD DEVIATION OF THE SERIES . . .        .9755
 T-VALUE OF MEAN (AGAINST ZERO) . . . .       1.7442
```

```
AUTOCORRELATIONS

   1- 12      .01  .01  .08  .01 -.01  .01  .00  .06  .01 -.02  .05 -.05
   ST.E.      .04  .04  .04  .05  .05  .05  .05  .05  .05  .05  .05  .05
    Q          .1   .1  3.4  3.5  3.5  3.5  3.5  5.1  5.2  5.4  6.6  7.8


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1    .01                         + I +
   2    .01                         + I +
   3    .08                         + IXX
   4    .01                         + I +
   5   -.01                         + I +
   6    .01                         + I +
   7    .00                         + I +
   8    .06                         + IX+
   9    .01                         + I +
  10   -.02                         +XI +
  11    .05                         + IX+
  12   -.05                         +XI +

 PARTIAL AUTOCORRELATIONS

   1- 12      .01  .01  .08  .01 -.01 -.00 -.00  .06  .01 -.02  .04 -.05
   ST.E.      .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1    .01                         + I +
   2    .01                         + I +
   3    .08                         + IXX
   4    .01                         + I +
   5   -.01                         + I +
   6    .00                         + I +
   7    .00                         + I +
   8    .06                         + IX+
   9    .01                         + I +
  10   -.02                         +XI +
  11    .04                         + IX+
  12   -.05                         +XI +
 --


 graph z,dz,d2z
 THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE  7
 --


 stop
```


12000
9000
Z 6000
3000
0
12 24 36 48 60 72 84 96 108 120 132 144 156 168 180 192 204 216 228 240 252 264 276 288 300 312 324 336 348 360 372 384 396 408 420 432 444 456 468 480 492
Time


48
36
24
12
0
12 24 36 48 60 72 84 96 108 120 132 144 156 168 180 192 204 216 228 240 252 264 276 288 300 312 324 336 348 360 372 384 396 408 420 432 444 456 468 480 492
Time


3.4
1.8
.2
-1.4
-3.0
12 24 36 48 60 72 84 96 108 120 132 144 156 168 180 192 204 216 228 240 252 264 276 288 300 312 324 336 348 360 372 384 396 408 420 432 444 456 468 480 492
Time

AO and IO Outliers for an AR(1) Model

Summary of Commands

tsm m1. model (1-ph*b)z=noise.simu
ph=.5
simu model m1.noise n(0.,1.).nobs 150.seed 123356
select old z.new z1.span(51,150)
pacf z1.maxl 10
z1(50)=z1(50)+5
pacf z1.maxl 10
tsm mz1.model (1)z1=c+noise
estim mz1.hold resi(rz1).output level(brief)
z2=z1
z2(50)=z2(50)-5
x=z2-z2
x(50)=5
tsm mm.model x=(1-t1*b)noise
t1=.5
filter mm.old x.new xx
print xx
z2=z2+xx
pacf z2.maxl 10
tsm mz2.model (1)z2=c+noise
estim mz2.hold resi(rz2).output level(brief)
graph z1,rz1,z2,rz2.
stop

SCA OUTPUT

```
tsm m1. model (1-ph*b)z=noise.simu
ph=.5
simu model m1.noise n(0.,1.).nobs 150.seed 123356

THE UNIVARIATE TIME SERIES    Z     IS SIMULATED USING MODEL M1
select old z.new z1.span(51,150)

VARIABLE    Z     IS EDITED, THE RESULT IS STORED IN VARIABLE    Z1
VARIABLE    Z1    IS  A  100  BY   1  MATRIX

pacf z1.maxl 10

NAME OF THE SERIES . . . . . . . . . .           Z1
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
MEAN OF THE (DIFFERENCED) SERIES . . .        .1099
STANDARD DEVIATION OF THE SERIES . . .       1.0676
T-VALUE OF MEAN (AGAINST ZERO) . . . .       1.0295


PARTIAL AUTOCORRELATIONS

  1- 10     .51 -.03 -.12 -.05 -.07 -.06  .07 -.18 -.21  .04
  ST.E.     .10  .10  .10  .10  .10  .10  .10  .10  .10  .10
```

```
          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .51                      +    IXXXX+XXXXXXXX
  2   -.03                      +   XI    +
  3   -.12                      + XXXI    +
  4   -.05                      +   XI    +
  5   -.07                      +  XXI    +
  6   -.06                      +   XI    +
  7    .07                      +    IXX  +
  8   -.18                      +XXXXI    +
  9   -.21                      XXXXXI    +
 10    .04                      +    IX   +
```

```
z1(50)=z1(50)+5
pacf z1.maxl 10
```

```
NAME OF THE SERIES . . . . . . . . . .           Z1
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
MEAN OF THE (DIFFERENCED) SERIES . . .        .1599
STANDARD DEVIATION OF THE SERIES . . .       1.1764
T-VALUE OF MEAN (AGAINST ZERO) . . . .       1.3593


PARTIAL AUTOCORRELATIONS

  1- 10     .37  .07 -.12 -.12  .06 -.08 -.07 -.08 -.24 -.13
  ST.E.     .10  .10  .10  .10  .10  .10  .10  .10  .10  .10


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .37                      +    IXXXX+XXXX
  2    .07                      +    IXX  +
  3   -.12                      + XXXI    +
  4   -.12                      + XXXI    +
  5    .06                      +    IXX  +
  6   -.08                      +  XXI    +
  7   -.07                      +  XXI    +
  8   -.08                      +  XXI    +
  9   -.24                     X+XXXXI    +
 10   -.13                      + XXXI    +
```

```
tsm mz1.model (1)z1=c+noise
estim mz1.hold resi(rz1).output level(brief)
```

```
THE FOLLOWING ANALYSIS IS BASED ON TIME SPAN   1  THRU  100
THE RECIPROCAL CONDITION VALUE FOR THE CROSS PRODUCT MATRIX OF
THE PARAMETER PARTIAL DERIVATIVES IS  .777004D+00



SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   MZ1
--------------------------------------------------------------------------
```

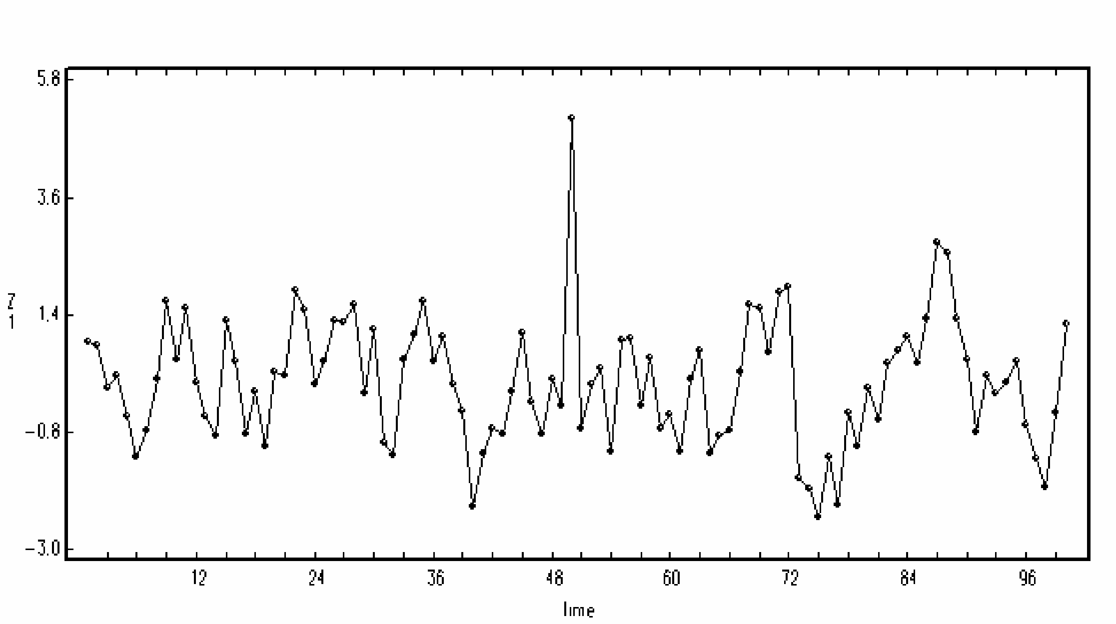
```
VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
          VARIABLE   OR CENTERED

   Z1      RANDOM     ORIGINAL     NONE
-----------------------------------------------------------------------


 PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
   LABEL       NAME    DENOM.                 TRAINT               ERROR  VALUE

  1    C               CNST      1      0     NONE      .0968     .1109    .87
  2             Z1      AR       1      1     NONE      .3729     .0935   3.99


TOTAL SUM OF SQUARES . . . . . . . .   .138386E+03
TOTAL NUMBER OF OBSERVATIONS . . . .           100
RESIDUAL SUM OF SQUARES. . . . . . .   .118747E+03
R-SQUARE . . . . . . . . . . . . . .          .133
EFFECTIVE NUMBER OF OBSERVATIONS . .            99
RESIDUAL VARIANCE ESTIMATE . . . . .   .119946E+01
RESIDUAL STANDARD ERROR. . . . . . .   .109520E+01
```

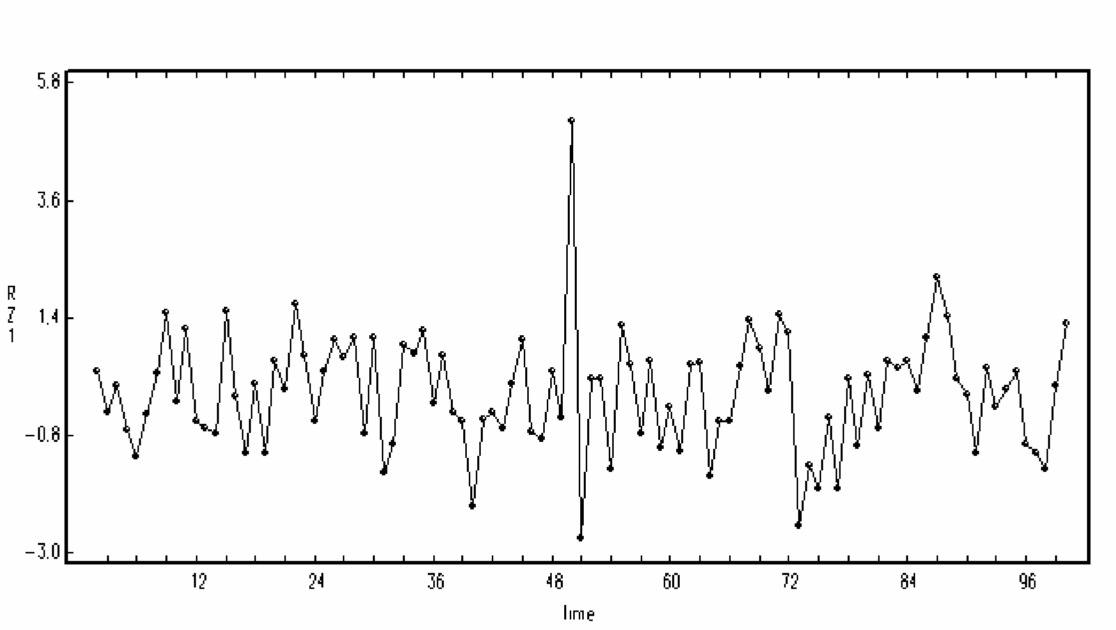
```
z2=z1
z2(50)=z2(50)-5
x=z2-z2
x(50)=5
tsm mm.model x=(1-t1*b)noise
t1=.5
filter mm.old x.new xx


THE FOLLOWING ANALYSIS IS BASED ON TIME SPAN   1  THRU  100

SERIES    X     IS FILTERED USING MODEL    MM   , THE RESULT IS IN XX
print xx
```

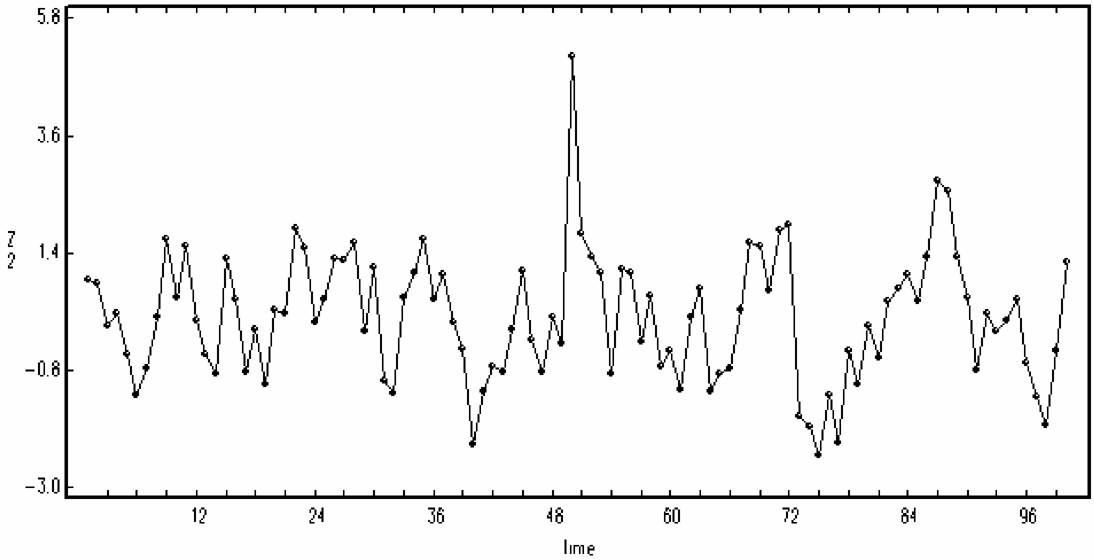
```
XX       IS  A   100  BY     1  VARIABLE
     .000      .000      .000      .000      .000      .000      .000
     .000      .000      .000      .000      .000      .000      .000
     .000      .000      .000      .000      .000      .000      .000
     .000      .000      .000      .000      .000      .000      .000
     .000      .000      .000      .000      .000      .000      .000
     .000      .000      .000      .000      .000      .000      .000
     .000      .000      .000      .000      .000      .000      .000
    5.000     2.500     1.250      .625      .313      .156      .078
     .039      .020      .010      .005      .002      .001  .610E-03
 .305E-03  .153E-03  .763E-04  .381E-04  .191E-04  .954E-05  .477E-05
 .238E-05  .119E-05  .596E-06  .298E-06  .149E-06  .745E-07  .373E-07
 .186E-07  .931E-08  .466E-08  .233E-08  .116E-08  .582E-09  .291E-09
 .146E-09  .728E-10  .364E-10  .182E-10  .909E-11  .455E-11  .227E-11
 .114E-11      .000      .000      .000      .000      .000      .000
     .000      .000
```

```
z2=z2+xx
pacf z2.maxl 10

NAME OF THE SERIES . . . . . . . . . .            Z2
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   100
MEAN OF THE (DIFFERENCED) SERIES . . .         .2099
STANDARD DEVIATION OF THE SERIES . . .        1.1902
T-VALUE OF MEAN (AGAINST ZERO) . . . .        1.7637
```

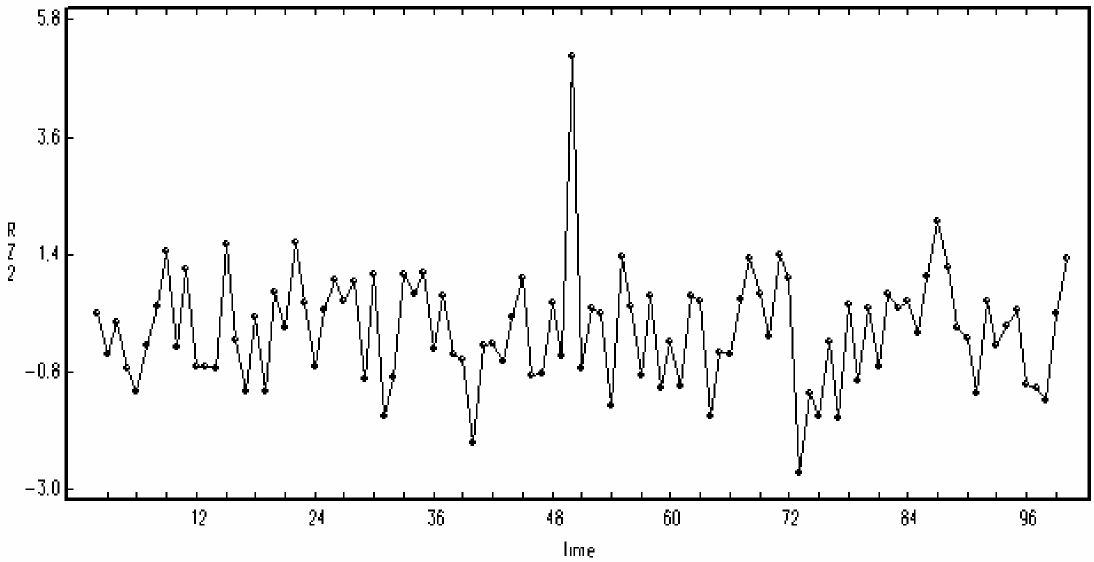
```
 PARTIAL AUTOCORRELATIONS

   1- 10      .47  .02 -.15 -.09  .08 -.10 -.07 -.12 -.26 -.11
   ST.E.      .10  .10  .10  .10  .10  .10  .10  .10  .10  .10


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1    .47                      +    IXXXX+XXXXXXX
   2    .02                      +    IX   +
   3   -.15                      +XXXXI    +
   4   -.09                      +  XXI    +
   5    .08                      +    IXX  +
   6   -.10                      + XXXI    +
   7   -.07                      +  XXI    +
   8   -.12                      + XXXI    +
   9   -.26                     X+XXXXI    +
  10   -.11                      + XXXI    +

tsm mz2.model (1)z2=c+noise
estim mz2.hold resi(rz2).output level(brief)


 THE FOLLOWING ANALYSIS IS BASED ON TIME SPAN   1  THRU  100
 SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   MZ2
 -----------------------------------------------------------------------
 VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
            VARIABLE   OR CENTERED

    Z2      RANDOM     ORIGINAL     NONE
 -----------------------------------------------------------------------


  PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
    LABEL       NAME    DENOM.                 TRAINT               ERROR  VALUE

   1    C               CNST      1      0     NONE      .1094     .1075   1.02
   2             Z2      AR       1      1     NONE      .4688     .0890   5.27


 TOTAL SUM OF SQUARES . . . . . . . .   .141648E+03
 TOTAL NUMBER OF OBSERVATIONS . . . .           100
 RESIDUAL SUM OF SQUARES. . . . . . .   .110267E+03
 R-SQUARE . . . . . . . . . . . . . .          .214
 EFFECTIVE NUMBER OF OBSERVATIONS . .            99
 RESIDUAL VARIANCE ESTIMATE . . . . .   .111381E+01
 RESIDUAL STANDARD ERROR. . . . . . .   .105537E+01

 graph z1,rz1,z2,rz2.
 stop
```


5.8
3.6
Z
1
1.4
-0.8
-3.0
12
24
36
48
60
72
84
96
Time


5.8
3.6
R
Z
1
1.4
-0.8
-3.0
12
24
36
48
60
72
84
96
Time



5.8
3.6
1.4
-0.8
-3.0
Z
2
12
24
36
48
60
72
84
96
Time



5.8
3.6
1.4
-0.8
-3.0
R
Z
2
12
24
36
48
60
72
84
96
Time

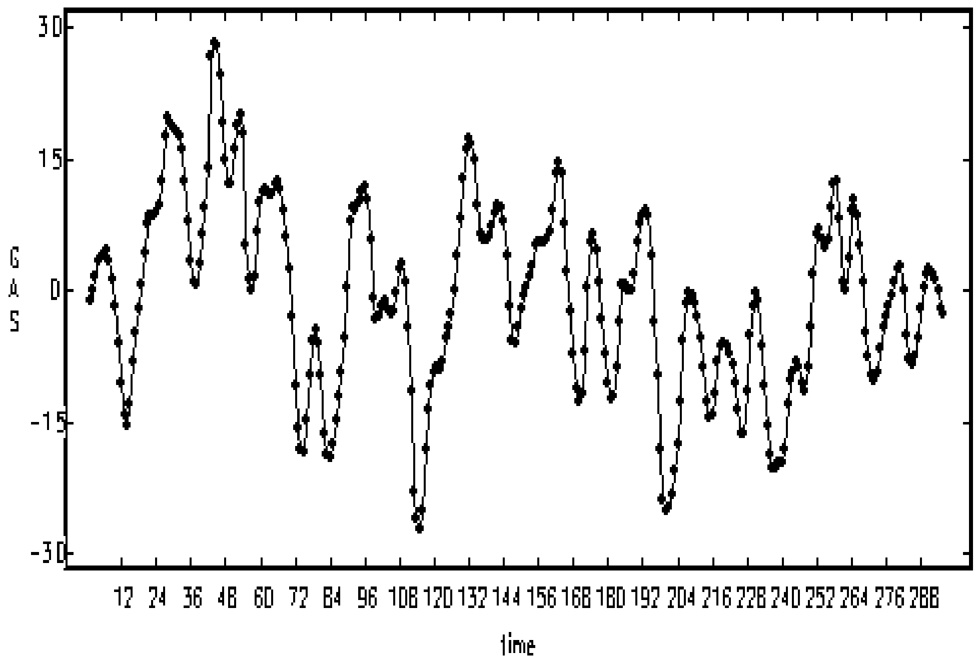
EXAMPLE 2. THE GAS FURNACE DATA

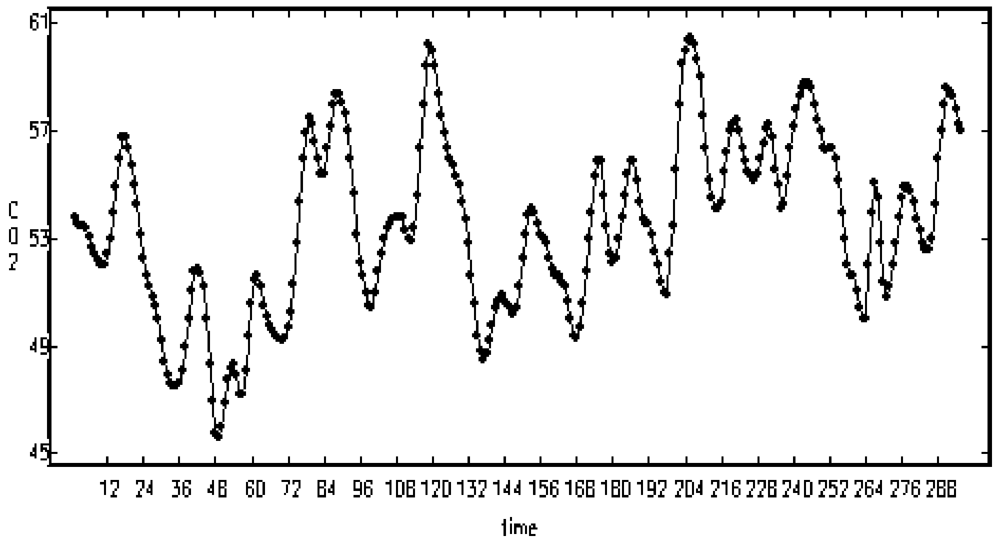
This data set consists of two time series from a chemical reactor:

gas--input gas rate and co2--output carbon dioxide.

The data was used by Box and Jenkins(1976) to introduce their transfer function modeling technique. We use here to illustrate first the multiple time series method and then compare the results with the transfer function modeling.

```
input gas.file'mts1.dat'.dataset gasrate
GAS     , A  296  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
input co2.file'mts1.dat'.dataset co2
CO2     , A  296  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
fsave gas,co2.file'gas.fmt'
```






```
miden gas,co2.arfits 1 to 8

TIME PERIOD ANALYZED . . . . . . . . . . . . .  1  TO   296
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .            296


SERIES   NAME           MEAN       STD. ERROR

   1    GAS             -.0568      1.0710
   2    CO2            53.5091      3.1967


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .05812


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
  -.48  1.00

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX
OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   ++++++++++++   ----------..
 1   ++..........   ............
```

```
  2   ------------   ++++++++++++
  2   ----------..   ++++++...+++


 CROSS CORRELATION MATRICES IN TERMS OF +,-,.




 LAGS  1 THROUGH  6

      + -      + -      + -      + -      + -      + -
      - +      - +      - +      - +      - +      - +

 LAGS  7 THROUGH 12

      + -      + -      + -      + -      + .      + .
      - +      - +      - +      - +      - +      - +

 LAGS 13 THROUGH 18

      + .      + .      . .      . .      . .      . .
      - +      - +      - +      - +      - +      - +

 LAGS 19 THROUGH 24

      . .      . .      . .      . .      . .      . .
      - .      - .      - .      - +      . +      . +

 DETERMINANT OF S(0) =  .943952E+01
 NOTE: S(0) IS THE SAMPLE COVARIANCE MATRIX OF W(MAXLAG+1),...,W
(NOBE)
==========  STEPWISE AUTOREGRESSION SUMMARY  ==========

----------------------------------------------------------------------
-
     I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
 LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR
COEFF.

----+----------+----------+---------+----------+--------------------
-
   1 I .102E+00 I .724E-01 I 1666.31 I   -3.585 I + +
     I .343E+00 I .373E+00 I         I          I - +

----+----------+----------+---------+----------+--------------------
-
   2 I .367E-01 I .362E-01 I  672.92 I   -5.940 I - +
     I .683E-01 I .688E-01 I         I          I - -

----+----------+----------+---------+----------+--------------------
-
   3 I .356E-01 I .354E-01 I   32.14 I   -6.028 I + .
     I .626E-01 I .628E-01 I         I          I . +

----+----------+----------+---------+----------+--------------------
-
   4 I .354E-01 I .350E-01 I   22.76 I   -6.082 I . .
     I .582E-01 I .586E-01 I         I          I + +
```

```
----+----------+----------+---------+----------+--------------------
-
   5 I .352E-01 I .347E-01 I    5.69 I   -6.076 I . .
     I .573E-01 I .578E-01 I         I          I . .

----+----------+----------+---------+----------+--------------------
-
   6 I .343E-01 I .341E-01 I   12.83 I   -6.096 I - .
     I .560E-01 I .563E-01 I         I          I + .

----+----------+----------+---------+----------+--------------------
-
   7 I .342E-01 I .340E-01 I    1.76 I   -6.075 I . .
     I .558E-01 I .560E-01 I         I          I . .

----+----------+----------+---------+----------+--------------------
-
   8 I .339E-01 I .337E-01 I    8.24 I   -6.078 I . .
     I .546E-01 I .548E-01 I         I          I . +

--------------------------------------------------------------------
-

 NOTE:  CHI-SQUARED CRITICAL VALUES WITH   4  DEGREES OF FREEDOM ARE

              5 PERCENT:   9.5     1 PERCENT:  13.3


 NOTE:  THE PARTIAL AUTOREGRESSION COEFFICIENT MATRIX FOR LAG L IS
THE
        ESTIMATED PHI(L) FROM THE FIT WHERE THE MAXIMUM LAG USED IS L
        (I.E. THE LAST COEFFICIENT MATRIX). THE ELEMENTS ARE
        STANDARDIZED BY DIVIDING EACH BY ITS STANDARD ERROR.
 --


 Mtsm m1.model (1-p1*b-p2*b**2-p3*b**3-p4*b**4-p5*b**5-p6*b**6)@
 series=c+noise.series gas,co2.constraints p1(cp1),p2(cp2),p3(cp3),@
 p4(cp4),p5(cp5),p6(cp6)


 SUMMARY FOR MULTIVARIATE ARMA MODEL --    M1

 VARIABLE   DIFFERENCING

  GAS
  CO2

  PARAMETER     FACTOR     ORDER    CONSTRAINT

  1        C   CONSTANT      0        --
  2       P1    REG AR       1        CP1
  3       P2    REG AR       2        CP2
  4       P3    REG AR       3        CP3
  5       P4    REG AR       4        CP4
  6       P5    REG AR       5        CP5
  7       P6    REG AR       6        CP6

 mestim m1.hold resi(r1,r2)

 SUMMARY FOR THE MULTIVARIATE ARMA MODEL

 SERIES    NAME            MEAN         STD DEV    DIFFERENCE ORDER
```

(S)

1	GAS	-.0568	1.0710
2	CO2	53.5091	3.1967

NUMBER OF OBSERVATIONS = 296 (EFFECTIVE NUMBER = NOBE = 290)

MODEL SPECIFICATION WITH PARAMETER VALUES

PARAMETER NUMBER	PARAMETER DESCRIPTION	PARAMETER VALUE
1	CONSTANT(1)	-32.128207
2	CONSTANT(2)	21.437749
3	AUTOREGRESSIVE (1, 1, 1)	.100000
4	AUTOREGRESSIVE (1, 1, 2)	.100000
5	AUTOREGRESSIVE (1, 2, 1)	.100000
6	AUTOREGRESSIVE (1, 2, 2)	.100000
7	AUTOREGRESSIVE (2, 1, 1)	.100000
8	AUTOREGRESSIVE (2, 1, 2)	.100000
9	AUTOREGRESSIVE (2, 2, 1)	.100000
10	AUTOREGRESSIVE (2, 2, 2)	.100000
11	AUTOREGRESSIVE (3, 1, 1)	.100000
12	AUTOREGRESSIVE (3, 1, 2)	.100000
13	AUTOREGRESSIVE (3, 2, 1)	.100000
14	AUTOREGRESSIVE (3, 2, 2)	.100000
15	AUTOREGRESSIVE (4, 1, 1)	.100000
16	AUTOREGRESSIVE (4, 1, 2)	.100000
17	AUTOREGRESSIVE (4, 2, 1)	.100000
18	AUTOREGRESSIVE (4, 2, 2)	.100000
19	AUTOREGRESSIVE (5, 1, 1)	.100000
20	AUTOREGRESSIVE (5, 1, 2)	.100000
21	AUTOREGRESSIVE (5, 2, 1)	.100000
22	AUTOREGRESSIVE (5, 2, 2)	.100000
23	AUTOREGRESSIVE (6, 1, 1)	.100000
24	AUTOREGRESSIVE (6, 1, 2)	.100000
25	AUTOREGRESSIVE (6, 2, 1)	.100000
26	AUTOREGRESSIVE (6, 2, 2)	.100000

ERROR COVARIANCE MATRIX

	1	2
1	3.753038	
2	-2.136201	6.956255

ITERATIONS TERMINATED DUE TO:
RELATIVE CHANGE IN DETERMINANT OF COVARIANCE MATRIX .LE. .100E-03
TOTAL NUMBER OF ITERATIONS IS 8

FINAL MODEL SUMMARY WITH CONDITIONAL LIKELIHOOD PARAMETER ESTIMATES

----- CONSTANT VECTOR (STD ERROR) -----

.770 (.654)
3.824 (.836)

----- PHI MATRICES -----

```
ESTIMATES OF    PHI(  1 ) MATRIX AND SIGNIFICANCE

     1.931    -.051      + .
      .063    1.545      . +
 STANDARD ERRORS
      .058     .046
      .074     .058

 ESTIMATES OF    PHI(  2 ) MATRIX AND SIGNIFICANCE

    -1.204     .100      - .
     -.133    -.593      . -
 STANDARD ERRORS
      .126     .084
      .161     .108

 ESTIMATES OF    PHI(  3 ) MATRIX AND SIGNIFICANCE

      .170    -.080      . .
     -.441    -.171      - .
 STANDARD ERRORS
      .144     .088
      .184     .113

 ESTIMATES OF    PHI(  4 ) MATRIX AND SIGNIFICANCE

     -.160     .027      . .
      .152     .132      . .
 STANDARD ERRORS
      .145     .088
      .186     .112

 ESTIMATES OF    PHI(  5 ) MATRIX AND SIGNIFICANCE

      .380    -.041      + .
     -.120     .057      . .
 STANDARD ERRORS
      .137     .077
      .175     .099

 ESTIMATES OF    PHI(  6 ) MATRIX AND SIGNIFICANCE

     -.214     .031      - .
      .249    -.042      + .
 STANDARD ERRORS
      .084     .033
      .107     .042


 -----------------------
 ERROR COVARIANCE MATRIX
 -----------------------

               1           2
   1      .034085
   2     -.002295     .055650



 ------------------------------------------
 REDUCED CORRELATION MATRIX OF THE PARAMETERS
 ------------------------------------------
```

```
          1     2     3     4     5     6     7     8     9    10
11
   1   1.00
   2    .    1.00
   3    .     .    1.00
   4   -.34   .     .    1.00
   5    .     .     .     .    1.00
   6    .    -.34   .     .     .    1.00
   7    .     .    -.89   .     .     .    1.00
   8    .     .     .    -.84   .     .     .    1.00
   9    .     .     .     .    -.89   .     .     .    1.00
  10    .     .     .     .     .    -.84   .     .     .    1.00
  11    .     .     .48   .     .     .    -.79   .     .     .
1.00
  12    .     .     .     .31   .     .     .    -.70   .     .     .
  13    .     .     .     .     .48   .     .     .    -.79   .     .
  14    .     .     .     .     .     .31   .     .     .    -.70   .
  15    .     .     .     .13   .     .     .26   .     .     .
-.70
  16    .     .     .     .     .     .     .     .     .     .     .
  17    .     .     .     .     .     .13   .     .     .26   .     .
  18    .     .     .     .     .     .     .     .     .     .     .
  19    .     .     .     .     .     .     .     .     .     .
.24
  20    .     .     .     .     .     .     .     .15   .     .     .
  21    .     .     .     .     .     .     .     .     .     .     .
  22    .     .     .     .     .     .     .     .     .     .15   .
  23   -.42   .     .     .     .     .     .     .     .     .     .
  24    .     .     .     .     .     .     .    -.14   .     .     .
  25    .    -.42   .     .     .     .     .     .     .     .     .
  26    .     .     .     .     .     .     .     .     .    -.14   .

         12    13    14    15    16    17    18    19    20    21
22
  12   1.00
  13    .    1.00
  14    .     .    1.00
  15    .     .     .    1.00
  16   -.63   .     .     .    1.00
  17    .    -.70   .     .     .    1.00
  18    .     .    -.63   .     .     .    1.00
  19    .     .     .    -.72   .     .     .    1.00
  20    .14   .     .     .    -.78   .     .     .    1.00
  21    .     .24   .     .     .    -.72   .     .     .    1.00
  22    .     .     .14   .     .     .    -.78   .     .     .
1.00
  23    .14   .     .     .33   .     .     .    -.80  -.13   .     .
  24    .     .     .     .     .49   .     .     .    -.90   .     .
  25    .     .     .14   .     .     .33   .     .     .    -.80
-.13
  26    .     .     .     .     .     .     .49   .     .     .
-.90

         23    24    25    26
  23   1.00
  24    .    1.00
  25    .     .    1.00
  26    .     .     .    1.00

 miden r1,r2

 TIME PERIOD ANALYZED . . . . . . . . . . . . .  7  TO   296
 EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .         290
```

```
SERIES   NAME           MEAN       STD. ERROR

   1    R1               .0000       .1846
   2    R2               .0000       .2359


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .05872


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
  -.05  1.00

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX
OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I

          1              2

 1   ...........-   ............
 1   ............   ............


 2   ............   ............
 2   ............   ......-.....


CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     . .      . .      . .      . .      . .      - .
     . .      . .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . -      . .      . .      . .      . .      . .
--
p1(1,2)=0
cp1(1,2)=1
p1(2,1)=0
cp1(2,1)=1
p2(1,2)=0
cp2(1,2)=1
p2(2,1)=0
```

```
cp2(2,1)=1
p3(1,2)=0
cp3(1,2)=1
p4(1,2)=0
cp4(1,2)=1
p5(1,2)=0
cp5(1,2)=1
p6(1,2)=0
cp6(1,2)=1

mestim m1.hold resi(rr1,rr2)

SUMMARY FOR THE MULTIVARIATE ARMA MODEL

SERIES    NAME            MEAN         STD DEV    DIFFERENCE ORDER
(S)

  1     GAS              -.0568         1.0710
  2     CO2             53.5091         3.1967

NUMBER OF OBSERVATIONS =  296  (EFFECTIVE NUMBER = NOBE =  290)
MODEL SPECIFICATION WITH PARAMETER VALUES


PARAMETER              PARAMETER                 PARAMETER
 NUMBER               DESCRIPTION                 VALUE
----------    -----------------------------     ------------

    1               CONSTANT( 1)                   .769973

    2               CONSTANT( 2)                  3.824136

    3         AUTOREGRESSIVE ( 1, 1, 1)           1.931320

 *FIXED*      AUTOREGRESSIVE ( 1, 1, 2)            .000000

 *FIXED*      AUTOREGRESSIVE ( 1, 2, 1)            .000000

    4         AUTOREGRESSIVE ( 1, 2, 2)           1.545215

    5         AUTOREGRESSIVE ( 2, 1, 1)          -1.204463

 *FIXED*      AUTOREGRESSIVE ( 2, 1, 2)            .000000

 *FIXED*      AUTOREGRESSIVE ( 2, 2, 1)            .000000

    6         AUTOREGRESSIVE ( 2, 2, 2)           -.592904

    7         AUTOREGRESSIVE ( 3, 1, 1)            .169695

 *FIXED*      AUTOREGRESSIVE ( 3, 1, 2)            .000000

    8         AUTOREGRESSIVE ( 3, 2, 1)           -.441241

    9         AUTOREGRESSIVE ( 3, 2, 2)           -.171083

   10         AUTOREGRESSIVE ( 4, 1, 1)           -.160197

 *FIXED*      AUTOREGRESSIVE ( 4, 1, 2)            .000000

   11         AUTOREGRESSIVE ( 4, 2, 1)            .152006

   12         AUTOREGRESSIVE ( 4, 2, 2)            .132391
```

```
   13          AUTOREGRESSIVE ( 5, 1, 1)           .380197

 *FIXED*       AUTOREGRESSIVE ( 5, 1, 2)           .000000

   14          AUTOREGRESSIVE ( 5, 2, 1)          -.120358

   15          AUTOREGRESSIVE ( 5, 2, 2)           .056875

   16          AUTOREGRESSIVE ( 6, 1, 1)          -.213661

 *FIXED*       AUTOREGRESSIVE ( 6, 1, 2)           .000000

   17          AUTOREGRESSIVE ( 6, 2, 1)           .249303

   18          AUTOREGRESSIVE ( 6, 2, 2)          -.042088


-----------------------
ERROR COVARIANCE MATRIX
-----------------------

              1           2
  1      .034085
  2     -.002295     .055650


ITERATIONS TERMINATED DUE TO:
  RELATIVE CHANGE IN DETERMINANT OF COVARIANCE MATRIX .LE.  .100E-03
  TOTAL NUMBER OF ITERATIONS IS    4

FINAL MODEL SUMMARY WITH CONDITIONAL LIKELIHOOD PARAMETER ESTIMATES


----- CONSTANT VECTOR (STD ERROR) -----

    -.004  (      .011  )
    3.871  (      .834  )

----- PHI MATRICES -----

ESTIMATES OF    PHI(  1 ) MATRIX AND SIGNIFICANCE

    1.935     .000      + .
     .000    1.544      . +

STANDARD ERRORS

     .058      --
      --      .058

ESTIMATES OF    PHI(  2 ) MATRIX AND SIGNIFICANCE

   -1.215     .000      - .
     .000    -.592      . -

STANDARD ERRORS

     .126      --
      --      .107

ESTIMATES OF    PHI(  3 ) MATRIX AND SIGNIFICANCE
```

```
     .192     .000      . .
    -.541    -.171      - .

STANDARD ERRORS

     .145      --
     .074     .112

ESTIMATES OF    PHI(  4 ) MATRIX AND SIGNIFICANCE

    -.128     .000      . .
     .176     .132      . .

STANDARD ERRORS

     .145      --
     .164     .112

ESTIMATES OF    PHI(  5 ) MATRIX AND SIGNIFICANCE

     .280     .000      + .
    -.117     .056      . .

STANDARD ERRORS

     .127      --
     .175     .098

ESTIMATES OF    PHI(  6 ) MATRIX AND SIGNIFICANCE

    -.116     .000      . .
     .247    -.041      + .

STANDARD ERRORS

     .058      --
     .107     .042

-----------------------
ERROR COVARIANCE MATRIX
-----------------------

              1            2
  1      .034793
  2     -.002323     .055797


--------------------------------------------
REDUCED CORRELATION MATRIX OF THE PARAMETERS
--------------------------------------------

          1     2     3     4     5     6     7     8     9    10
11
   1   1.00
   2    .    1.00
   3    .     .    1.00
   4    .    -.35   .    1.00
   5    .     .    -.89   .    1.00
   6    .     .     .    -.84   .    1.00
   7    .     .     .49   .    -.79   .    1.00
   8    .     .     .     .     .     .     .    1.00
   9    .     .     .     .32   .    -.70   .     .    1.00
  10    .     .     .     .     .28   .    -.72   .     .    1.00
  11    .     .     .     .18   .    -.12   .    -.88   .     .
```

```
1.00
  12    .     .     .     .     .     .     .     .    -.64   .     .
  13    .     .     .     .     .     .    .28    .     .    -.79   .
  14    .     .     .     .     .     .     .    .52    .     .
-.80
  15    .     .     .     .     .    .15    .     .    .14    .     .
  16    .     .     .     .     .     .     .     .     .    .49    .
  17    .   -.42    .     .     .     .     .   -.14   .14    .
.41
  18    .     .     .     .     .   -.14    .     .     .     .     .

         12    13    14    15    16    17    18
  12   1.00
  13    .    1.00
  14    .     .    1.00
  15   -.78   .     .    1.00
  16    .    -.89   .     .    1.00
  17    .     .    -.81  -.12   .    1.00
  18   .49    .     .    -.90   .     .    1.00
 --


 miden rr1,rr2

 TIME PERIOD ANALYZED . . . . . . . . . . . . .  7  TO   296
 EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .           290


 SERIES   NAME           MEAN       STD. ERROR

    1    RR1              .0000       .1865
    2    RR2              .0000       .2362


 NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
       IS (1/NOBE**.5) =     .05872

 SAMPLE CORRELATION MATRIX OF THE SERIES

   1.00
   -.05  1.00

 SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
      + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
      - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
      . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


 BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX
OVER
 ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


  1   ..........+.   ............
  1   ............   ............


  2   ............   ............
  2   ......-.....   ......-.....
```

```
 CROSS CORRELATION MATRICES IN TERMS OF +,-,.

 LAGS  1 THROUGH  6

      . .      . .      . .      . .      . .      . .
      . .      . .      . .      . .      . .      . .

 LAGS  7 THROUGH 12

      . .      . .      . .      . .      + .      . .
      . .      . .      . .      . .      . .      . .

 LAGS 13 THROUGH 18

      . .      . .      . .      . .      . .      . .
      . .      . .      . .      . .      . .      . .

 LAGS 19 THROUGH 24

      . .      . .      . .      . .      . .      . .
      - -      . .      . .      . .      . .      . .
 --
```

We now apply the transfer function method to this data set.

```
 uiden gas.maxlag 12

 TIME PERIOD ANALYZED . . . . . . . . . .  1  TO   296
 NAME OF THE SERIES . . . . . . . . . .            GAS
 EFFECTIVE NUMBER OF OBSERVATIONS . . .            296
 STANDARD DEVIATION OF THE SERIES . . .         1.0710
 MEAN OF THE (DIFFERENCED) SERIES . . .         -.0568
 STANDARD DEVIATION OF THE MEAN . . . .          .0622
 T-VALUE OF MEAN (AGAINST ZERO) . . . .         -.9130

 AUTOCORRELATIONS


   1- 12      .95  .83  .68  .53  .41  .32  .26  .23  .21  .21  .20
.19
   ST.E.      .06  .10  .12  .13  .14  .14  .15  .15  .15  .15  .15
.15
    Q         271  480  620  705  756  786  807  823  837  850  863
874


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1    .95                        +  IXX+XXXXXXXXXXXXXXXXXXXXX
   2    .83                      +    IXXXX+XXXXXXXXXXXXXXXX
   3    .68                     +     IXXXXX+XXXXXXXXXXX
   4    .53                     +     IXXXXX+XXXXXXX
   5    .41                    +      IXXXXXX+XXX
   6    .32                    +      IXXXXXX+X
   7    .26                    +      IXXXXXXX
   8    .23                    +      IXXXXXX+
   9    .21                    +      IXXXXX +
  10    .21                    +      IXXXXX +
  11    .20                    +      IXXXXX +
  12    .19                    +      IXXXXX +
```

```
 PARTIAL AUTOCORRELATIONS


   1- 12      .95 -.79  .34  .12  .06 -.11  .05  .10  .02 -.07 -.09
.04
   ST.E.      .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06
```

```
           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
   1    .95                      +   IXX+XXXXXXXXXXXXXXXXXXXXX
   2   -.79       XXXXXXXXXXXXXXXXX+XXI   +
   3    .34                      +   IXX+XXXXX
   4    .12                      +   IXXX
   5    .06                      +   IX +
   6   -.11                      XXXI   +
   7    .05                      +   IX +
   8    .10                      +   IXX+
   9    .02                      +   I  +
  10   -.07                      +XXI   +
  11   -.09                      +XXI   +
  12    .04                      +   IX +
 --
```

```
 utsm ugas.model (1-pp1*b-pp2*b**2-pp3*b**3)gas=cc+noise.no show
 uestim ugas.hold resi(ra)
```

```
 SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   UGAS

  PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE
STD     T
    LABEL       NAME    DENOM.                TRAINT
ERROR  VALUE

   1    CC              CNST      1      0     NONE     -.0038
.0110  -.35
   2   PP1      GAS      AR       1      1     NONE     1.9750
.0549 35.99
   3   PP2      GAS      AR       1      2     NONE    -1.3732
.0995-13.81
   4   PP3      GAS      AR       1      3     NONE      .3424
.0549  6.24


 TOTAL SUM OF SQUARES . . . . . . . .   .339494E+03
 TOTAL NUMBER OF OBSERVATIONS . . . .           296
 RESIDUAL SUM OF SQUARES. . . . . . .   .104347E+02
 R-SQUARE . . . . . . . . . . . . . .          .969
 EFFECTIVE NUMBER OF OBSERVATIONS . .           293
 RESIDUAL VARIANCE ESTIMATE . . . . .   .356132E-01
 RESIDUAL STANDARD ERROR. . . . . . .   .188715E+00
 --
```

```
acf ra.maxlag 12

TIME PERIOD ANALYZED . . . . . . . . .  4  TO   296
NAME OF THE SERIES . . . . . . . . . .           RA
EFFECTIVE NUMBER OF OBSERVATIONS . . .          293
STANDARD DEVIATION OF THE SERIES . . .        .1887
MEAN OF THE (DIFFERENCED) SERIES . . .        .0000
STANDARD DEVIATION OF THE MEAN . . . .        .0110
T-VALUE OF MEAN (AGAINST ZERO) . . . .        .0000


AUTOCORRELATIONS


  1- 12    -.04  .07  .06 -.15 -.01  .06  .02  .00 -.05  .04  .14
-.08
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06
   Q         .5  1.9  2.9  9.2  9.2 10.3 10.4 10.4 11.2 11.7 18.0
19.9


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1   -.04                        + XI  +
  2    .07                        +  IXX+
  3    .06                        +  IX +
  4   -.15                       X+XXI  +
  5   -.01                        +  I  +
  6    .06                        +  IXX+
  7    .02                        +  I  +
  8    .00                        +  I  +
  9   -.05                        + XI  +
 10    .04                        +  IX +
 11    .14                        +  IXX+X
 12   -.08                        +XXI  +

ufilter model ugas.old co2.new rb

THE FOLLOWING ANALYSIS IS BASED ON TIME SPAN   1  THRU  296
SERIES   CO2    IS FILTERED USING MODEL   UGAS  , THE RESULT IS IN
RB

ccf ra,rb

TIME PERIOD ANALYZED . . . . . . . . .  4  TO   296
NAMES OF THE SERIES  . . . . . . . . .           RA            RB
EFFECTIVE NUMBER OF OBSERVATIONS . . .          293           293
STANDARD DEVIATION OF THE SERIES . . .        .1887         .3629
MEAN OF THE (DIFFERENCED) SERIES . . .        .0000        2.9943
STANDARD DEVIATION OF THE MEAN . . . .        .0110         .0212
T-VALUE OF MEAN (AGAINST ZERO) . . . .        .0000      141.2260



CORRELATION       BETWEEN       RB     AND       RA IS    .00


CROSS CORRELATION BETWEEN       RA(T)  AND       RB(T-L)


  1- 12    -.03  .01 -.05 -.02 -.00 -.12 -.03 -.09  .00  .02  .01
-.00
```

```
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 13- 24     .01  .01  .03 -.03 -.06 -.05  .04 -.06  .03 -.05 -.02
-.03
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 25- 36     .02  .04  .00 -.12 -.03 -.05 -.06  .04  .04 -.07  .07
-.00
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06


 CROSS CORRELATION BETWEEN       RB(T)  AND       RA(T-L)


  1- 12     .05 -.03 -.28 -.33 -.46 -.27 -.17 -.03  .03 -.05 -.03
-.02
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 13- 24    -.07  .00 -.01 -.06 -.05 -.03 -.12  .01 -.05 -.02  .06
.01
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 25- 36     .02  .01 -.09  .00  .02 -.13  .05  .01  .00  .04 -.02
.01
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
 -36    .01                        +  I  +
 -35   -.02                        +  I  +
 -34    .04                        +  IX +
 -33    .00                        +  I  +
 -32    .01                        +  I  +
 -31    .05                        +  IX +
 -30   -.13                        XXXI  +
 -29    .02                        +  I  +
 -28    .00                        +  I  +
 -27   -.09                        +XXI  +
 -26    .01                        +  I  +
 -25    .02                        +  I  +
 -24    .01                        +  I  +
 -23    .06                        +  IXX+
 -22   -.02                        + XI  +
 -21   -.05                        + XI  +
 -20    .01                        +  I  +
 -19   -.12                        XXXI  +
 -18   -.03                        + XI  +
 -17   -.05                        + XI  +
 -16   -.06                        + XI  +
 -15   -.01                        +  I  +
 -14    .00                        +  I  +
 -13   -.07                        +XXI  +
 -12   -.02                        +  I  +
 -11   -.03                        + XI  +
 -10   -.05                        + XI  +
  -9    .03                        +  IX +
```

```
 -8   -.03                      + XI  +
 -7   -.17                     X+XXI  +
 -6   -.27                  XXXX+XXI  +
 -5   -.46              XXXXXXXX+XXI  +
 -4   -.33                 XXXXX+XXI  +
 -3   -.28                  XXXX+XXI  +
 -2   -.03                      + XI  +
 -1    .05                      +  IX +
  0    .00                      +  I  +
  1   -.03                      + XI  +
  2    .01                      +  I  +
  3   -.05                      + XI  +
  4   -.02                      +  I  +
  5    .00                      +  I  +
  6   -.12                      XXXI  +
  7   -.03                      + XI  +
  8   -.09                      +XXI  +
  9    .00                      +  I  +
 10    .02                      +  IX +
 11    .01                      +  I  +
 12    .00                      +  I  +
 13    .01                      +  I  +
 14    .01                      +  I  +
 15    .03                      +  IX +
 16   -.03                      + XI  +
 17   -.06                      + XI  +
 18   -.05                      + XI  +
 19    .04                      +  IX +
 20   -.06                      +XXI  +
 21    .03                      +  IX +
 22   -.05                      + XI  +
 23   -.02                      + XI  +
 24   -.03                      + XI  +
 25    .02                      +  I  +
 26    .04                      +  IX +
 27    .00                      +  I  +
 28   -.12                      XXXI  +
 29   -.03                      + XI  +
 30   -.05                      + XI  +
 31   -.06                      + XI  +
 32    .04                      +  IX +
 33    .04                      +  IX +
 34   -.07                      +XXI  +
 35    .07                      +  IXX+
 36    .00                      +  I  +
```

```
tsm tfm.model co2=c0+(3,4,5,6,7;w3,w4,w5,w6,w7)gas+noise.no show
estim tfm.hold resi(rtf1)
```

```
 SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   TFM
  PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE
STD     T
    LABEL       NAME    DENOM.                TRAINT
ERROR  VALUE

   1    C0              CNST      1      0     NONE    53.3296
.05011063.56
   2    W3      GAS     NUM.      1      3     NONE     -.6983
.2656 -2.63
   3    W4      GAS     NUM.      1      4     NONE     -.5530
.5770  -.96
   4    W5      GAS     NUM.      1      5     NONE    -1.0118
.6592 -1.53
```

```
  5    W6      GAS     NUM.      1      6     NONE      .1347
.5770   .23
  6    W7      GAS     NUM.      1      7     NONE    -1.0130
.2655 -3.82


 TOTAL SUM OF SQUARES . . . . . . . .   .302481E+04
 TOTAL NUMBER OF OBSERVATIONS . . . .           296
 RESIDUAL SUM OF SQUARES. . . . . . .   .209094E+03
 R-SQUARE . . . . . . . . . . . . . .          .929
 EFFECTIVE NUMBER OF OBSERVATIONS . .           289
 RESIDUAL VARIANCE ESTIMATE . . . . .   .723508E+00
 RESIDUAL STANDARD ERROR. . . . . . .   .850593E+00

 uiden rtf1.maxlag 12

 TIME PERIOD ANALYZED . . . . . . . . .  8  TO   296
 NAME OF THE SERIES . . . . . . . . . .         RTF1
 EFFECTIVE NUMBER OF OBSERVATIONS . . .          289
 STANDARD DEVIATION OF THE SERIES . . .        .8506
 MEAN OF THE (DIFFERENCED) SERIES . . .        .0000
 STANDARD DEVIATION OF THE MEAN . . . .        .0500
 T-VALUE OF MEAN (AGAINST ZERO) . . . .        .0000

 AUTOCORRELATIONS

  1- 12      .89  .70  .49  .31  .18  .10  .04  .00 -.02 -.01  .00
-.01
  ST.E.      .06  .09  .11  .12  .12  .12  .12  .12  .12  .12  .12
.12
   Q         229  372  444  472  482  485  485  485  485  485  485
485


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
  1    .89                      +   IXX+XXXXXXXXXXXXXXXXXX
  2    .70                    +     IXXXX+XXXXXXXXXXX
  3    .49                    +     IXXXX+XXXXXXX
  4    .31                   +      IXXXXX+XX
  5    .18                   +      IXXXXX+
  6    .10                   +      IXX   +
  7    .04                   +      IX    +
  8    .00                   +      I     +
  9   -.02                   +      I     +
 10   -.01                   +      I     +
 11    .00                   +      I     +
 12   -.01                   +      I     +


 PARTIAL AUTOCORRELATIONS


  1- 12      .89 -.38 -.13  .04  .05 -.01 -.03 -.01  .03  .08 -.06
-.12
  ST.E.      .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06


           -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
  1    .89                      +   IXX+XXXXXXXXXXXXXXXXXX
```

```
   2   -.38                 XXXXXXX+XXI  +
   3   -.13                        XXXI  +
   4    .04                        +  IX +
   5    .05                        +  IX +
   6   -.01                        +  I  +
   7   -.03                        + XI  +
   8   -.01                        +  I  +
   9    .03                        +  IX +
  10    .08                        +  IXX+
  11   -.06                        + XI  +
  12   -.12                        XXXI  +
 --


 tsm tfm.change 1/(1,2;g1,g2)noise.no show
 estim tfm

 SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   TFM

  PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE
STD      T
    LABEL       NAME    DENOM.                TRAINT
ERROR  VALUE

   1    C0              CNST      1      0     NONE    53.3871
.1611331.49
   2    W3      GAS     NUM.      1      3     NONE     -.5547
.0761 -7.29
   3    W4      GAS     NUM.      1      4     NONE     -.6434
.0806 -7.98
   4    W5      GAS     NUM.      1      5     NONE     -.8598
.0805-10.68
   5    W6      GAS     NUM.      1      6     NONE     -.4836
.0805 -6.00
   6    W7      GAS     NUM.      1      7     NONE     -.3620
.0759 -4.77
   7    G1      CO2     D-AR      1      1     NONE     1.5416
.0471 32.70
   8    G2      CO2     D-AR      1      2     NONE     -.6322
.0496-12.74

 TOTAL SUM OF SQUARES . . . . . . . .   .302481E+04
 TOTAL NUMBER OF OBSERVATIONS . . . .           296
 RESIDUAL SUM OF SQUARES. . . . . . .   .168090E+02
 R-SQUARE . . . . . . . . . . . . . .          .994
 EFFECTIVE NUMBER OF OBSERVATIONS . .           287
 RESIDUAL VARIANCE ESTIMATE . . . . .   .585680E-01
 RESIDUAL STANDARD ERROR. . . . . . .   .242008E+00
 tsm tfm.change (3,4,5;w3,w4,w5)/(1;d1)gas.no show
 estim tfm.hold resi(rtf2)

 SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   TFM

  PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE
STD      T
    LABEL       NAME    DENOM.                TRAINT
ERROR  VALUE

   1    C0              CNST      1      0     NONE    53.3571
.1434372.08
   2    W3      GAS     NUM.      1      3     NONE     -.5271
.0748 -7.05
   3    W4      GAS     NUM.      1      4     NONE     -.3790
.1029 -3.69
```

```
   4    W5      GAS     NUM.      1      5     NONE     -.5211
.1070 -4.87
   5    D1      GAS     DENM      1      1     NONE      .5495
.0372 14.79
   6    G1      CO2     D-AR      1      1     NONE     1.5308
.0475 32.23
   7    G2      CO2     D-AR      1      2     NONE     -.6313
.0500-12.62


 TOTAL SUM OF SQUARES . . . . . . . .   .302481E+04
 TOTAL NUMBER OF OBSERVATIONS . . . .           296
 RESIDUAL SUM OF SQUARES. . . . . . .   .165232E+02
 R-SQUARE . . . . . . . . . . . . . .          .994
 EFFECTIVE NUMBER OF OBSERVATIONS . .           283
 RESIDUAL VARIANCE ESTIMATE . . . . .   .583860E-01
 RESIDUAL STANDARD ERROR. . . . . . .   .241632E+00
 --


 acf rtf2.maxlag 12

 TIME PERIOD ANALYZED . . . . . . . . . 14  TO   296
 NAME OF THE SERIES . . . . . . . . . .         RTF2
 EFFECTIVE NUMBER OF OBSERVATIONS . . .          283
 STANDARD DEVIATION OF THE SERIES . . .        .2416
 MEAN OF THE (DIFFERENCED) SERIES . . .        .0000
 STANDARD DEVIATION OF THE MEAN . . . .        .0144
 T-VALUE OF MEAN (AGAINST ZERO) . . . .       -.0001


 AUTOCORRELATIONS


   1- 12      .02  .05 -.07 -.06 -.05  .13  .03  .03 -.08  .05  .03
.10
   ST.E.      .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06
    Q          .2  1.0  2.5  3.4  4.3  9.2  9.5  9.8 11.8 12.7 12.9
15.8




          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
   1    .02                      +  IX +
   2    .05                      +  IX +
   3   -.07                      +XXI  +
   4   -.06                      + XI  +
   5   -.05                      + XI  +
   6    .13                      +  IXXX
   7    .03                      +  IX +
   8    .03                      +  IX +
   9   -.08                      +XXI  +
  10    .05                      +  IX +
  11    .03                      +  IX +
  12    .10                      +  IXX+
```

```
ccf ra,rtf2

TIME PERIOD ANALYZED . . . . . . . . . 14  TO   296
NAMES OF THE SERIES  . . . . . . . . .          RA           RTF2
EFFECTIVE NUMBER OF OBSERVATIONS . . .         283            283
STANDARD DEVIATION OF THE SERIES . . .       .1910          .2416
MEAN OF THE (DIFFERENCED) SERIES . . .       .0022          .0000
STANDARD DEVIATION OF THE MEAN . . . .       .0114          .0144
T-VALUE OF MEAN (AGAINST ZERO) . . . .       .1905         -.0001


CORRELATION       BETWEEN     RTF2     AND       RA IS   -.05


CROSS CORRELATION BETWEEN       RA(T)  AND     RTF2(T-L)


  1- 12    -.04  .04 -.05 -.01 -.00 -.10  .01 -.03  .05  .05 -.01
-.02
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 13- 24    -.00 -.02 -.02 -.05 -.02  .02  .08 -.05  .03 -.08 -.07
-.05
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 25- 36     .04  .08  .02 -.07  .04 -.01 -.03  .09 -.00 -.12  .07
-.01
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06


CROSS CORRELATION BETWEEN     RTF2(T)  AND       RA(T-L)


  1- 12     .03 -.01  .00  .00  .01 -.00 -.05  .03  .08 -.03 -.02
-.03
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 13- 24    -.11  .01  .05  .04  .01  .01 -.15 -.03 -.07 -.08  .02
-.02
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06

 25- 36     .02  .06 -.07 -.00  .03 -.16  .02  .03 -.02  .01 -.04
.03
  ST.E.     .06  .06  .06  .06  .06  .06  .06  .06  .06  .06  .06
.06


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
-36    .03                        +  IX +
-35   -.04                        + XI  +
-34    .01                        +  I  +
-33   -.02                        + XI  +
-32    .03                        +  IX +
-31    .02                        +  IX +
-30   -.16                       X+XXI  +
-29    .03                        +  IX +
```

```
-28    .00                        +  I  +
-27   -.07                        +XXI  +
-26    .06                        +  IXX+
-25    .02                        +  IX +
-24   -.02                        +  I  +
-23    .02                        +  IX +
-22   -.08                        +XXI  +
-21   -.07                        +XXI  +
-20   -.03                        + XI  +
-19   -.15                       X+XXI  +
-18    .01                        +  I  +
-17    .01                        +  I  +
-16    .04                        +  IX +
-15    .05                        +  IX +
-14    .01                        +  I  +
-13   -.11                        XXXI  +
-12   -.03                        + XI  +
-11   -.02                        + XI  +
-10   -.03                        + XI  +
 -9    .08                        +  IXX+
 -8    .03                        +  IX +
 -7   -.05                        + XI  +
 -6    .00                        +  I  +
 -5    .01                        +  I  +
 -4    .00                        +  I  +
 -3    .00                        +  I  +
 -2   -.01                        +  I  +
 -1    .03                        +  IX +
  0   -.05                        + XI  +
  1   -.04                        + XI  +
  2    .04                        +  IX +
  3   -.05                        + XI  +
  4   -.01                        +  I  +
  5    .00                        +  I  +
  6   -.10                        +XXI  +
  7    .01                        +  I  +
  8   -.03                        + XI  +
  9    .05                        +  IX +
 10    .05                        +  IX +
 11   -.01                        +  I  +
 12   -.02                        + XI  +
 13    .00                        +  I  +
 14   -.02                        + XI  +
 15   -.02                        +  I  +
 16   -.05                        + XI  +
 17   -.02                        + XI  +
 18    .02                        +  I  +
 19    .08                        +  IXX+
 20   -.05                        + XI  +
 21    .03                        +  IX +
 22   -.08                        +XXI  +
 23   -.07                        +XXI  +
 24   -.05                        + XI  +
 25    .04                        +  IX +
 26    .08                        +  IXX+
 27    .02                        +  I  +
 28   -.07                        +XXI  +
 29    .04                        +  IX +
 30   -.01                        +  I  +
 31   -.03                        + XI  +
 32    .09                        +  IXX+
 33    .00                        +  I  +
 34   -.12                        XXXI  +
 35    .07                        +  IXX+
```

```
  36   -.01                          +  I
```

See Tiao and Box(JASA,1981) for discussion.

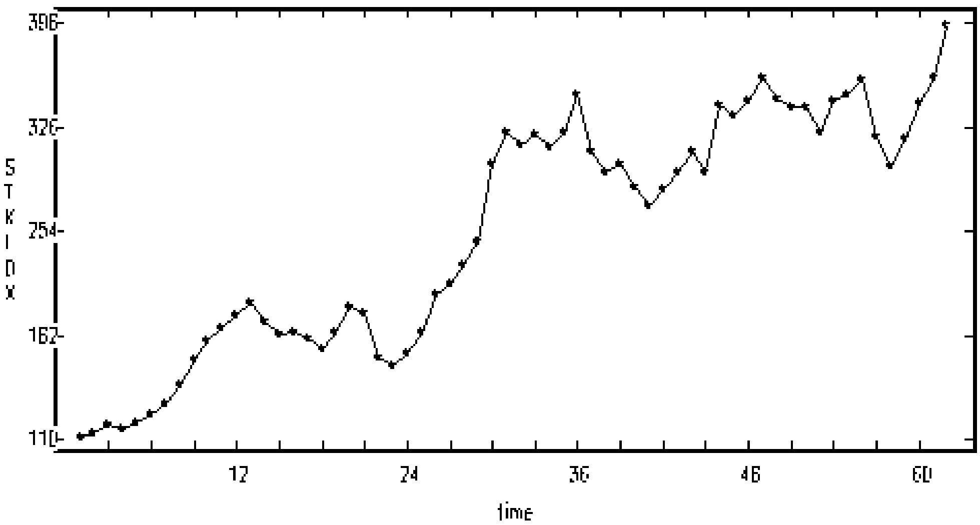
EXAMPLE 1. THE FINANCIAL TIMES DATA

This data set consists of three quarterly series 3/52 to 4/67:

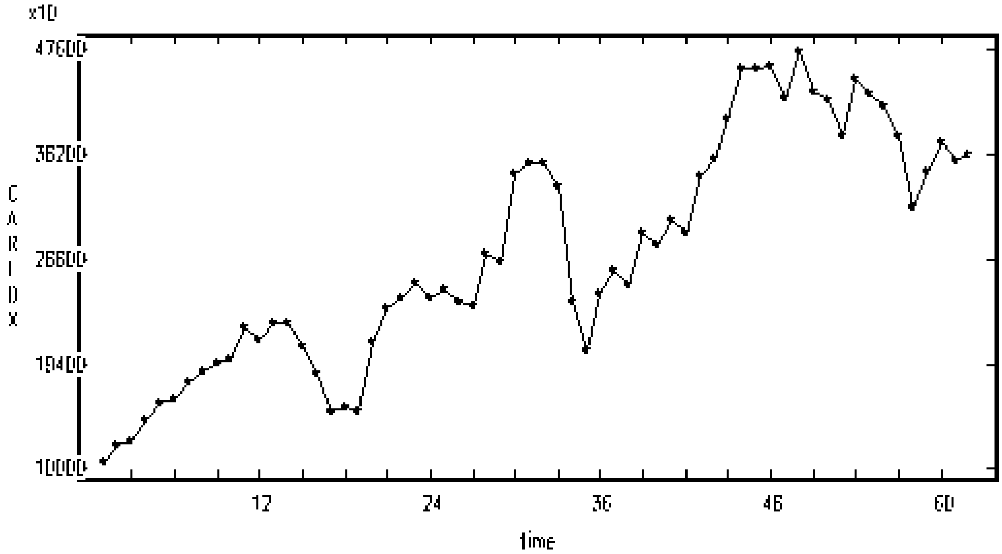
> ***stkidx--the Financial Times stock index***
> ***caridx--the car production index***
> ***comidx--the commodity index***

In a paper by Coen,Gomme and Kendall(JRSSA,1969), an ordinary linear regression model is used to relate stkidx to caridx lagged 6 periods and comidx lagged 7 periods. The model gives a high R^2 value and all the coefficients are highly significant. The results have led to a great deal of criticisms in the literature. See e.g. Box and Newbold(JRSSA,1971). We now analyze the data via regression and multiple time series techniques.

```
 input stkidx.file'mts1.dat'.dataset orgcgk1

 STKIDX  , A   62  BY   1 VARIABLE, IS STORED IN THE WORKSPACE

 input caridx.file'mts1.dat'.dataset orgcgk2

 CARIDX  , A   62  BY   1 VARIABLE, IS STORED IN THE WORKSPACE

 input comidx.file'mts1.dat'.dataset orgcgk3

 COMIDX  , A   62  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
 --                                                                    fsave
stkidx,caridx,comidx.file'cgk.fmt'
```

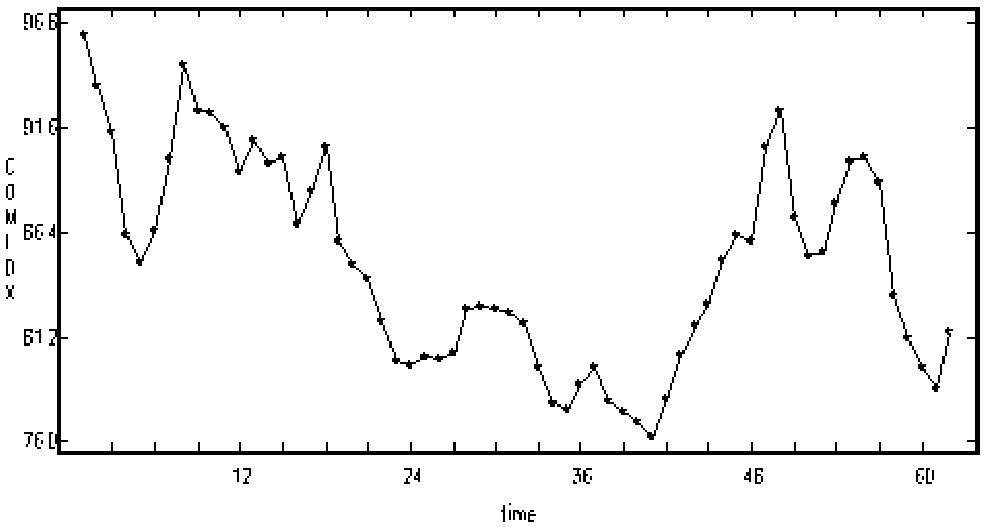


1



2

```
lag caridx.new caridx6.lag 6
THE ORIGINAL SERIES IS CARIDX
THE LAG  6 SERIES IS STORED IN VARIABLE CARIDX6 ,  WHICH HAS   68  ENTRIES
```




```
--

lag comidx.new comidx7.lag 7
THE ORIGINAL SERIES IS COMIDX
THE LAG  7 SERIES IS STORED IN VARIABLE COMIDX7 ,  WHICH HAS   69  ENTRIES
print stkidx,caridx6,comidx7


STKIDX   IS  A    62  BY     1  VARIABLE
CARIDX6  IS  A    68  BY     1  VARIABLE
COMIDX7  IS  A    69  BY     1  VARIABLE
```

3

VARIABLE	STKIDX	CARIDX6	COMIDX7
COLUMN-->	1	1	1
ROW			
1	112.700	***	***
2	115.000	***	***
3	121.400	***	***
4	118.400	***	***
5	122.700	***	***
6	128.800	***	***
7	135.700	.11E+06	***
8	148.500	.12E+06	96.210
9	165.400	.13E+06	93.740
10	178.500	.15E+06	91.370
11	187.300	.16E+06	86.310
12	195.900	.16E+06	84.980
13	205.200	.18E+06	86.460
14	191.500	.19E+06	90.040
15	183.000	.20E+06	94.740
16	184.500	.20E+06	92.430
17	181.100	.23E+06	92.410
18	173.700	.22E+06	91.650
19	185.400	.23E+06	89.380
20	201.800	.23E+06	91.050
21	198.000	.21E+06	89.890
22	168.000	.19E+06	90.160
23	161.600	.15E+06	86.780
24	170.200	.16E+06	88.450
25	184.500	.15E+06	90.690
26	211.000	.21E+06	86.030
27	218.300	.24E+06	84.850
28	231.700	.25E+06	84.070
29	247.400	.27E+06	81.960
30	301.900	.25E+06	80.030
31	323.800	.26E+06	79.800
32	314.100	.25E+06	80.190
33	321.000	.25E+06	80.130
34	312.900	.29E+06	80.420
35	323.700	.29E+06	82.670
36	349.300	.37E+06	82.780
37	310.400	.37E+06	82.610
38	295.800	.38E+06	82.470
39	301.200	.35E+06	81.860
40	285.800	.25E+06	79.700
41	271.700	.21E+06	77.890
42	283.600	.26E+06	77.610
43	295.700	.28E+06	78.900
44	309.300	.26E+06	79.720
45	295.700	.31E+06	78.080
46	342.000	.30E+06	77.540
47	335.100	.32E+06	76.990
48	344.400	.31E+06	76.250
49	360.900	.36E+06	78.130
50	346.500	.38E+06	80.380
51	340.600	.41E+06	81.780
52	340.300	.46E+06	82.810
53	323.300	.46E+06	84.990
54	345.600	.46E+06	86.310

```
   55      349.300   .43E+06    85.950
   56      359.700   .48E+06    90.730
   57      320.000   .44E+06    92.420
   58      299.900   .43E+06    87.180
   59      318.500   .40E+06    85.200
   60      343.100   .45E+06    85.440
   61      360.800   .44E+06    87.850
   62      397.800   .43E+06    89.950
   63                .40E+06    90.200
   64                .33E+06    88.890
   65                .37E+06    83.250
   66                .39E+06    81.210
   67                .38E+06    79.700
   68                .38E+06    78.700
   69                           81.500

regr stkidx,caridx6,comidx7.hold resi(rstk).dw.span 8,62


REGRESSION ANALYSIS FOR THE VARIABLE   STKIDX

PREDICTOR       COEFFICIENT    STD. ERROR     T-VALUE
INTERCEPT        594.51062      60.65120        9.80
 CARIDX6            .00051        .00003       15.10
 COMIDX7          -5.54391        .67274       -8.24

CORRELATION MATRIX OF REGRESSION COEFFICIENTS

 CARIDX6      1.00
 COMIDX7       .26     1.00
           CARIDX6  COMIDX7

S =       25.0618      R**2 =  88.2%      R**2(ADJ) =  87.8%



---------------------------
ANALYSIS OF VARIANCE TABLE
---------------------------

  SOURCE     SUM OF SQUARES    DF    MEAN SQUARE     F-RATIO
REGRESSION       244274.076     2     122137.038     194.457
 RESIDUAL         32660.808    52        628.092
ADJ. TOTAL       276934.884    54

  SOURCE      SEQUENTIAL SS    DF    MEAN SQUARE     F-RATIO
  CARIDX6        201620.490     1     201620.490     321.004
  COMIDX7         42653.574     1      42653.574      67.910

DURBIN-WATSON STATISTIC =  .87


uiden rstk.maxlag 10

TIME PERIOD ANALYZED . . . . . . . . . .  1  TO    55
NAME OF THE SERIES . . . . . . . . . . .         RSTK
EFFECTIVE NUMBER OF OBSERVATIONS . . .             55
```

```
STANDARD DEVIATION OF THE SERIES . . .      24.3687
MEAN OF THE (DIFFERENCED) SERIES . . .        .0000
STANDARD DEVIATION OF THE MEAN . . . .       3.2859
T-VALUE OF MEAN (AGAINST ZERO) . . . .        .0000


AUTOCORRELATIONS


  1- 10     .45  .09 -.11 -.17  .05  .01 -.09 -.13 -.22 -.10
  ST.E.     .13  .16  .16  .16  .17  .17  .17  .17  .17  .17
   Q       11.9 12.4 13.2 14.9 15.1 15.1 15.6 16.8 20.2 20.9


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .45                    +      IXXXXXX+XXXX
  2    .09                   +       IXX     +
  3   -.11                   +    XXXI       +
  4   -.17                   +   XXXXI       +
  5    .05                   +       IX      +
  6    .01                   +       I       +
  7   -.09                   +     XXI       +
  8   -.13                   +    XXXI       +
  9   -.22                   + XXXXXXI       +
 10   -.10                  +     XXXI        +

PARTIAL AUTOCORRELATIONS


  1- 10     .45 -.14 -.12 -.07  .22 -.17 -.11 -.03 -.13  .00
  ST.E.     .13  .13  .13  .13  .13  .13  .13  .13  .13  .13


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .45                    +      IXXXXXX+XXXX
  2   -.14                    +  XXXXI      +
  3   -.12                    +   XXXI      +
  4   -.07                    +    XXI      +
  5    .22                    +      IXXXXX +
  6   -.17                    +  XXXXI      +
  7   -.11                    +   XXXI      +
  8   -.03                    +     XI      +
  9   -.13                    +   XXXI      +
 10    .00                    +      I      +
--

tsm stk.model stkidx=c0+(c1)caridx6+(c2)comidx7+1/(1)noise.no show

estim stk.hold resi(rrstk).span 8,62

SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   STK
```

```
----------------------------------------------------------------------
VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
          VARIABLE   OR CENTERED

 STKIDX    RANDOM     ORIGINAL     NONE

CARIDX6    RANDOM     ORIGINAL     NONE

COMIDX7    RANDOM     ORIGINAL     NONE
----------------------------------------------------------------------


 PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
   LABEL       NAME    DENOM.                TRAINT               ERROR  VALUE

  1    C0              CNST      1      0     NONE   365.2028  168.2835  2.17
  2    C1    CARIDX6   NUM.      1      0     NONE      .0002 .8165E-04  1.85
  3    C2    COMIDX7   NUM.      1      0     NONE    -.6005    1.1973  -.50
  4           STKIDX   D-AR      1      1     NONE      .9584     .0459 20.87


TOTAL SUM OF SQUARES . . . . . . . .   .276935E+06
TOTAL NUMBER OF OBSERVATIONS . . . .            55
RESIDUAL SUM OF SQUARES. . . . . . .   .169105E+05
R-SQUARE . . . . . . . . . . . . . .          .938
EFFECTIVE NUMBER OF OBSERVATIONS . .            54
RESIDUAL VARIANCE ESTIMATE . . . . .   .313158E+03
RESIDUAL STANDARD ERROR. . . . . . .   .176963E+02
--


acf rrstk.maxlag 10

TIME PERIOD ANALYZED . . . . . . . . .  2  TO    55
NAME OF THE SERIES . . . . . . . . . .         RRSTK
EFFECTIVE NUMBER OF OBSERVATIONS . . .            54
STANDARD DEVIATION OF THE SERIES . . .       17.6963
MEAN OF THE (DIFFERENCED) SERIES . . .         .0001
STANDARD DEVIATION OF THE MEAN . . . .        2.4082
T-VALUE OF MEAN (AGAINST ZERO) . . . .         .0001


AUTOCORRELATIONS


  1- 10      .12 -.03 -.01 -.11  .07  .03 -.14 -.07 -.31  .04
  ST.E.      .14  .14  .14  .14  .14  .14  .14  .14  .14  .16
   Q          .8   .8   .8  1.6  1.9  2.0  3.1  3.4  9.9 10.0


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .12                     +     IXXX   +
  2   -.03                     +    XI      +
```

7

```
  3   -.01                   +      I      +
  4   -.11                   +   XXXI      +
  5    .07                   +      IXX    +
  6    .03                   +      IX     +
  7   -.14                   +   XXXI      +
  8   -.07                   +    XXI      +
  9   -.31                  X+XXXXXXI      +
 10    .04                  +       IX      +
```

We see that once the AR(1) model is formulated for the error term of the regression, the residual standard error is vastly reduced and the importance of caridx6 and comidx7 essentially disappears.

```
miden stkidx,caridx,comidx.arfits 1 to 5

TIME PERIOD ANALYZED . . . . . . . . . . . . .  1  TO    62
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .             62


SERIES   NAME           MEAN       STD. ERROR

   1    STKIDX        254.0468     81.7844
   2    CARIDX     293205.6290 103368.4058
   3    COMIDX         84.8687      5.1289


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .12700


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
   .88  1.00
  -.40  -.24  1.00

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2              3


 1   ++++++++++++   ++++++++++++   ------------
 1   +++++.......   ++..........   ---------...


 2   ++++++++++++   ++++++++++++   ------------
 2   ++++++......   +++.........   ------------
```

```
 3   -------.....   ............   ++++++++....
 3   ...+++++++++   .....+++++++   .........---


CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     + + -      + + -      + + -      + + -      + + -      + + -
     + + -      + + -      + + -      + + -      + + -      + + -
     - . +      - . +      - . +      - . +      - . +      - . +

LAGS  7 THROUGH 12

     + + -      + + -      + + -      + + -      + + -      + + -
     + + -      + + -      + + -      + + -      + + -      + + -
     - . +      . . +      . . .      . . .      . . .      . . .

LAGS 13 THROUGH 18

     + + -      + + -      + . -      + . -      + . -      . . -
     + + -      + + -      + + -      + . -      + . -      + . -
     . . .      . . .      . . .      + . .      + . .      + + .

LAGS 19 THROUGH 24

     . . -      . . -      . . -      . . .      . . .      . . .
     . . -      . . -      . . -      . . -      . . -      . . -
     + + .      + + .      + + .      + + -      + + -      + + -

DETERMINANT OF S(0) =  .278189E+15

NOTE: S(0) IS THE SAMPLE COVARIANCE MATRIX OF W(MAXLAG+1),...,W(NOBE)


==========  STEPWISE AUTOREGRESSION SUMMARY  ==========

-------------------------------------------------------------------
    I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR COEFF.
----+----------+----------+---------+----------+----------------------
  1 I .296E+03 I .378E+01 I  301.28 I   27.811 I + . .
    I .946E+09 I .250E+03 I         I          I . + .
    I .426E+01 I .946E+09 I         I          I . . +
----+----------+----------+---------+----------+----------------------
  2 I .269E+03 I .302E+01 I   18.64 I   27.725 I - . .
    I .892E+09 I .228E+03 I         I          I . . .
    I .323E+01 I .892E+09 I         I          I - + -
----+----------+----------+---------+----------+----------------------
  3 I .245E+03 I .297E+01 I    9.57 I   27.809 I . . .
    I .861E+09 I .196E+03 I         I          I . . .
    I .319E+01 I .861E+09 I         I          I . . .
----+----------+----------+---------+----------+----------------------
  4 I .243E+03 I .288E+01 I    3.60 I   28.017 I . . .
```

9

```
    I .841E+09 I .190E+03 I         I          I . . .
    I .313E+01 I .841E+09 I         I          I . . .
----+----------+----------+---------+----------+---------------------
  5 I .213E+03 I .262E+01 I   11.93 I   28.013 I . + .
    I .749E+09 I .175E+03 I         I          I . + .
    I .292E+01 I .749E+09 I         I          I . . .
---------------------------------------------------------------------
```

```
NOTE:  CHI-SQUARED CRITICAL VALUES WITH   9  DEGREES OF FREEDOM ARE

             5 PERCENT:  16.9     1 PERCENT:  21.7

NOTE:  THE PARTIAL AUTOREGRESSION COEFFICIENT MATRIX FOR LAG L IS THE
       ESTIMATED PHI(L) FROM THE FIT WHERE THE MAXIMUM LAG USED IS L
       (I.E. THE LAST COEFFICIENT MATRIX). THE ELEMENTS ARE
       STANDARDIZED BY DIVIDING EACH BY ITS STANDARD ERROR.
```

It is clear from the partial autoregression table (STEPWISE AUTOREGRESSION) that these three series are basically uncoupled. In particular, the stkidx is not related to past values of caridx and comidx.

10

```
CLASSAR AND CLASSMA IDEN

 input y1,y2.file '\b521\classar.dat'

 Y1      , A  150  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
 Y2      , A  150  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
 --


 miden y1,y2.max 24.arfit 1 to 5

 TIME PERIOD ANALYZED . . . . . . . . . . . . .  1  TO   150
 EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .         150


 SERIES   NAME           MEAN       STD. ERROR

    1    Y1              4.9624      2.7685
    2    Y2             10.0616      3.0082


 NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
       IS (1/NOBE**.5) =     .08165


 SAMPLE CORRELATION MATRIX OF THE SERIES

   1.00
    .56  1.00

 SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
      + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
      - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
      . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


 BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
 ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


           1              2


  1   +.....--....   ++++++......
  1   ............   ............


  2   ...-..-.....   +++++.......
  2   ............   ............



 CROSS CORRELATION MATRICES IN TERMS OF +,-,.

 LAGS  1 THROUGH  6

      + +      . +      . +      . +      . +      . +
      . +      . +      . +      - +      . +      . .

 LAGS  7 THROUGH 12

      - .      - .      . .      . .      . .      . .
      - .      . .      . .      . .      . .      . .

 LAGS 13 THROUGH 18

      . .      . .      . .      . .      . .      . .
      . .      . .      . .      . .      . .      . .
```

```
LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

DETERMINANT OF S(0) =  .494426E+02

NOTE: S(0) IS THE SAMPLE COVARIANCE MATRIX OF W(MAXLAG+1),...,W(NOBE)


==========  PARTIAL AUTOREGRESSION SUMMARY  ==========

----------------------------------------------------------------------
    I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR COEFF.
----+----------+----------+---------+----------+----------------------
  1 I .530E+01 I .699E+00 I  356.96 I    1.431 I . +
    I .108E+01 I .568E+01 I         I          I - +
----+----------+----------+---------+----------+----------------------
  2 I .516E+01 I .685E+00 I    7.04 I    1.434 I . .
    I .103E+01 I .550E+01 I         I          I . +
----+----------+----------+---------+----------+----------------------
  3 I .507E+01 I .684E+00 I    2.63 I    1.469 I . .
    I .103E+01 I .541E+01 I         I          I . .
----+----------+----------+---------+----------+----------------------
  4 I .501E+01 I .668E+00 I    4.38 I    1.490 I . .
    I .102E+01 I .536E+01 I         I          I . .
----+----------+----------+---------+----------+----------------------
  5 I .495E+01 I .664E+00 I    2.42 I    1.525 I . .
    I .101E+01 I .530E+01 I         I          I . .
----------------------------------------------------------------------


NOTE:  CHI-SQUARED CRITICAL VALUES WITH   4  DEGREES OF FREEDOM ARE

             5 PERCENT:   9.5     1 PERCENT:  13.3


NOTE:  THE PARTIAL AUTOREGRESSION COEFFICIENT MATRIX FOR LAG L IS THE
       ESTIMATED PHI(L) FROM THE FIT WHERE THE MAXIMUM LAG USED IS L
       (I.E. THE LAST COEFFICIENT MATRIX). THE ELEMENTS ARE
       STANDARDIZED BY DIVIDING EACH BY ITS STANDARD ERROR.
--


input x1,x2.file '\b521\classma.dat'

X1       , A  250  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
X2       , A  250  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
--


miden x1,x2.max 24.arfits 1 to 8

TIME PERIOD ANALYZED . . . . . . . . . . . . .  1  TO   250
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .            250


SERIES   NAME           MEAN       STD. ERROR

   1    X1             17.0664      2.3334
   2    X2             25.0393      1.5875


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .06325
```

```
SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
   .30  1.00

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   -...........   -.....-.....
 1   ............   ...........+


 2   +...........   -......-....
 2   ............   ............


CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     - -      . .      . .      . .      . .      . .
     + -      . .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     . -      . .      . .      . .      . .      . .
     . .      . -      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . +
     . .      . .      . .      . .      . .      . .

DETERMINANT OF S(0) =  .121867E+02

NOTE: S(0) IS THE SAMPLE COVARIANCE MATRIX OF W(MAXLAG+1),...,W(NOBE)


==========  PARTIAL AUTOREGRESSION SUMMARY  ==========

-------------------------------------------------------------------
    I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR COEFF.
----+----------+----------+---------+----------+---------------------
  1 I .478E+01 I .138E+01 I  123.18 I    2.016 I - -
    I .188E+01 I .528E+01 I         I          I + -
----+----------+----------+---------+----------+---------------------
  2 I .475E+01 I .102E+01 I   75.86 I    1.727 I . .
    I .143E+01 I .515E+01 I         I          I + -
```

```
----+----------+----------+---------+----------+----------------------
  3 I .463E+01 I .918E+00 I   35.17 I    1.609 I + .
    I .123E+01 I .495E+01 I         I          I + -
----+----------+----------+---------+----------+----------------------
  4 I .463E+01 I .785E+00 I   37.46 I    1.480 I . .
    I .108E+01 I .492E+01 I         I          I + .
----+----------+----------+---------+----------+----------------------
  5 I .461E+01 I .732E+00 I   16.58 I    1.440 I . .
    I .104E+01 I .492E+01 I         I          I + .
----+----------+----------+---------+----------+----------------------
  6 I .453E+01 I .705E+00 I   13.47 I    1.413 I . .
    I .983E+00 I .481E+01 I         I          I + .
----+----------+----------+---------+----------+----------------------
  7 I .438E+01 I .679E+00 I   16.51 I    1.372 I . -
    I .942E+00 I .464E+01 I         I          I + .
----+----------+----------+---------+----------+----------------------
  8 I .431E+01 I .668E+00 I    8.05 I    1.368 I . .
    I .913E+00 I .455E+01 I         I          I . -
----------------------------------------------------------------------

NOTE:  CHI-SQUARED CRITICAL VALUES WITH   4  DEGREES OF FREEDOM ARE

             5 PERCENT:   9.5     1 PERCENT:  13.3


NOTE:  THE PARTIAL AUTOREGRESSION COEFFICIENT MATRIX FOR LAG L IS THE
       ESTIMATED PHI(L) FROM THE FIT WHERE THE MAXIMUM LAG USED IS L
       (I.E. THE LAST COEFFICIENT MATRIX). THE ELEMENTS ARE
       STANDARDIZED BY DIVIDING EACH BY ITS STANDARD ERROR.
--


stop
```

CLASS AR EXAMPLE

```
input y1,y2.file'\b521\classar.dat'

Y1       , A  150  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
Y2       , A  150  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
--


acf y1.max 24.output noprint(value,lbq)

NAME OF THE SERIES . . . . . . . . . .           Y1
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   150
MEAN OF THE (DIFFERENCED) SERIES . . .       4.9624
STANDARD DEVIATION OF THE SERIES . . .       2.7685
T-VALUE OF MEAN (AGAINST ZERO) . . . .      21.9527


AUTOCORRELATIONS


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .39                       +   IXXX+XXXXXX
  2    .08                      +    IXX  +
  3    .02                      +    IX   +
  4   -.08                      +  XXI    +
  5   -.04                      +   XI    +
  6    .02                      +    IX   +
  7   -.19                      XXXXXI    +
  8   -.22                     X+XXXXI    +
  9   -.10                      + XXXI    +
 10   -.03                      +   XI    +
 11    .04                      +    IX   +
 12    .07                      +    IXX  +
 13    .00                      +    I    +
 14   -.03                      +   XI    +
 15    .00                      +    I    +
 16    .07                      +    IXX  +
 17    .08                      +    IXX  +
 18    .05                      +    IX   +
 19    .08                      +    IXX  +
 20   -.01                      +    I    +
```

```
 21   -.05                     +   XI    +
 22   -.06                     +   XI    +
 23   -.06                     +  XXI    +
 24   -.03                     +   XI    +
--
```

```
acf y2.max 24.output noprint(value,lbq)

NAME OF THE SERIES . . . . . . . . . .            Y2
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   150
MEAN OF THE (DIFFERENCED) SERIES . . .       10.0616
STANDARD DEVIATION OF THE SERIES . . .        3.0082
T-VALUE OF MEAN (AGAINST ZERO) . . . .       40.9638


AUTOCORRELATIONS


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .81                       +   IXXX+XXXXXXXXXXXXXXXXX
  2    .64                     +     IXXXXX+XXXXXXXXXXX
  3    .47                    +      IXXXXXX+XXXXX
  4    .32                   +       IXXXXXXXX
  5    .23                   +       IXXXXXX +
  6    .16                   +       IXXXX   +
  7    .05                   +       IX      +
  8   -.01                   +       I       +
  9   -.04                   +      XI       +
 10   -.06                   +      XI       +
 11   -.03                   +      XI       +
 12    .00                   +       I       +
 13    .00                   +       I       +
 14    .00                   +       I       +
 15    .01                   +       I       +
 16    .00                   +       I       +
 17    .00                   +       I       +
 18    .01                   +       I       +
 19    .00                   +       I       +
 20   -.02                   +      XI       +
 21   -.05                   +      XI       +
```

```
 22   -.09                 +     XXI       +
 23   -.11                 +    XXXI       +
 24   -.08                 +     XXI       +
--


ccf y1,y2.max 24. output noprint(value)

TIME PERIOD ANALYZED . . . . . . . . .  1  TO   150
NAMES OF THE SERIES  . . . . . . . . .          Y1            Y2
EFFECTIVE NUMBER OF OBSERVATIONS . . .         150           150
STANDARD DEVIATION OF THE SERIES . . .      2.7685        3.0082
MEAN OF THE (DIFFERENCED) SERIES . . .      4.9624       10.0616
STANDARD DEVIATION OF THE MEAN . . . .       .2261         .2456
T-VALUE OF MEAN (AGAINST ZERO) . . . .     21.9527       40.9638



CORRELATION       BETWEEN       Y2     AND       Y1 IS    .56


CROSS CORRELATION BETWEEN       Y1(T)  AND       Y2(T-L)



          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
-24   -.07                     + XXI   +
-23   -.11                     +XXXI   +
-22   -.14                     +XXXI   +
-21   -.13                     +XXXI   +
-20   -.11                     +XXXI   +
-19   -.03                     +  XI   +
-18    .00                     +   I   +
-17    .02                     +   IX  +
-16    .05                     +   IX  +
-15    .05                     +   IX  +
-14    .03                     +   IX  +
-13    .03                     +   IX  +
-12    .06                     +   IX  +
-11    .05                     +   IX  +
-10    .00                     +   I   +
 -9   -.06                     +  XI   +
```

```
 -8   -.14                     XXXXI   +
 -7   -.19                    X+XXXI   +
 -6   -.11                     +XXXI   +
 -5   -.15                     XXXXI   +
 -4   -.19                    X+XXXI   +
 -3   -.16                     XXXXI   +
 -2   -.12                     +XXXI   +
 -1    .06                     +   IXX +
  0    .56                     +   IXXX+XXXXXXXXXXX
  1    .54                     +   IXXX+XXXXXXXXXX
  2    .49                     +   IXXX+XXXXXXXXX
  3    .42                     +   IXXX+XXXXXXXX
  4    .29                     +   IXXX+XXX
  5    .25                     +   IXXX+XX
  6    .23                     +   IXXX+XX
  7    .10                     +   IXX +
  8    .02                     +   IX  +
  9   -.02                     +   I   +
 10   -.05                     +  XI   +
 11   -.01                     +   I   +
 12    .03                     +   IX  +
 13    .04                     +   IX  +
 14    .01                     +   I   +
 15    .01                     +   I   +
 16    .01                     +   I   +
 17    .03                     +   IX  +
 18    .05                     +   IX  +
 19    .05                     +   IX  +
 20    .05                     +   IX  +
 21    .04                     +   IX  +
 22    .00                     +   I   +
 23   -.04                     +  XI   +
 24   -.03                     +  XI   +
--


miden y1,y2.max 24.desc.output level(detailed).@
arfit 1 to 5.rccm 1,2

TIME PERIOD ANALYZED . . . . . . . . . . . . .   1   TO    150
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .               150


SERIES    NAME            MEAN        STD. ERROR
```

```
   1    Y1              4.9624      2.7685
   2    Y2             10.0616      3.0082


 SAMPLE COVARIANCE MATRIX OF THE SERIES

   .766E+01
   .462E+01   .905E+01


EIGENVALUES OF THE COVARIANCE MATRIX
      3.681     13.033


EIGENVECTORS OF THE COVARIANCE MATRIX

       .758       .653
      -.653       .758


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .08165


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
   .56  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE ORIGINAL SERIES.
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

   .39   .54
   .06   .81

LAG = 2

   .08   .49
```

```
  -.12   .64

LAG = 3

   .02   .42
  -.16   .47

LAG = 4

  -.08   .29
  -.19   .32

LAG = 5

  -.04   .25
  -.15   .23

LAG = 6

   .02   .23
  -.11   .16

LAG = 7

  -.19   .10
  -.19   .05

LAG = 8

  -.22   .02
  -.14  -.01

LAG = 9

  -.10  -.02
  -.06  -.04

LAG =10

  -.03  -.05
   .00  -.06

LAG =11
```

```
   .04  -.01
   .05  -.03

LAG =12

   .07   .03
   .06   .00

LAG =13

   .00   .04
   .03   .00

LAG =14

  -.03   .01
   .03   .00

LAG =15

   .00   .01
   .05   .01

LAG =16

   .07   .01
   .05   .00

LAG =17

   .08   .03
   .02   .00

LAG =18

   .05   .05
   .00   .01

LAG =19

   .08   .05
  -.03   .00

LAG =20
```

```
  -.01   .05
  -.11  -.02

LAG =21

  -.05   .04
  -.13  -.05

LAG =22

  -.06   .00
  -.14  -.09

LAG =23

  -.06  -.04
  -.11  -.11

LAG =24

  -.03  -.03
  -.07  -.08

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   +.....--....   ++++++......
 1   ............   ............


 2   ...-..-.....   +++++.......
 2   ............   ............
```

```
CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     + +      . +      . +      . +      . +      . +
     . +      . +      . +      - +      . +      . .

LAGS  7 THROUGH 12

     - .      - .      . .      . .      . .      . .
     - .      . .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

DETERMINANT OF S(0) =  .494426E+02

NOTE: S(0) IS THE SAMPLE COVARIANCE MATRIX OF W(MAXLAG+1),...,W(NOBE)


AUTOREGRESSIVE FITTING ON LAG(S)  1


=== PHI( 1) ===

     .140     .436      . +
    -.604    1.119      - +

STANDARD ERRORS

     .082     .075
     .037     .034
```

```
RESIDUAL COVARIANCE MATRIX S( 1)

  .530E+01
  .132E+01  .108E+01


RESIDUAL CORRELATION MATRIX RS( 1)

  1.00
   .55  1.00


EIGENVALUES AND EIGENVECTORS OF S( 1)

EIGENVALUES

      .699     5.678

EIGENVECTORS

     -.277      .961
      .961      .277


DETERMINANT OF S(J) =  .396750E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =  356.96
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                                  -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2
  1   1.00
  2   -.55  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.
```

```
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

  -.01  -.15
  -.04  -.19

LAG = 2

  -.05   .03
   .01   .03

LAG = 3

   .08   .12
   .01  -.03

LAG = 4

  -.03   .00
  -.05   .08

LAG = 5

  -.02  -.03
  -.05  -.05

LAG = 6

   .18   .10
   .13   .13

LAG = 7

  -.17  -.18
  -.10  -.07

LAG = 8

  -.16   .02
  -.11  -.05
```

```
LAG = 9

  -.01   .06
  -.06   .09

LAG =10

   .02  -.01
   .05  -.07

LAG =11

   .00  -.08
   .04  -.03

LAG =12

   .07   .01
   .04  -.05

LAG =13

  -.06  -.02
  -.01   .08

LAG =14

  -.05   .00
  -.06  -.02

LAG =15

  -.06   .04
   .02   .08

LAG =16

   .04   .06
  -.05   .02

LAG =17

   .04  -.01
  -.01  -.10
```

```
LAG =18

  -.03   .09
   .02   .06

LAG =19

   .10   .03
  -.02  -.07

LAG =20

   .00   .04
  -.09   .04

LAG =21

  -.03   .03
  -.02  -.04

LAG =22

   .03   .13
  -.05   .03

LAG =23

   .02  -.05
   .01  -.08

LAG =24

  -.04   .00
   .05   .09

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I
```

```
          1              2


 1   .....+-.....   ......-.....
 1   ............   ............


 2   ............   -...........
 2   ............   ............



CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     . .      . .      . .      . .      . .      + .
     . -      . .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     - -      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .


AUTOREGRESSIVE FITTING ON LAG(S)  1  2


=== PHI( 1) ===

     .210     .037      + .
    -.581     .909      - +
```

```
STANDARD ERRORS

     .097     .216
     .044     .097


=== PHI( 2) ===

    -.272     .464      . .
    -.119     .251      . +

STANDARD ERRORS

     .162     .234
     .073     .105


RESIDUAL COVARIANCE MATRIX S( 2)

  .516E+01
  .125E+01  .103E+01


RESIDUAL CORRELATION MATRIX RS( 2)

  1.00
   .54  1.00


EIGENVALUES AND EIGENVECTORS OF S( 2)

EIGENVALUES

      .685     5.505

EIGENVECTORS

     -.269      .963
      .963      .269


DETERMINANT OF S(J) =  .377215E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =    7.04
```

```
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.



                                          -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2     3     4
  1   1.00
  2   -.55  1.00
  3   -.53   .86  1.00
  4    .39  -.94  -.88  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

   .02  -.01
   .02  -.02

LAG = 2

  -.03   .02
  -.05  -.05

LAG = 3

   .11   .12
   .05  -.02

LAG = 4

  -.02  -.03
  -.02   .05
```

LAG = 5

```
  -.01   .00
  -.02  -.01
```

LAG = 6

```
   .17   .08
   .16   .14
```

LAG = 7

```
  -.13  -.15
  -.06  -.04
```

LAG = 8

```
  -.15   .03
  -.09  -.04
```

LAG = 9

```
  -.02   .05
  -.07   .07
```

LAG =10

```
   .00  -.03
   .04  -.08
```

LAG =11

```
   .00  -.11
   .03  -.08
```

LAG =12

```
   .07   .00
   .06  -.05
```

LAG =13

```
  -.05  -.02
  -.01   .07
```

LAG =14

-.04 .03
-.06 .01

LAG =15

-.06 .05
.02 .08

LAG =16

.04 .06
-.06 .00

LAG =17

.04 -.01
-.02 -.11

LAG =18

-.05 .07
-.01 .02

LAG =19

.11 .04
-.01 -.05

LAG =20

.00 .03
-.10 .02

LAG =21

-.02 .06
-.03 -.01

LAG =22

.02 .12

```
  -.06   .01

LAG =23

   .01  -.06
   .02  -.06

LAG =24

  -.04  -.01
   .04   .06

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2



 1   .....+......   ............
 1   ............   ............



 2   ............   ............
 2   ............   ............




CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     . .      . .      . .      . .      . .      + .
     . .      . .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     . .      . .      . .      . .      . .      . .
```

```
     . .      . .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3


=== PHI( 1) ===

     .220     .021      + .
    -.579     .901      - +

STANDARD ERRORS

     .097     .217
     .044     .098


=== PHI( 2) ===

    -.360     .611      - +
    -.136     .256      . .

STANDARD ERRORS

     .170     .291
     .076     .131


=== PHI( 3) ===

     .194    -.128      . .
     .022     .010      . .

STANDARD ERRORS
```

```
     .162     .237
     .073     .107


RESIDUAL COVARIANCE MATRIX S( 3)

  .507E+01
  .123E+01  .103E+01


RESIDUAL CORRELATION MATRIX RS( 3)

  1.00
   .54  1.00


EIGENVALUES AND EIGENVECTORS OF S( 3)

EIGENVALUES

      .684     5.411

EIGENVECTORS

     -.270      .963
      .963      .270


DETERMINANT OF S(J) =  .370064E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =     2.63
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                                -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1      2      3      4      5      6
```

```
 1   1.00
 2   -.53  1.00
 3   -.52   .81  1.00
 4    .36  -.68  -.77  1.00
 5    .09   .06  -.25   .58  1.00
 6   -.08  -.12   .13  -.59  -.88  1.00


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3  4


=== PHI( 1) ===

     .220     .027      + .
    -.576     .899      - +

STANDARD ERRORS

     .097     .216
     .044     .098


=== PHI( 2) ===

    -.348     .640      - +
    -.142     .253      . .

STANDARD ERRORS

     .169     .291
     .077     .131


=== PHI( 3) ===

     .185     .075      . .
     .039    -.056      . .

STANDARD ERRORS

     .171     .295
     .077     .133
```

```
=== PHI( 4) ===

     .145    -.294      . .
    -.062     .080      . .

STANDARD ERRORS

     .162     .235
     .073     .106


RESIDUAL COVARIANCE MATRIX S( 4)

  .501E+01
  .124E+01  .102E+01


RESIDUAL CORRELATION MATRIX RS( 4)

  1.00
   .55  1.00


EIGENVALUES AND EIGENVECTORS OF S( 4)

EIGENVALUES

      .668     5.361

EIGENVECTORS

     -.275      .961
      .961      .275


DETERMINANT OF S(J) =  .358298E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =    4.38
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.
```

-1

```
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2     3     4     5     6     7     8
  1   1.00
  2   -.53  1.00
  3   -.53   .81  1.00
  4    .36  -.68  -.77  1.00
  5    .13   .05  -.27   .55  1.00
  6   -.10  -.08   .15  -.44  -.78  1.00
  7   -.09   .03   .09   .02  -.27   .58  1.00
  8    .05  -.03  -.07  -.06   .14  -.59  -.88  1.00


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3  4  5


=== PHI( 1) ===

     .225     .025      + .
    -.578     .908      - +

STANDARD ERRORS

     .098     .217
     .044     .098


=== PHI( 2) ===

    -.361     .642      - +
    -.143     .247      . .

STANDARD ERRORS

     .170     .290
     .077     .131


=== PHI( 3) ===

     .195     .047      . .
```

```
     .048    -.063      . .

STANDARD ERRORS

     .171     .295
     .077     .133


=== PHI( 4) ===

     .083    -.259      . .
    -.098     .147      . .

STANDARD ERRORS

     .170     .292
     .077     .132


=== PHI( 5) ===

     .083     .023      . .
     .082    -.060      . .

STANDARD ERRORS

     .164     .235
     .074     .106


RESIDUAL COVARIANCE MATRIX S( 5)

  .495E+01
  .122E+01  .101E+01


RESIDUAL CORRELATION MATRIX RS( 5)

  1.00
   .54  1.00


EIGENVALUES AND EIGENVECTORS OF S( 5)
```

```
EIGENVALUES

      .664     5.298

EIGENVECTORS

     -.273      .962
      .962      .273


DETERMINANT OF S(J) =  .351866E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =     2.42
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                          -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:    CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


        1     2     3     4     5     6     7     8     9    10
  1   1.00
  2   -.54  1.00
  3   -.53   .81  1.00
  4    .36  -.68  -.77  1.00
  5    .11   .06  -.26   .54  1.00
  6   -.12  -.07   .16  -.44  -.77  1.00
  7   -.08   .00   .09   .03  -.28   .56  1.00
  8   -.06   .07  -.00  -.09   .17  -.44  -.78  1.00
  9   -.14   .14   .07  -.06   .10   .02  -.25   .58  1.00
 10    .18  -.16  -.11   .07  -.08  -.06   .12  -.58  -.88  1.00


==========  STEPWISE AUTOREGRESSION SUMMARY  ==========

---------------------------------------------------------------------------
    I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR COEFF.
----+----------+----------+---------+----------+----------------------
  1 I .530E+01 I .699E+00 I  356.96 I    1.431 I . +
```

```
    I .108E+01 I .568E+01 I          I          I - +
----+----------+----------+---------+----------+---------------------
  2 I .516E+01 I .685E+00 I    7.04 I    1.434 I . .
    I .103E+01 I .550E+01 I          I          I . +
----+----------+----------+---------+----------+---------------------
  3 I .507E+01 I .684E+00 I    2.63 I    1.469 I . .
    I .103E+01 I .541E+01 I          I          I . .
----+----------+----------+---------+----------+---------------------
  4 I .501E+01 I .668E+00 I    4.38 I    1.490 I . .
    I .102E+01 I .536E+01 I          I          I . .
----+----------+----------+---------+----------+---------------------
  5 I .495E+01 I .664E+00 I    2.42 I    1.525 I . .
    I .101E+01 I .530E+01 I          I          I . .
---------------------------------------------------------------------


NOTE:  CHI-SQUARED CRITICAL VALUES WITH   4  DEGREES OF FREEDOM ARE

             5 PERCENT:   9.5     1 PERCENT:  13.3


NOTE:  THE PARTIAL AUTOREGRESSION COEFFICIENT MATRIX FOR LAG L IS THE
       ESTIMATED PHI(L) FROM THE FIT WHERE THE MAXIMUM LAG USED IS L
       (I.E. THE LAST COEFFICIENT MATRIX). THE ELEMENTS ARE
       STANDARDIZED BY DIVIDING EACH BY ITS STANDARD ERROR.
--


mtsm name classar.series y1,y2.@
model (1-p1*b)series=c+noise


SUMMARY FOR MULTIVARIATE ARMA MODEL -- CLASSAR

VARIABLE   DIFFERENCING

 Y1
 Y2

 PARAMETER     FACTOR     ORDER    CONSTRAINT

 1         C    CONSTANT      0         CC
 2        P1     REG AR       1         CP1
--
```

```
mestim model classar.hold resi(ry1,ry2)

SUMMARY FOR THE MULTIVARIATE ARMA MODEL

SERIES    NAME            MEAN         STD DEV    DIFFERENCE ORDER(S)

  1     Y1               4.9624         2.7685
  2     Y2              10.0616         3.0082

NUMBER OF OBSERVATIONS =  150  (EFFECTIVE NUMBER = NOBE =  149)

MODEL SPECIFICATION WITH PARAMETER VALUES


PARAMETER              PARAMETER                  PARAMETER
 NUMBER               DESCRIPTION                  VALUE
----------    -----------------------------     ------------


    1                 CONSTANT( 1)                 3.460012

    2                 CONSTANT( 2)                 8.559217

    3           AUTOREGRESSIVE ( 1, 1, 1)           .100000

    4           AUTOREGRESSIVE ( 1, 1, 2)           .100000

    5           AUTOREGRESSIVE ( 1, 2, 1)           .100000

    6           AUTOREGRESSIVE ( 1, 2, 2)           .100000



-----------------------
ERROR COVARIANCE MATRIX
-----------------------


               1           2
  1     6.359586
  2     3.263111    7.681468
```

```
ITERATIONS TERMINATED DUE TO:
  RELATIVE CHANGE IN DETERMINANT OF COVARIANCE MATRIX .LE.  .100E-03
  TOTAL NUMBER OF ITERATIONS IS    4

FINAL MODEL SUMMARY WITH CONDITIONAL LIKELIHOOD PARAMETER ESTIMATES


----- CONSTANT VECTOR (STD ERROR) -----

     .032  (     .655  )
    1.878  (     .300  )

----- PHI MATRICES -----

ESTIMATES OF    PHI(  1 ) MATRIX AND SIGNIFICANCE

     .133     .428      . +
    -.609    1.116      - +

STANDARD ERRORS

     .082     .075
     .037     .034



-----------------------
ERROR COVARIANCE MATRIX
-----------------------

              1           2
  1     5.271828
  2     1.331015    1.105583



--------------------------------------------
REDUCED CORRELATION MATRIX OF THE PARAMETERS
--------------------------------------------

        1     2     3     4     5     6
  1   1.00
  2    .55  1.00
  3    .     .    1.00
```

```
  4   -.81  -.45  -.55  1.00
  5    .     .     .55  -.31  1.00
  6   -.45  -.81  -.31   .55  -.55  1.00
--


miden ry1,ry2.max 24.output print(ccm)

TIME PERIOD ANALYZED . . . . . . . . . . . .  2  TO   150
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .          149


SERIES   NAME           MEAN       STD. ERROR

   1    RY1              .0000      2.2960
   2    RY2              .0000      1.0515


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .08192


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
   .55  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE ORIGINAL SERIES.
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

  -.02  -.15
  -.05  -.17

LAG = 2

  -.05   .04
   .01   .04

LAG = 3
```

```
  .09   .15
  .02  -.02

LAG = 4

 -.06  -.03
 -.09   .04

LAG = 5

 -.02  -.03
 -.04  -.02

LAG = 6

  .18   .09
  .14   .13

LAG = 7

 -.18  -.19
 -.12  -.10

LAG = 8

 -.15   .01
 -.12  -.05

LAG = 9

  .00   .08
 -.04   .11

LAG =10

  .01  -.04
  .04  -.07

LAG =11

  .01  -.05
  .04  -.03
```

LAG =12

```
  .07   .00
  .06  -.04
```

LAG =13

```
 -.05   .00
 -.01   .04
```

LAG =14

```
 -.07   .00
 -.07  -.02
```

LAG =15

```
 -.03   .06
  .03   .09
```

LAG =16

```
  .05   .03
 -.03   .04
```

LAG =17

```
  .02  -.05
 -.03  -.12
```

LAG =18

```
 -.01   .11
  .03   .06
```

LAG =19

```
  .11   .04
  .00  -.07
```

LAG =20

```
 -.02   .02
 -.11   .00
```

```
LAG =21

  -.02   .05
  -.01  -.01

LAG =22

   .03   .14
  -.04   .05

LAG =23

  -.01  -.06
   .00  -.08

LAG =24

  -.03   .02
   .04   .08

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   .....+-.....   ......-.....
 1   ............   ............


 2   ............   -...........
 2   ............   ............


CROSS CORRELATION MATRICES IN TERMS OF +,-,.
```

```
LAGS  1 THROUGH  6

     . .      . .      . .      . .      . .      + .
     . -      . .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     - -      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .
--


save ry1,ry2. file '\b521\clsarr.dat'.format '2f10.5'


** THE SPECIFIED FILE DOES NOT EXIST. A NEW FILE IS OPENED.
   THE FILE NAME IS \B521\CLSARR.DAT
THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE  7
--


mforecast model classar.nofs 12.nopsiweights 10

----------------------------------------
 12 FORECASTS, BEGINNING AT ORIGIN = 150
----------------------------------------

SERIES:       Y1                  Y2

TIME    FORECAST  STD ERR   FORECAST  STD ERR

 151       5.966    2.296      9.686    1.051
 152       4.970    2.392      9.057    1.621
```

153	4.568	2.450	8.961	2.182
154	4.473	2.545	9.099	2.564
155	4.520	2.631	9.311	2.786
156	4.617	2.688	9.519	2.902
157	4.718	2.721	9.692	2.957
158	4.806	2.738	9.823	2.982
159	4.874	2.745	9.916	2.991
160	4.923	2.748	9.979	2.995
161	4.956	2.749	10.019	2.996
162	4.978	2.750	10.044	2.996

PSI MATRICES USED IN THE COMPUTATION OF FORECAST ERROR MATRICES

PSI 1

.133	.428
-.609	1.116

PSI 2

-.243	.534
-.761	.986

PSI 3

-.358	.493
-.702	.775

PSI 4

-.348	.397
-.565	.565

PSI 5

-.288	.294
-.419	.388

```
PSI   6

    -.218     .205
    -.293     .254


PSI   7

    -.154     .136
    -.194     .159


PSI   8

    -.103     .086
    -.123     .094


PSI   9

    -.066     .052
    -.074     .053


PSI  10

    -.040     .030
    -.042     .027
--


stop
```

1

1

CLASS MA EXAMPLE

```
 input x1,x2. file '\b521\classma.dat'

 X1       , A  250  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
 X2       , A  250  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
 --


 acf x1.max 24.output noprint(value,lbq)

 NAME OF THE SERIES . . . . . . . . . .           X1
 TIME PERIOD ANALYZED . . . . . . . . .  1  TO   250
 MEAN OF THE (DIFFERENCED) SERIES . . .      17.0664
 STANDARD DEVIATION OF THE SERIES . . .       2.3334
 T-VALUE OF MEAN (AGAINST ZERO) . . . .     115.6431


 AUTOCORRELATIONS


          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
   1   -.28                   XXXX+XXI  +
   2    .03                       +  IX +
   3    .04                       +  IX +
   4   -.11                       XXXI  +
   5   -.02                       +  I  +
   6    .10                       +  IXX+
   7   -.11                       XXXI  +
   8   -.09                       +XXI  +
   9    .01                       +  I  +
  10    .00                       +  I  +
  11    .03                       +  IX +
  12    .06                       +  IX +
  13    .05                       +  IX +
  14   -.07                       +XXI  +
  15    .05                       +  IX +
  16   -.06                      +  XI   +
  17    .07                      +   IXX +
  18   -.04                      +  XI   +
  19    .08                      +   IXX +
```

```
 20   -.03                      +  XI   +
 21    .01                      +   I   +
 22   -.02                      +   I   +
 23    .01                      +   I   +
 24   -.02                      +   I   +
--
```

```
acf x2.max 24.output noprint(value,lbq)
```

```
NAME OF THE SERIES . . . . . . . . . .           X2
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   250
MEAN OF THE (DIFFERENCED) SERIES . . .      25.0393
STANDARD DEVIATION OF THE SERIES . . .       1.5875
T-VALUE OF MEAN (AGAINST ZERO) . . . .     249.3909
```

```
AUTOCORRELATIONS

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
             +----+----+----+----+----+----+----+----+----+----+
                                      I
  1   -.19                       XX+XXI  +
  2    .01                         +  I  +
  3   -.08                         +XXI  +
  4    .09                         +  IXX+
  5   -.08                         +XXI  +
  6    .00                         +  I  +
  7   -.06                         +XXI  +
  8   -.16                        X+XXI  +
  9    .10                         +  IXXX
 10   -.04                         + XI  +
 11    .08                         +  IXX+
 12    .01                         +  I  +
 13    .05                         +  IX +
 14   -.06                         + XI  +
 15    .05                         +  IX +
 16    .00                         +  I  +
 17    .01                         +  I  +
 18   -.02                         + XI  +
 19   -.03                         + XI  +
 20    .07                         +  IXX+
 21   -.05                         + XI  +
```

```
 22    .02                      +  IX +
 23    .04                      +  IX +
 24    .03                      +  IX +
--
```

```
ccf x1,x2.max 24. output noprint(value)

TIME PERIOD ANALYZED . . . . . . . . . .  1  TO   250
NAMES OF THE SERIES  . . . . . . . . .           X1             X2
EFFECTIVE NUMBER OF OBSERVATIONS . . .          250            250
STANDARD DEVIATION OF THE SERIES . . .       2.3334         1.5875
MEAN OF THE (DIFFERENCED) SERIES . . .      17.0664        25.0393
STANDARD DEVIATION OF THE MEAN . . . .        .1476          .1004
T-VALUE OF MEAN (AGAINST ZERO) . . . .     115.6432       249.3909



CORRELATION       BETWEEN       X2     AND       X1 IS    .30


CROSS CORRELATION BETWEEN       X1(T)  AND       X2(T-L)



          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
-24    .07                      +  IXX+
-23    .01                      +  I  +
-22   -.04                      + XI  +
-21    .08                      +  IXX+
-20   -.04                      + XI  +
-19   -.04                      + XI  +
-18    .08                      +  IXX+
-17   -.05                      + XI  +
-16    .00                      +  I  +
-15    .00                      +  I  +
-14    .00                      +  I  +
-13    .04                      +  IX +
-12    .04                      +  IX +
-11    .00                      +  I  +
-10    .02                      +  I  +
 -9   -.06                      + XI  +
```

```
 -8   -.12                      XXXI  +
 -7    .01                      +  I  +
 -6    .01                      +  I  +
 -5   -.09                      +XXI  +
 -4    .04                      +  IX +
 -3   -.03                      + XI  +
 -2    .08                      +  IXX+
 -1    .37                      +  IXX+XXXXXX
  0    .30                      +  IXX+XXXX
  1   -.21                    XX+XXI  +
  2    .02                      +  IX +
  3   -.01                      +  I  +
  4   -.03                      + XI  +
  5   -.02                      + XI  +
  6    .01                      +  I  +
  7   -.17                     X+XXI  +
  8   -.03                      + XI  +
  9    .08                      +  IXX+
 10    .01                      +  I  +
 11    .08                      +  IXX+
 12   -.01                      +  I  +
 13    .03                      +  IX +
 14   -.07                      +XXI  +
 15    .05                      +  IX +
 16    .03                      +  IX +
 17   -.07                      +XXI  +
 18    .07                      +  IXX+
 19    .03                      +  IX +
 20   -.04                      + XI  +
 21   -.03                      + XI  +
 22    .09                      +  IXX+
 23   -.09                      +XXI  +
 24    .15                      +  IXX+X
--

miden x1,x2. max 24. desc.output level(detailed)

TIME PERIOD ANALYZED . . . . . . . . . . . . .  1  TO   250
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .            250


SERIES   NAME          MEAN       STD. ERROR
```

```
   1    X1              17.0664      2.3334
   2    X2              25.0393      1.5875


 SAMPLE COVARIANCE MATRIX OF THE SERIES

   .544E+01
   .110E+01   .252E+01


EIGENVALUES OF THE COVARIANCE MATRIX
      2.152       5.813


EIGENVECTORS OF THE COVARIANCE MATRIX

      -.317       .948
       .948       .317


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .06325


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
   .30  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE ORIGINAL SERIES.
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

  -.28  -.21
   .37  -.19

LAG = 2

   .03   .02
   .08   .01
```

LAG = 3

```
  .04  -.01
 -.03  -.08
```

LAG = 4

```
 -.11  -.03
  .04   .09
```

LAG = 5

```
 -.02  -.02
 -.09  -.08
```

LAG = 6

```
  .10   .01
  .01   .00
```

LAG = 7

```
 -.11  -.17
  .01  -.06
```

LAG = 8

```
 -.09  -.03
 -.12  -.16
```

LAG = 9

```
  .01   .08
 -.06   .10
```

LAG =10

```
  .00   .01
  .02  -.04
```

LAG =11

```
  .03   .08
```

```
   .00   .08

LAG =12

   .06  -.01
   .04   .01

LAG =13

   .05   .03
   .04   .05

LAG =14

  -.07  -.07
   .00  -.06

LAG =15

   .05   .05
   .00   .05

LAG =16

  -.06   .03
   .00   .00

LAG =17

   .07  -.07
  -.05   .01

LAG =18

  -.04   .07
   .08  -.02

LAG =19

   .08   .03
  -.04  -.03

LAG =20
```

```
  -.03  -.04
  -.04   .07

LAG =21

   .01  -.03
   .08  -.05

LAG =22

  -.02   .09
  -.04   .02

LAG =23

   .01  -.09
   .01   .04

LAG =24

  -.02   .15
   .07   .03

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   -...........   -.....-.....
 1   ............   ...........+



 2   +...........   -......-....
 2   ............   ............
```

```
CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     - -      . .      . .      . .      . .      . .
     + -      . .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     . -      . .      . .      . .      . .      . .
     . .      . -      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . +
     . .      . .      . .      . .      . .      . .
--


miden x1,x2. no ccm.arfit 1 to 8.rccm 1,2.@
output level(detailed).noprint(ccm_values)

** NOPRINT  SENTENCE IS NOT USED

TIME PERIOD ANALYZED . . . . . . . . . . . .  1  TO   250
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .          250


SERIES   NAME           MEAN       STD. ERROR

   1     X1            17.0664      2.3334
   2     X2            25.0393      1.5875

DETERMINANT OF S(0) =  .121867E+02

NOTE: S(0) IS THE SAMPLE COVARIANCE MATRIX OF W(MAXLAG+1),...,W(NOBE)
```

```
AUTOREGRESSIVE FITTING ON LAG(S)  1


=== PHI( 1) ===

    -.213    -.206      - -
     .354    -.345      + -

STANDARD ERRORS

     .065     .092
     .041     .058


RESIDUAL COVARIANCE MATRIX S( 1)

  .478E+01
  .131E+01  .188E+01


RESIDUAL CORRELATION MATRIX RS( 1)

  1.00
   .44  1.00


EIGENVALUES AND EIGENVECTORS OF S( 1)

EIGENVALUES

     1.377     5.279

EIGENVECTORS

     -.358      .934
      .934      .358


DETERMINANT OF S(J) =  .727084E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =  123.18
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
```

```
   W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                           -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2
  1   1.00
  2   -.30  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

   .00  -.04
   .10  -.21

LAG = 2

   .04  -.08
   .39   .07

LAG = 3

   .01   .02
  -.03  -.11

LAG = 4

  -.11  -.03
   .03   .10

LAG = 5

  -.03  -.05
  -.05  -.07
```

LAG = 6

```
  .02  -.05
 -.03  -.04
```

LAG = 7

```
 -.12  -.15
 -.08  -.08
```

LAG = 8

```
 -.14  -.05
 -.08  -.06
```

LAG = 9

```
 -.03   .08
 -.07   .05
```

LAG =10

```
  .03   .07
  .03  -.04
```

LAG =11

```
  .05   .06
  .02   .07
```

LAG =12

```
  .10   .00
  .04   .00
```

LAG =13

```
  .04   .02
  .02   .07
```

LAG =14

```
 -.01  -.08
  .01  -.08
```

```
LAG =15

    .02   .08
    .06   .10

LAG =16

   -.06  -.01
   -.04  -.04

LAG =17

    .08   .00
    .02  -.01

LAG =18

   -.04   .05
    .01   .00

LAG =19

    .07   .02
   -.02  -.01

LAG =20

    .01  -.05
   -.07   .03

LAG =21

   -.01   .00
    .09   .01

LAG =22

   -.02   .03
    .00   .01

LAG =23

    .05   .00
```

```
   .04   .01

LAG =24

  -.04   .10
   .04   .09

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   .......-....   ......-.....
 1   ............   ............


 2   .+..........   -...........
 2   ............   ............


CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     . .      . .      . .      . .      . .      . .
     . -      + .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     . -      - .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
```

```
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .


AUTOREGRESSIVE FITTING ON LAG(S)  1  2


=== PHI( 1) ===

    -.194    -.279      - -
     .497    -.648      + -

STANDARD ERRORS

     .071     .114
     .039     .062


=== PHI( 2) ===

     .076    -.110      . .
     .382    -.131      + -

STANDARD ERRORS

     .080     .099
     .044     .054


RESIDUAL COVARIANCE MATRIX S( 2)

  .475E+01
  .123E+01  .143E+01


RESIDUAL CORRELATION MATRIX RS( 2)

  1.00
   .47  1.00
```

```
EIGENVALUES AND EIGENVECTORS OF S( 2)

EIGENVALUES

     1.024     5.152

EIGENVECTORS

     -.313      .950
      .950      .313


DETERMINANT OF S(J) =  .527577E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   75.86
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                          -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2     3     4
  1   1.00
  2   -.44  1.00
  3    .40  -.57  1.00
  4   -.03   .33  -.40  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE RESIDUAL SERIES.
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

  -.01  -.07
  -.02  -.17
```

```
LAG = 2

   .02  -.04
   .06  -.20

LAG = 3

   .07   .07
   .24   .05

LAG = 4

  -.12  -.09
   .00   .05

LAG = 5

  -.04  -.05
  -.03  -.05

LAG = 6

   .02  -.01
   .00   .01

LAG = 7

  -.12  -.13
  -.10  -.06

LAG = 8

  -.14  -.02
  -.11  -.05

LAG = 9

  -.02   .11
  -.02   .08

LAG =10

   .02   .05
   .06  -.06
```

LAG =11

.04	.01
.01	.00

LAG =12

.10	.00
.03	.03

LAG =13

.04	.00
-.01	.03

LAG =14

-.01	-.03
-.02	-.06

LAG =15

.02	.06
.07	.10

LAG =16

-.05	.01
-.02	-.04

LAG =17

.08	.00
.01	-.06

LAG =18

-.04	.04
.03	.07

LAG =19

.06	.02

```
  -.05  -.03

LAG =20

   .02  -.04
  -.08  -.01

LAG =21

  -.03   .00
   .04   .03

LAG =22

  -.01   .07
   .03   .00

LAG =23

   .05   .00
   .05   .01

LAG =24

  -.05   .06
   .05   .10

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   .......-....   ......-.....
 1   ............   ............
```

```
 2   ..+.........   --..........
 2   ............   ............



CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     . .      . .      . .      . .      . .      . .
     . -      . -      + .      . .      . .      . .

LAGS  7 THROUGH 12

     . -      - .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3


=== PHI( 1) ===

    -.160    -.443      - -
     .544    -.870      + -

STANDARD ERRORS

     .072     .132
     .037     .068


=== PHI( 2) ===
```

```
     .220    -.341      + -
     .578    -.434      + -

STANDARD ERRORS

     .101     .136
     .052     .070


=== PHI( 3) ===

     .222    -.120      + .
     .294    -.129      + -

STANDARD ERRORS

     .092     .099
     .047     .051


RESIDUAL COVARIANCE MATRIX S( 3)

  .463E+01
  .108E+01  .123E+01


RESIDUAL CORRELATION MATRIX RS( 3)

  1.00
   .45  1.00


EIGENVALUES AND EIGENVECTORS OF S( 3)

EIGENVALUES

      .918     4.947

EIGENVECTORS

     -.279      .960
      .960      .279
```

```
DETERMINANT OF S(J) =  .454103E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   35.17
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.



                                          -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2     3     4     5     6
  1   1.00
  2   -.48  1.00
  3    .44  -.71  1.00
  4   -.15   .56  -.63  1.00
  5    .20  -.52   .60  -.69  1.00
  6   -.01   .14  -.11   .36  -.41  1.00


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3  4


=== PHI( 1) ===

    -.159    -.462       - -
     .550    -.994       + -

STANDARD ERRORS

     .072     .140
     .035     .068


=== PHI( 2) ===

     .233    -.378       + -
     .688    -.715       + -

STANDARD ERRORS
```

```
     .109     .171
     .053     .083


=== PHI( 3) ===

     .248    -.168      . .
     .525    -.397      + -

STANDARD ERRORS

     .124     .146
     .060     .071


=== PHI( 4) ===

     .039    -.056      . .
     .257    -.023      + .

STANDARD ERRORS

     .098     .100
     .048     .048


RESIDUAL COVARIANCE MATRIX S( 4)

  .463E+01
  .107E+01  .108E+01


RESIDUAL CORRELATION MATRIX RS( 4)

  1.00
   .48  1.00


EIGENVALUES AND EIGENVECTORS OF S( 4)

EIGENVALUES

      .785     4.924
```

```
EIGENVECTORS

     -.267      .964
      .964      .267


DETERMINANT OF S(J) =  .386533E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   37.46
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                          -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


        1     2     3     4     5     6     7     8
  1   1.00
  2   -.46  1.00
  3    .42  -.74  1.00
  4   -.14   .62  -.69  1.00
  5    .17  -.59   .66  -.82  1.00
  6   -.02   .34  -.33   .63  -.68  1.00
  7    .02  -.34   .36  -.60   .66  -.73  1.00
  8    .00   .14  -.10   .21  -.18   .39  -.43  1.00


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3  4  5


=== PHI( 1) ===

    -.173    -.399       - -
     .569   -1.082       + -

STANDARD ERRORS

     .073     .153
     .035     .073
```

```
=== PHI( 2) ===

     .195    -.278       . .
     .742    -.852       + -

STANDARD ERRORS

     .115     .196
     .055     .093


=== PHI( 3) ===

     .170    -.029       . .
     .633    -.586       + -

STANDARD ERRORS

     .145     .199
     .069     .094


=== PHI( 4) ===

    -.070     .065       . .
     .408    -.182       + -

STANDARD ERRORS

     .144     .158
     .068     .075


=== PHI( 5) ===

    -.108     .019       . .
     .144    -.008       + .

STANDARD ERRORS

     .107     .100
     .051     .048
```

```
RESIDUAL COVARIANCE MATRIX S( 5)

  .461E+01
  .109E+01  .104E+01


RESIDUAL CORRELATION MATRIX RS( 5)

  1.00
   .50  1.00


EIGENVALUES AND EIGENVECTORS OF S( 5)

EIGENVALUES

      .732     4.917

EIGENVECTORS

     -.272      .962
      .962      .272


DETERMINANT OF S(J) =  .359709E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   16.58
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                                  -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2     3     4     5     6     7     8     9    10
  1   1.00
  2   -.48  1.00
  3    .45  -.77  1.00
  4   -.20   .69  -.72  1.00
```

```
  5    .23  -.67   .70  -.86  1.00
  6   -.13   .49  -.44   .74  -.78  1.00
  7    .15  -.50   .47  -.72   .77  -.86  1.00
  8   -.12   .36  -.28   .49  -.48   .70  -.73  1.00
  9    .16  -.37   .30  -.49   .51  -.67   .72  -.77  1.00
 10    .02   .03  -.02   .13  -.11   .21  -.17   .38  -.42  1.00


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3  4  5  6


=== PHI( 1) ===

    -.148    -.473       . -
     .594   -1.155       + -

STANDARD ERRORS

     .074     .157
     .035     .073


=== PHI( 2) ===

     .261    -.450       + -
     .808   -1.020       + -

STANDARD ERRORS

     .120     .218
     .056     .101


=== PHI( 3) ===

     .280    -.264       . .
     .741    -.807       + -

STANDARD ERRORS

     .157     .233
     .073     .109
```

```
=== PHI( 4) ===

     .108    -.230      . .
     .577    -.455      + -

STANDARD ERRORS

     .173     .220
     .080     .102


=== PHI( 5) ===

     .117    -.229      . .
     .355    -.225      + -

STANDARD ERRORS

     .160     .161
     .075     .075


=== PHI( 6) ===

     .215    -.101      . .
     .191    -.047      + .

STANDARD ERRORS

     .110     .099
     .051     .046


RESIDUAL COVARIANCE MATRIX S( 6)

  .453E+01
  .103E+01  .983E+00


RESIDUAL CORRELATION MATRIX RS( 6)

  1.00
   .49  1.00
```

```
EIGENVALUES AND EIGENVECTORS OF S( 6)

EIGENVALUES

      .705     4.813

EIGENVECTORS

     -.260      .966
      .966      .260


DETERMINANT OF S(J) =  .339115E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   13.47
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                         -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2     3     4     5     6     7     8     9    10    11
  1   1.00
  2   -.51  1.00
  3    .48  -.79  1.00
  4   -.26   .71  -.75  1.00
  5    .29  -.69   .74  -.89  1.00
  6   -.21   .54  -.52   .79  -.81  1.00
  7    .23  -.55   .54  -.78   .81  -.90  1.00
  8   -.22   .43  -.41   .62  -.59   .79  -.82  1.00
  9    .25  -.44   .42  -.62   .60  -.78   .81  -.89  1.00
 10   -.13   .21  -.24   .40  -.35   .52  -.51   .71  -.74  1.00
 11    .18  -.25   .30  -.42   .38  -.53   .54  -.70   .74  -.78  1.00
 12   -.02   .02  -.01   .04  -.04   .12  -.10   .19  -.16   .37  -.41

        12
 12   1.00
```

```
AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3  4  5  6  7


=== PHI( 1) ===

    -.155    -.491       - -
     .592   -1.198       + -

STANDARD ERRORS

     .073     .158
     .034     .073


=== PHI( 2) ===

     .275    -.486       + -
     .852   -1.126       + -

STANDARD ERRORS

     .123     .228
     .057     .106


=== PHI( 3) ===

     .303    -.307       . .
     .826    -.971       + -

STANDARD ERRORS

     .167     .261
     .078     .121


=== PHI( 4) ===

     .136    -.321       . .
     .686    -.652       + -

STANDARD ERRORS
```

```
     .189      .259
     .088      .120


=== PHI( 5) ===

     .172     -.348       . .
     .503     -.449       + -

STANDARD ERRORS

     .191      .224
     .089      .104


=== PHI( 6) ===

     .288     -.265       . .
     .362     -.232       + -

STANDARD ERRORS

     .165      .158
     .077      .073


=== PHI( 7) ===

     .125     -.285       . -
     .160     -.087       + .

STANDARD ERRORS

     .109      .097
     .051      .045


RESIDUAL COVARIANCE MATRIX S( 7)

  .438E+01
  .985E+00  .942E+00


RESIDUAL CORRELATION MATRIX RS( 7)
```

```
 1.00
  .49  1.00


EIGENVALUES AND EIGENVECTORS OF S( 7)

EIGENVALUES

     .679     4.641

EIGENVECTORS

    -.257      .966
     .966      .257


DETERMINANT OF S(J) =  .315279E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =   16.51
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.


                                          -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)
```

	1	2	3	4	5	6	7	8	9	10	11
1	1.00										
2	-.50	1.00									
3	.46	-.80	1.00								
4	-.25	.73	-.77	1.00							
5	.26	-.70	.76	-.90	1.00						
6	-.19	.56	-.57	.82	-.84	1.00					
7	.20	-.57	.59	-.81	.84	-.92	1.00				
8	-.18	.46	-.48	.67	-.67	.84	-.86	1.00			
9	.20	-.47	.49	-.68	.68	-.83	.85	-.92	1.00		
10	-.09	.29	-.35	.51	-.50	.66	-.63	.81	-.83	1.00	
11	.12	-.31	.39	-.52	.52	-.66	.65	-.80	.83	-.90	1.00
12	-.00	.17	-.21	.28	-.31	.42	-.38	.52	-.51	.71	-.74

```
13   -.01  -.20   .26  -.33   .37  -.46   .42  -.54   .55  -.70   .74
14    .03   .04  -.04   .06  -.05   .06  -.06   .13  -.11   .19  -.16

        12    13    14
12   1.00
13   -.79  1.00
14    .36  -.40  1.00


AUTOREGRESSIVE FITTING ON LAG(S)  1  2  3  4  5  6  7  8


=== PHI( 1) ===

    -.180    -.479       - -
     .579   -1.206       + -

STANDARD ERRORS

     .073     .159
     .034     .073


=== PHI( 2) ===

     .259    -.471       + -
     .851   -1.150       + -

STANDARD ERRORS

     .123     .235
     .057     .108


=== PHI( 3) ===

     .283    -.273       . .
     .843   -1.007       + -

STANDARD ERRORS

     .175     .281
     .081     .129
```

```
=== PHI( 4) ===

     .103    -.259      . .
     .710    -.691      + -

STANDARD ERRORS

     .206     .298
     .095     .137


=== PHI( 5) ===

     .130    -.311      . .
     .528    -.507      + -

STANDARD ERRORS

     .215     .273
     .099     .126


=== PHI( 6) ===

     .248    -.239      . .
     .400    -.302      + -

STANDARD ERRORS

     .203     .230
     .094     .106


=== PHI( 7) ===

     .091    -.314      . .
     .207    -.168      + -

STANDARD ERRORS

     .171     .160
     .079     .074
```

```
=== PHI( 8) ===

     .008    -.181      . .
     .061    -.124      . -

STANDARD ERRORS

     .110     .098
     .051     .045


RESIDUAL COVARIANCE MATRIX S( 8)

  .431E+01
  .944E+00  .913E+00


RESIDUAL CORRELATION MATRIX RS( 8)

  1.00
   .48  1.00


EIGENVALUES AND EIGENVECTORS OF S( 8)

EIGENVALUES

      .668     4.553

EIGENVECTORS

     -.251      .968
      .968      .251


DETERMINANT OF S(J) =  .304180E+01
LEADING TO A VALUE OF THE TEST STATISTIC M = -W*LN(U) =    8.05
APPROXIMATELY DISTRIBUTED AS A CHI SQUARE WITH  4 DF,
WHERE U = DET(S(J))/DET(S(J-1))
   S(J) = RESIDUAL COVARIANCE MATRIX AFTER JTH FIT
      W = (NOBE-MAXARF-1)-J*K-.5, AND DF = K*K.
```

```
                                             -1
RV(J) = CORRELATION FORM OF  V(J) = (X'X)
NOTE:   CORR(PHI(LAG(L1),I1,J1),PHI(LAG(L2),I2,J2))
      = RS(I1,I2)*RV(K*(L1-1)+J1,K*(L2-1)+J2)


         1     2     3     4     5     6     7     8     9    10    11
  1   1.00
  2   -.49  1.00
  3    .47  -.81  1.00
  4   -.25   .74  -.78  1.00
  5    .27  -.72   .76  -.91  1.00
  6   -.19   .58  -.58   .83  -.86  1.00
  7    .21  -.59   .59  -.82   .85  -.93  1.00
  8   -.19   .49  -.48   .70  -.71   .87  -.88  1.00
  9    .21  -.49   .50  -.70   .72  -.86   .88  -.94  1.00
 10   -.10   .34  -.37   .56  -.57   .72  -.71   .86  -.87  1.00
 11    .13  -.36   .40  -.57   .59  -.72   .72  -.85   .87  -.93  1.00
 12   -.04   .25  -.25   .39  -.43   .55  -.54   .67  -.65   .82  -.84
 13    .04  -.27   .28  -.42   .47  -.57   .57  -.69   .67  -.81   .84
 14    .01   .17  -.13   .25  -.28   .33  -.35   .45  -.41   .54  -.53
 15    .03  -.19   .14  -.27   .32  -.38   .41  -.49   .46  -.57   .58
 16    .14   .03   .01   .07  -.06   .09  -.08   .09  -.08   .15  -.13

        12    13    14    15    16
 12   1.00
 13   -.90  1.00
 14    .72  -.75  1.00
 15   -.73   .76  -.80  1.00
 16    .22  -.19   .38  -.41  1.00


==========  STEPWISE AUTOREGRESSION SUMMARY  ==========

----------------------------------------------------------------------
    I RESIDUAL I EIGENVAL.I CHI-SQ  I          I SIGNIFICANCE
LAG I VARIANCESI OF SIGMA I  TEST   I   AIC    I OF PARTIAL AR COEFF.
----+----------+----------+---------+----------+----------------------
  1 I .478E+01 I .138E+01 I  123.18 I    2.016 I - -
    I .188E+01 I .528E+01 I         I          I + -
----+----------+----------+---------+----------+----------------------
  2 I .475E+01 I .102E+01 I   75.86 I    1.727 I . .
    I .143E+01 I .515E+01 I         I          I + -
----+----------+----------+---------+----------+----------------------
```

```
  3 I .463E+01 I .918E+00 I   35.17 I     1.609 I + .
    I .123E+01 I .495E+01 I         I           I + -
----+----------+----------+---------+-----------+----------------------
  4 I .463E+01 I .785E+00 I   37.46 I     1.480 I . .
    I .108E+01 I .492E+01 I         I           I + .
----+----------+----------+---------+-----------+----------------------
  5 I .461E+01 I .732E+00 I   16.58 I     1.440 I . .
    I .104E+01 I .492E+01 I         I           I + .
----+----------+----------+---------+-----------+----------------------
  6 I .453E+01 I .705E+00 I   13.47 I     1.413 I . .
    I .983E+00 I .481E+01 I         I           I + .
----+----------+----------+---------+-----------+----------------------
  7 I .438E+01 I .679E+00 I   16.51 I     1.372 I . -
    I .942E+00 I .464E+01 I         I           I + .
----+----------+----------+---------+-----------+----------------------
  8 I .431E+01 I .668E+00 I    8.05 I     1.368 I . .
    I .913E+00 I .455E+01 I         I           I . -
-----------------------------------------------------------------------


NOTE:  CHI-SQUARED CRITICAL VALUES WITH   4  DEGREES OF FREEDOM ARE

             5 PERCENT:   9.5     1 PERCENT:  13.3


NOTE:  THE PARTIAL AUTOREGRESSION COEFFICIENT MATRIX FOR LAG L IS THE
       ESTIMATED PHI(L) FROM THE FIT WHERE THE MAXIMUM LAG USED IS L
       (I.E. THE LAST COEFFICIENT MATRIX). THE ELEMENTS ARE
       STANDARDIZED BY DIVIDING EACH BY ITS STANDARD ERROR.
--


mtsmodel name classma. series x1,x2.@
model series=c+(1-t1*b)noise


SUMMARY FOR MULTIVARIATE ARMA MODEL -- CLASSMA

VARIABLE   DIFFERENCING

 X1
 X2

 PARAMETER     FACTOR     ORDER    CONSTRAINT
```

```
 1        C   CONSTANT      0        CC
 2       T1    REG MA       1        CT1
--


mestim model classma.hold resi(rx1,rx2)

SUMMARY FOR THE MULTIVARIATE ARMA MODEL

SERIES    NAME            MEAN        STD DEV    DIFFERENCE ORDER(S)


  1     X1              17.0664         2.3334
  2     X2              25.0393         1.5875

NUMBER OF OBSERVATIONS =  250  (EFFECTIVE NUMBER = NOBE =  250)

MODEL SPECIFICATION WITH PARAMETER VALUES


PARAMETER              PARAMETER                 PARAMETER
 NUMBER               DESCRIPTION                 VALUE
----------    -----------------------------     ------------


    1               CONSTANT( 1)                 17.066418

    2               CONSTANT( 2)                 25.039344

    3         MOVING AVERAGE ( 1, 1, 1)           .100000

    4         MOVING AVERAGE ( 1, 1, 2)           .100000

    5         MOVING AVERAGE ( 1, 2, 1)           .100000

    6         MOVING AVERAGE ( 1, 2, 2)           .100000



-----------------------
ERROR COVARIANCE MATRIX
-----------------------


              1           2
  1     5.095206
```

```
  2     1.069981    2.807718



ITERATIONS TERMINATED DUE TO:
  RELATIVE CHANGE IN DETERMINANT OF COVARIANCE MATRIX .LE.  .100E-03
  TOTAL NUMBER OF ITERATIONS IS    8

FINAL MODEL SUMMARY WITH CONDITIONAL LIKELIHOOD PARAMETER ESTIMATES


----- CONSTANT VECTOR (STD ERROR) -----

   17.153  (     .102  )
   25.108  (     .076  )

----- THETA MATRICES -----

ESTIMATES OF  THETA(  1 ) MATRIX AND SIGNIFICANCE

     .192     .410      + +
    -.574    1.125      - +

STANDARD ERRORS

     .039     .093
     .029     .066



-----------------------
ERROR COVARIANCE MATRIX
-----------------------

              1           2
  1     4.877268
  2     1.108070    1.083262



--------------------------------------------
REDUCED CORRELATION MATRIX OF THE PARAMETERS
--------------------------------------------
```

```
        1     2     3     4     5     6
  1   1.00
  2    .99  1.00
  3    .     .    1.00
  4    .     .    -.49  1.00
  5    .     .     .85  -.42  1.00
  6    .     .    -.42   .88  -.49  1.00
--
```

```
miden rx1,rx2.max 24

TIME PERIOD ANALYZED . . . . . . . . . . . .  1  TO   250
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .          250


SERIES   NAME          MEAN       STD. ERROR


   1    RX1            -.0993      2.2062
   2    RX2             .0063      1.0408


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .06325


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
   .48  1.00

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2
```

```
 1   ...-........   ...........
 1   ...........    ...........


 2   .......-....   ...........
 2   .......-....   ...........



CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS  1 THROUGH  6

     . .      . .      . .      - .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS  7 THROUGH 12

     . .      . .      . .      . .      . .      . .
     . .      - .      . .      . .      . .      . .

LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      - .      . .      . .      . .      . .
--


mestim model classma.method exact.hold resi(rx1,rx2)

SUMMARY FOR THE MULTIVARIATE ARMA MODEL

SERIES    NAME             MEAN         STD DEV    DIFFERENCE ORDER(S)

  1     X1              17.0664         2.3334
  2     X2              25.0393         1.5875

NUMBER OF OBSERVATIONS =  250  (EFFECTIVE NUMBER = NOBE =  250)
```

```
MODEL SPECIFICATION WITH PARAMETER VALUES


PARAMETER             PARAMETER                  PARAMETER
 NUMBER              DESCRIPTION                   VALUE
----------    -----------------------------     ------------


    1              CONSTANT( 1)                  17.152502

    2              CONSTANT( 2)                  25.107558

    3        MOVING AVERAGE ( 1, 1, 1)             .192436

    4        MOVING AVERAGE ( 1, 1, 2)             .409898

    5        MOVING AVERAGE ( 1, 2, 1)            -.574372

    6        MOVING AVERAGE ( 1, 2, 2)            1.124618



-----------------------
ERROR COVARIANCE MATRIX
-----------------------


             1           2
  1      4.877268
  2      1.108070    1.083262



ITERATIONS TERMINATED DUE TO:
  CHANGE IN  (-2*LOG LIKELIHOOD)/NOBE  .LE.  .100E-03
  TOTAL NUMBER OF ITERATIONS IS    4

FINAL MODEL SUMMARY WITH MAXIMUM LIKELIHOOD PARAMETER ESTIMATES


----- CONSTANT VECTOR (STD ERROR) -----

   17.083  (     .104  )
   25.052  (     .078  )

----- THETA MATRICES -----
```

```
ESTIMATES OF  THETA(  1 ) MATRIX AND SIGNIFICANCE

     .173     .447      + +
    -.604    1.208      - +

STANDARD ERRORS

     .038     .095
     .028     .067



-----------------------
ERROR COVARIANCE MATRIX
-----------------------

              1           2
  1     4.887851
  2     1.125052    1.014659

-2*(LOG LIKELIHOOD AT FINAL ESTIMATES)  IS    .82972218E+03



--------------------------------------------
REDUCED CORRELATION MATRIX OF THE PARAMETERS
--------------------------------------------

        1     2     3     4     5     6
  1   1.00
  2   1.00  1.00
  3    .     .    1.00
  4    .     .    -.51  1.00
  5    .     .     .88  -.45  1.00
  6    .     .    -.47   .92  -.52  1.00
--


miden rx1,rx2.max 24.output print(ccm)

TIME PERIOD ANALYZED . . . . . . . . . . . .  1  TO   250
EFFECTIVE NUMBER OF OBSERVATIONS (NOBE). . .          250
```

```
SERIES   NAME          MEAN      STD. ERROR

   1    RX1           -.0137      2.2055
   2    RX2            .0002      1.0072


NOTE: THE APPROX. STD. ERROR FOR THE ESTIMATED CORRELATIONS BELOW
      IS (1/NOBE**.5) =     .06325


SAMPLE CORRELATION MATRIX OF THE SERIES

  1.00
   .51  1.00


SAMPLE CROSS CORRELATION MATRICES FOR THE ORIGINAL SERIES.
THE (I,J) ELEMENT OF THE LAG L MATRIX IS THE ESTIMATE OF
THE LAG L CROSS CORRELATION WHEN SERIES J LEADS SERIES I


LAG = 1

  -.01  -.02
  -.05  -.03

LAG = 2

   .03   .08
   .07   .06

LAG = 3

   .02   .07
   .02  -.02

LAG = 4

  -.12  -.09
  -.01   .05

LAG = 5
```

-.02	-.04
.01	.00

LAG = 6

.06	.05
.03	.04

LAG = 7

-.11	-.09
-.09	-.07

LAG = 8

-.12	.03
-.13	-.06

LAG = 9

-.04	.09
-.08	.05

LAG =10

-.01	.01
.00	-.08

LAG =11

.05	.01
.04	-.02

LAG =12

.10	-.01
.08	-.05

LAG =13

.07	.05
.04	.04

LAG =14

```
  -.04  -.04
  -.03  -.04

LAG =15

   .02   .07
   .01   .09

LAG =16

  -.04  -.02
  -.04  -.01

LAG =17

   .06  -.01
  -.03  -.06

LAG =18

  -.02   .04
   .02   .01

LAG =19

   .08   .03
  -.02  -.05

LAG =20

   .00  -.03
  -.14  -.07

LAG =21

  -.01   .04
  -.03  -.01

LAG =22

  -.03   .06
  -.05   .00
```

```
LAG =23

   .01  -.03
   .01  -.02

LAG =24

  -.01   .05
   .07   .06

SUMMARIES OF CROSS CORRELATION MATRICES USING +,-,., WHERE
     + DENOTES A VALUE GREATER THAN 2/SQRT(NOBE)
     - DENOTES A VALUE LESS THAN -2/SQRT(NOBE)
     . DENOTES A NON-SIGNIFICANT VALUE BASED ON THE ABOVE CRITERION


BEHAVIOR OF VALUES IN (I,J)TH POSITION OF CROSS CORRELATION MATRIX OVER
ALL OUTPUTTED LAGS WHEN SERIES J LEADS SERIES I


          1              2


 1   ............   ............
 1   ............   ............


 2   .......-....   ............
 2   .......-....   ............



CROSS CORRELATION MATRICES IN TERMS OF +,-,.

LAGS   1 THROUGH   6

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS   7 THROUGH  12

     . .      . .      . .      . .      . .      . .
     . .      - .      . .      . .      . .      . .
```

```
LAGS 13 THROUGH 18

     . .      . .      . .      . .      . .      . .
     . .      . .      . .      . .      . .      . .

LAGS 19 THROUGH 24

     . .      . .      . .      . .      . .      . .
     . .      - .      . .      . .      . .      . .
--
```

```
save rx1,rx2.file '\b521\clsmar.dat'.format '2f10.5'
```

```
** THE SPECIFIED FILE DOES NOT EXIST. A NEW FILE IS OPENED.
   THE FILE NAME IS \B521\CLSMAR.DAT
THE SPECIFIED VARIABLE(S) ARE SAVED ON FILE  7
--
```

```
mforecast model classma.nofs 12.nopsiweights 3.

----------------------------------------
 12 FORECASTS, BEGINNING AT ORIGIN = 250
----------------------------------------

SERIES:       X1                   X2

TIME    FORECAST  STD ERR   FORECAST  STD ERR

 251     18.019    2.211     27.965    1.007
 252     17.083    2.326     25.052    1.624
 253     17.083    2.326     25.052    1.624
 254     17.083    2.326     25.052    1.624
 255     17.083    2.326     25.052    1.624
 256     17.083    2.326     25.052    1.624
 257     17.083    2.326     25.052    1.624
 258     17.083    2.326     25.052    1.624
 259     17.083    2.326     25.052    1.624
 260     17.083    2.326     25.052    1.624
 261     17.083    2.326     25.052    1.624
 262     17.083    2.326     25.052    1.624
```

```
PSI MATRICES USED IN THE COMPUTATION OF FORECAST ERROR MATRICES


PSI   1

    -.173    -.447
     .604   -1.208


PSI   2

     .000     .000
     .000     .000


PSI   3

     .000     .000
     .000     .000
--


stop


38




1
```

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Estimation of Time Series Parameters in the Presence of Outliers

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Outliers in time series can be regarded as being generated by dynamic intervention models at unknown time points. Two special cases, innovational outlier (IO) and additive outlier (AO), are studied in this article. The likelihood ratio criteria for testing the existence of outliers of both types, and the criteria for distinguishing between them are derived. An iterative procedure is proposed for detecting IO and AO in practice and for estimating the time series parameters in autoregressive-integrated-moving-average models in the presence of outliers. The powers of the procedure in detecting outliers are investigated by simulation experiments. The performance of the proposed procedure for estimating the autoregressive coefficient of a simple AR(1) model compares favorably with robust estimation procedures proposed in the literature. Two real examples are presented.

KEY WORDS: Additive outlier; Innovational outlier; ARIMA model; Intervention; Robust estimate.

1. INTRODUCTION

Unexpected extraordinary observations that look discordant from most observations in a data set are often encountered in various kinds of data analysis. Time series analysis is no exception. In addition to possible gross errors, time series observations are often subject to the influence of nonrepetitive exogenous interventions—for example, strikes, outbreak of wars, sudden changes in the market structure of a commodity, unexpected changes of certain conditions in a physical system, and so forth—and as a result some observations become outliers.

Let x_t be a stochastic process following an autoregressive-integrated-moving average (ARIMA) model or order p, d, and q; that is,

$$\phi(B)\alpha(B)x_t = \theta(B)a_t, \qquad (1.1)$$

where B is the backshift operator such that $Bx_t = x_{t-1}$; $\phi(B) = (1 - \phi_1 B - \cdots - \phi_p B^p)$ and $\theta(B) = (1 - \theta_1 B - \cdots - \theta_q B^q)$ are two polynomials in B with zeros all outside the unit circle; $\alpha(B) = (1 - B)^{d_1}(1 - B^s)^{d_2}$; $d = d_1 + sd_2$; and $\{a_t\}$ are $\overset{\text{iid}}{\sim} N(0, \sigma_a^2)$. The effect of an exogenous intervention on a time series can often be represented by a dynamic model (see Box and Tiao 1975) as follows:

$$z_t = \frac{\omega(B)}{\beta(B)}\,\zeta_t^{(T)} + x_t, \qquad (1.2)$$

where x_t follows the model (1.1),

$$\zeta_t^{(T)} = 1 \quad \text{for } t = T$$
$$= 0 \quad \text{otherwise}$$

signifies the time of occurrence of the intervention and $\omega(B) = (\omega_0 - \omega_1 B - \cdots - \omega_s B^s)$ and $\beta(B) = (\beta_0 - \beta_1 B - \cdots - \beta_r B^r)$ are two polynomials in B. The ratio $\omega(B)/\beta(B)$ describes the dynamic behavior of the intervention.

Guttman and Tiao (1978), Miller (1980), and Chang (1982) showed that the effect of such interventions may cause serious bias in estimating autocorrelations, partial autocorrelations, and autoregressive moving average (ARMA) parameters. It is, therefore, important to be able to identify these interventions and remove their effects from the observations to better understand the underlying structure of the series. When the time of occurrence T of an intervention is known, its effect can often be accounted for using the intervention analysis techniques proposed by Box and Tiao (1975). In practice, however, information about the timing is seldom available. In this article, we shall develop a procedure for estimating ARMA parameters under that circumstance.

Following Fox (1972), Denby and Martin (1979), and Abraham and Box (1979), we shall focus our attentions on two simple intervention models that

176

are believed to represent a large portion of the outliers likely to be found in practice. They are referred to as *innovational outlier* (IO) and *additive outlier* (AO), respectively. The model for an IO is

$$z_t = x_t + \frac{\theta(B)}{\phi(B)\alpha(B)} \omega \zeta_t^{(T)}, \tag{1.3}$$

and that of an AO is

$$z_t = x_t + \omega \zeta_t^{(T)}, \tag{1.4}$$

where $\theta(B)$, $\phi(B)$, and $\alpha(B)$ are defined as in (1.1). These models can be rewritten in terms of the innovation sequence a_t's as follows:

$$\text{(IO)} \quad z_t = \frac{\theta(B)}{\phi(B)\alpha(B)} \{a_t + \omega \zeta_t^{(T)}\}, \tag{1.5}$$

and

$$\text{(AO)} \quad z_t = \frac{\theta(B)}{\phi(B)\alpha(B)} a_t + \omega \zeta_t^{(T)}. \tag{1.6}$$

Thus the AO case may be called a *gross error* model, since only the level of the Tth observation is affected. On the other hand, an IO represents an extraordinary shock at time point T influencing $z_T, z_{T+1}, \ldots$ through the dynamic system described by $\theta(B)/[\phi(B)\alpha(B)]$. We hope that our study of these simple models might shed some light on the problems of more complex structure.

Section 2 discusses the likelihood ratio criteria for testing these two types of outliers. Section 3 presents a procedure for estimating ARMA parameters in the presence of these outliers. The power and performance of this procedure are examined and compared with those of some other approaches through simulation methods in Section 4. The proposed procedure is then applied to two examples in Section 5. Finally, some concluding remarks are given in Section 6.

2. LIKELIHOOD RATIO CRITERIA FOR DETECTING AN OUTLIER AND DETERMINING ITS NATURE

2.1 When ARMA Parameters and σ_a^2 Are Known

We shall first consider estimating the impact ω of an IO (1.3) and that of an AO (1.4), respectively, in a hypothetical situation in which all of the time series parameters, as well as the innovation variance σ_a^2 of the underlying process x_t, are known. Moreover, we assume that observations are available from $t = -J$ to $t = n$, where J is an integer much larger than $p + d + q$ and that $1 \le T \le n$. Let $\pi(B) = \phi(B)\alpha(B)/\theta(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots$. Because the zeros of $\theta(B)$ are all outside the unit circle, the weights π_j's for j beyond J would in practice become essentially equal to zero with J of moderate size.

Let $e_t = \pi(B)z_t$ for $t = 1, \ldots, n$. We can rewrite (1.5) and (1.6), respectively, as

$$\text{(IO)} \quad e_t = \omega \zeta_t^{(T)} + a_t,$$
$$\text{(AO)} \quad e_t = \omega \pi(B) \zeta_t^{(T)} + a_t. \tag{2.1}$$

In other words, the information about an IO is contained in the "residual" e_T right at that particular point T, whereas that of an AO is scattered over a string of "residuals" $e_T, e_{T+1}, \ldots$. Write $X_t = \pi(B)\zeta_t^{(T)}$ so that $X_t = 0$ for $t < T$, $X_t = 1$ for $t = T$, and $X_t = -\pi_{t-T}$ for $t > T$. From least squares theory, the estimators of the impact, ω, in these two models are

$$\text{(IO)} \quad \tilde{\omega}_I = e_T,$$
$$\text{(AO)} \quad \tilde{\omega}_A = \rho^2 \pi(F) e_T = \rho^2(1 - \pi_1 F - \pi_2 F^2 - \cdots - \pi_{n-T} F^{n-T}) e_T, \tag{2.2a}$$

respectively, where $\rho^2 = (1 + \pi_1^2 + \pi_2^2 + \cdots + \pi_{n-T}^2)^{-1}$ and F is the forward-shift operator such that $Fe_t = e_{t+1}$. Thus, not surprisingly, the effect of an IO at time T can be best estimated by the residual e_T at that point, but the best estimate of the AO effect is a linear combination of $e_T, e_{T+1}, \ldots$, with weights depending on the model from which the underlying process x_t is generated. We note that, in terms of the observations z_t, the estimate $\tilde{\omega}_A$ in (2.2a) could be written as

$$\tilde{\omega}_A = \rho^2 \pi(F) \pi(B) z_t. \tag{2.2b}$$

For the case of (autoregressive) AR(p) model, this expression reduces to equation (2.5) of Fox (1972).

The variances of these estimators are

$$\operatorname{var}(\tilde{\omega}_I) = \sigma_a^2, \qquad \operatorname{var}(\tilde{\omega}_A) = \rho^2 \sigma_a^2, \tag{2.3}$$

respectively. Note that since $\rho^2 \le 1$, the variance of $\tilde{\omega}_A$ is at most as large as that of $\tilde{\omega}_I$, and in some cases it can be much smaller than σ_a^2. For example, when the underlying process x_t follows a first-order moving average (MA) model, $x_t = (1 - \theta B)a_t$, the variance for $\tilde{\omega}_A$ would be nearly $\sigma_a^2(1 - \theta^2)$ if the outlier does not occur near the end of the series.

Let H_0 denote the hypothesis that $\omega = 0$ in (1.5) and (1.6), H_1 denote the situation $\omega \ne 0$ in (1.5), and H_2 denote the situation $\omega \ne 0$ in (1.6). The likelihood ratio criteria can be derived for testing one hypothesis versus another among the following:

H_0 vs. H_1: $\lambda_{1,T} = \tilde{\omega}_I/\sigma_a$,

H_0 vs. H_2: $\lambda_{2,T} = \tilde{\omega}_A/\rho\sigma_a$,

Table 1. Estimated Level of Significance of Tests (2.7) and (2.8): 1,000 Replications

Time series model	C	T = 50 IO	T = 50 AO	T = 100 IO	T = 100 AO	T = 150 IO	T = 150 AO
AR(1), $\phi = .6$	3.0	23.3	18.3	33.9	31.3	38.7	35.0
	3.5	5.6	3.9	8.4	4.8	8.4	7.4
	4.0	1.3	1.1	1.2	.7	1.2	1.0
MA(1), $\theta = .6$	3.0	19.9	10.9	31.5	23.3	39.2	32.4
	3.5	4.3	2.9	7.1	4.2	9.4	5.6
	4.0	1.1	.6	2.1	.7	1.5	.8

NOTE: The estimated standard deviation of the estimated level of significance is $(\hat{p}(1-\hat{p})/N)^{1/2}$, where N is the number of replications and $\hat{p}$ is the estimated significance level.

and

$$H_1 \text{ vs. } H_2: \quad \lambda_{3,T} = [\rho^{-2}\tilde{\omega}_A^2 - \tilde{\omega}_I^2]/[2\sigma_a^2(1-\rho^2)^{1/2}]. \tag{2.4}$$

Under hypothesis H_0, the statistics $\lambda_{1,T}$ and $\lambda_{2,T}$ both have the standard normal distribution. To derive the distribution of $\lambda_{3,T}$, let

$$u = [\rho^{-1}\tilde{\omega}_A + \tilde{\omega}_I]/[\sigma_a[2(1+\rho)]^{1/2}]$$

and

$$v = [\rho^{-1}\tilde{\omega}_A - \tilde{\omega}_I]/[\sigma_a[2(1-\rho)]^{1/2}].$$

Then we have that $\lambda_{3,T} = uv$. The vector $(u, v)'$ has a bivariate normal distribution with mean $(m_1, m_2)'$ and covariance matrix I, where $m_1 = \omega[(1+\rho)/(2\sigma_a^2)]^{1/2}$ and $m_2 = -\omega[(1-\rho)/(2\sigma_a^2)]^{1/2}$ under the hypothesis H_1, and where $m_1 = \omega[(1+\rho)/(2\rho\sigma_a^2)]^{1/2}$ and $m_2 = -\omega[(1-\rho)/(2\rho\sigma_a^2)]^{1/2}$ under the hypothesis H_2. It is noted that the distribution of $\lambda_{3,T}$ depends only on the two parameters, ρ and the ratio ω/σ_a, in either case. Craig (1936) showed that the distribution of the product of two such independent and normally distributed random variables can be expressed as a convergent expression of Bessel functions. The percentage points of the distribution of $\lambda_{3,T}$ under hypothesis H_1 were tabulated via numerical integration for a range of values of ω/σ_a and ρ by Chang (1982).

The likelihood ratio method further leads to the criteria

$$\text{(IO)} \quad \max_{t=1,\ldots,n} |\lambda_{1,t}|, \qquad \text{(AO)} \quad \max_{t=1,\ldots,n} |\lambda_{2,t}|, \tag{2.5}$$

for testing the possibility of an IO or an AO, respectively, at an unknown position in the series $z_1, \ldots, z_n$.

2.2 When ARMA Parameters and σ_a^2 Are Unknown

In practice, the ARMA parameters and σ_a^2 are usually unknown. Estimates of these parameters, together with that of ω under either the IO or the AO case, can be obtained by maximizing the likelihood function of $(\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, \omega, \sigma_a^2)$ in the same fashion as that described by Box and Jenkins (1976). Based on these estimates, the likelihood ratios can be computed accordingly for testing the hypotheses, one against another, in (2.4). The criteria for detecting an outlier at an unknown position then follow. Fox (1972) first discussed these ratio criteria in an AR context.

The maximum likelihood estimates for the parameters, as well as the likelihood ratios, do not, however, reduce to explicit expressions in general because

Table 2. Estimated Percentiles of the Test Statistics of Tests (2.7) and (2.8): 1,000 Replications

Time series model	%	T = 50 IO	T = 50 AO	T = 100 IO	T = 100 AO	T = 150 IO	T = 150 AO
AR(1), $\phi = .6$	10	3.29	3.21	3.45	3.36	3.45	3.43
	5	3.55	3.39	3.64	3.49	3.63	3.60
	1	4.11	4.05	4.03	3.88	4.13	4.00
MA(1), $\theta = .6$	10	3.25	3.02	3.40	3.24	3.48	3.36
	5	3.47	3.22	3.62	3.46	3.66	3.53
	1	4.02	3.80	4.27	3.88	4.04	3.95

NOTE: The standard deviation of the estimated pth percentile of the test statistics λ_p is approximately $(p(1-p)/[Nf^2(\lambda_p)])^{1/2}$, where N is the number of replications and $f(\lambda)$ is the probability density function of the test statistics. Taking the normal approximation of $f(\lambda)$, we have found that standard deviations of the estimated 10% and 5% percentiles are less than .03 and those of the estimated 1% percentiles are less than .06.

of the nonlinear nature of the AO model (2.1) and that of the general ARMA models. Nonlinear estimation algorithms are, therefore, necessary to implement the likelihood ratio tests in this circumstance, and the computational burden will become very heavy when the timing of the outlier is unknown. Nevertheless, several simpler statistics arise naturally as possible approximations to these likelihood ratio criteria.

Let $\hat{\phi}_1, \ldots, \hat{\phi}_p$; $\hat{\theta}_1, \ldots, \hat{\theta}_q$; and $\hat{\sigma}_a^2$ be the maximum likelihood estimate of the parameters $\phi_1, \ldots, \phi_p$; $\theta_1, \ldots, \theta_q$; and σ_a^2, respectively, obtained by treating the time series z_t as containing no outliers. Moreover, let $\hat{e}_t$ be the residuals computed from such an estimated model and $\hat{\pi}(B) = \hat{\phi}(B)\alpha(B)/\hat{\theta}(B)$. We shall consider the following statistics:

$$\hat{\lambda}_{1,T} = \hat{\omega}_I/\hat{\sigma}_a, \qquad \hat{\lambda}_{2,T} = \hat{\omega}_A/(\hat{\rho}\hat{\sigma}_a), \tag{2.6}$$

and

$$\hat{\lambda}_{3,T} = \frac{\hat{\rho}^{-2}\hat{\omega}_A^2 - \hat{\omega}_I^2}{2\hat{\sigma}_a^2(1 - \hat{\rho}^2)^{1/2}},$$

where

$$\hat{\omega}_I = \hat{e}_T,$$

$$\hat{\omega}_A = \hat{\rho}^2(1 - \hat{\pi}_1 F - \hat{\pi}_2 F^2 - \cdots - \hat{\pi}_{n-T} F^{n-T})\hat{e}_T,$$

and

$$\hat{\rho}^2 = (1 + \hat{\pi}_1^2 + \hat{\pi}_2^2 + \cdots + \hat{\pi}_{n-T}^2)^{-1}.$$

It can be shown that $\hat{\lambda}_{1,T}$, $\hat{\lambda}_{2,T}$, and $\hat{\lambda}_{3,T}$ are asymptotically equivalent to the likelihood ratio criteria $\lambda_{1,T}$, $\lambda_{2,T}$, and $\lambda_{3,T}$ in (2.4) for testing hypotheses H_0 versus H_1, H_0 versus H_2, and H_1 versus H_2, respectively, at a given point T (see Chang 1982).

To detect an IO or an AO at an unknown position, we can then scan through the sequence of $\hat{\lambda}_{1,t}$, $t = 1, \ldots, n$ or the sequence of $\hat{\lambda}_{2,t}$, $t = 1, \ldots, n$, respectively. In other words, the general possibility of an IO in the series or that of an AO in the series can be tested by

$$\hat{\eta}_{\text{IO}} = \max_{t=1,\ldots,n} |\lambda_{1,t}| > C \tag{2.7}$$

or

$$\hat{\eta}_{\text{AO}} = \max_{t=1,\ldots,n} |\lambda_{2,t}| > C, \tag{2.8}$$

respectively, where C is some suitably chosen positive constant. A simulation experiment was conducted on an AR model and an MA model to estimate the significance level of the tests (2.7) and (2.8) at critical values $C = 3$, 3.5, and 4 for sample sizes $n = 50$, 100, and 150. The results are given in Tables 1 and 2. Because of the extreme-value nature of the statistics (they are maximum of a set of random variables), the tail probability of the statistics is expected to increase as the sample size increases. This study shows that the estimated percentiles only increase moderately when n changes from 50 to 150; for instance, the 10% percentile of $\hat{\eta}_{\text{AO}}$ changes from 3.21 to 3.43 in the AR(1) case and from 3.02 to 3.36 in the MA(1) case. We also observe that the estimated 1%, 5%, and 10% percentiles of $\hat{\eta}_{\text{IO}}$ are slightly higher than those of $\hat{\eta}_{\text{AO}}$. In practice, we recommend using $C = 3.0$ for high sensitivity, $C = 3.5$ for medium sensitivity, and $C = 4.0$ for low sensitivity in the outlier-detecting procedure when the length of the series is less than 200.

2.3 Distinguishing an AO From an IO

There is usually little information available in practice about what type the possible outlier might be; thus it is unclear which detection test, (2.7) or (2.8), is more appropriate for a given situation. When a test of an inappropriate type is used, the detecting power of the test could be substantially reduced. Furthermore, even if it is known that an outlier has occurred at a particular point, the possibly adverse effect of the outlier may not be easy to remove unless its nature is properly identified. The statistic $\hat{\lambda}_{3,T}$ in (2.6) is derived for making the distinction between an IO and an AO at a given point T. When the position of the possible outlier is unknown, however, we may need to perform such a test repeatedly at various time points, and it can become a cumbersome task.

To simplify the problem, we consider a simple rule mentioned by Fox (1972) as a possible way to distinguish between an IO and an AO. At any suspected point T, the possible outlier is classified as an IO if $|\hat{\lambda}_{1,T}| > |\hat{\lambda}_{2,T}|$, and it is classified as an AO if $|\hat{\lambda}_{1,T}| \le |\hat{\lambda}_{2,T}|$.

From a simulation experiment similar to that described in Section 2.2 on AR series of moderate length, $n = 50$, we have noted that on the average this simple rule can correctly identify the nature of an outlier, either IO or AO, at a particular point 78.6% of the time when the magnitude ω of the outlier is $3\sigma_a$, whereas the test based on $\hat{\lambda}_{3,T}$ at the nominal 10% level has an average success rate of 87.3%. When the magnitude ω is increased to $5\sigma_a$, however, this simple rule makes correct identification, based on the simulation study, 92% of the time, which is about the same as the 92.4% achieved by the test based on $\hat{\lambda}_{3,T}$ at the 10% level (Chang 1982).

3. AN ITERATIVE PROCEDURE FOR OUTLIER DETECTION AND PARAMETER ESTIMATION

The considerations in Section 2 have led to the following iterative procedure to handle situations in which there may exist an unknown number of IO's or AO's. The procedure begins with modeling the original series z_t by supposing that there is no outlier. Then the outlier-detection steps and the parameter-estimation step will be alternatively followed. A detailed description of the procedure follows:

3.1 Outlier-Detection Stage

1. From the estimated model, compute the residuals $\hat{e}_t$, and let $\hat{\sigma}_a^2 = n^{-1} \sum_{t=1}^{n} \hat{e}_t^2$ be the estimate of σ_a^2. A possible robust alternative estimate of $\hat{\sigma}_a$ may be based on the median of the absolute values of residuals.

2. Compute $\hat{\lambda}_{1,T}$ and $\hat{\lambda}_{2,T}$ in (2.2a) and let $\eta_t = \max\{|\hat{\lambda}_{1,t}|, |\hat{\lambda}_{2,t}|\}$ for $t = 1, \ldots, n$. If $\max_t \eta_t = |\hat{\lambda}_{1,T}| > C$, where C is a predetermined positive constant, then there is the possibility of an IO at T. The impact ω of this possible IO is estimated by $\hat{\omega}_I$ in (2.2a). We then eliminate its effect by defining a new residual $\check{e}_T = \hat{e}_T - \hat{\omega}_I = 0$ at T. If $\max_t \eta_t = |\hat{\lambda}_{2,T}| > C$, then there is the possibility of an AO at T and its impact is estimated by $\hat{\omega}_A$ in (2.2a). The effect of this AO can be removed by defining new residuals $\check{e}_t = \hat{e}_t - \hat{\omega}_A \hat{\pi}(B)\zeta_t^{(T)}$ for $t \geq T$. In either of the preceding cases, a new estimate $\check{\sigma}_a^2$ is computed from the modified residuals.

3. If an IO or an AO is identified in step 2, recompute $\hat{\lambda}_{1,T}$ and $\hat{\lambda}_{2,T}$ based on the same initial estimates of the time series parameters, but using the modified residuals $\check{e}_t$'s and the estimate $\check{\sigma}_a^2$, and repeat step 2.

4. Continue to repeat steps 2 and 3 until no further outlier candidates can be identified.

3.2 Parameter Estimation Stage

5. Suppose at k time points $T_1, T_2, \ldots, T_k$ are pinpointed for possible IO's or AO's. Treat these times as known, and estimate the outlier parameters $\omega_1, \omega_2, \ldots, \omega_k$ and the time series parameters simultaneously, as described by Box and Tiao (1975), using models of the form

$$z_t = \sum_{j=1}^{k} \omega_j L_j(B)\zeta_t^{(T_j)} + \frac{\theta(B)}{\phi(B)\alpha(B)} a_t, \qquad (3.1)$$

where $L_j(B) = 1$ for an AO and $L_j(B) = \theta(B)/[\phi(B)\alpha(B)]$ for an IO at $t = T_j$.

Treating model (3.1) as the entertained model, we start the outlier detection stage again. The notations $\hat{\pi}_j$'s, $\hat{\omega}_j$'s and $\hat{e}_t$ represent the estimated values obtained from the joint estimation of all parameters of model (3.1). If no other outliers are found, we stop. Otherwise, the estimation stage is repeated, with the newly identified outliers incorporated into model (3.1), until no more outliers can be found and all of the outlier effects have been simultaneously estimated with the time series parameters.

4. PERFORMANCE OF THE ITERATIVE PROCEDURE

We have conducted simulation studies to obtain some information about the performance of the preceding procedure. Results on (a) the powers of detecting outliers and (b) the estimation of time series parameters will be reported in this section.

4.1 Power of the Iterative Outlier Detection Procedure

The proposed procedure is designed to detect the existence and to identify the types of outliers simultaneously. To assess the power of the procedure, we should be concerned with the probability of correctly detecting the location of outliers and the probability of correctly identifying the type of outliers. In this study, we focus on the issues of (a) single or multiple outliers, (b) time series structure of x_t, (c) types of outliers, (d) size of outliers, and (e) sample size of the series. For factor (a), we consider the cases of one outlier and two outliers.

In the one-outlier case, we will examine 24 situations that result from combinations of two series models for x_t, AR(1) with $\phi = .6$, $\sigma_a^2 = 1$ and MA(1) with $\theta = .6$, $\sigma_a^2 = 1$; two types of outliers, IO and AO; two different sizes of outliers, $\omega = 3\sigma_a$ and $\omega = 5\sigma_a$; and three sample sizes, $n = 50$, 100, and 150. The location of the outlier is set in the middle of the observational period, specifically $T = 26$ when $n = 50$, $T = 51$ for $n = 100$, and $T = 76$ for $n = 150$.

For the two outliers case, we consider the AR(1) model with $\phi = .6$, and $\sigma_a^2 = 1$ for x_t; three types of outliers, two AO's, two IO's, and one IO, one AO; two sizes, both outliers having $\omega = 3\sigma_a$ and both having $\omega = 5\sigma_a$; and three sample sizes, $n = 50$, 100, and 150. We assume that outliers occurred at the one-third point from the beginning period and the one-third point from the ending period, or $T_1 = 17$, $T_2 = 34$ for $n = 50$; $T_1 = 34$, $T_2 = 66$ for $n = 100$; and $T_1 = 51$, $T_2 = 101$ for $n = 150$.

In each case of the preceding design, time series data are generated according to the particular specification. Assuming that the model is known and that there is no outlier in the sample, the exact likelihood estimation method, considered by Hillmer and Tiao (1979), will be used to estimate time series parameters and to produce the residuals. Based on the estimated parameters and residuals, one iteration of the outlier-detection steps will be conducted with three critical values, $C = 3$, 3.5, and 4, individually. The outcomes of this detection procedure are then compared with the specification of the alternatives. Repeating such

Table 3. Frequency of Correct Detection of Outliers–Location (percentage of correct identification of types): Critical Value C = 4.0; 1,000 Replications

	$\omega = 3\sigma_a$			$\omega = 5\sigma_a$		
n	50	100	150	50	100	150
AR, 1 AO	236	266	278	857	945	948
	(.72)	(.80)	(.81)	(.91)	(.90)	(.92)
AR, 1 IO	188	185	174	777	837	820
	(.88)	(.87)	(.87)	(.95)	(.93)	(.91)
MA, 1 AO	172	248	283	814	914	952
	(.70)	(.82)	(.84)	(.83)	(.90)	(.93)
MA, 1 IO	157	194	185	769	832	837
	(.97)	(.92)	(.88)	(.99)	(.96)	(.97)
AR, 2 AO'S	39	70	75	681	854	847
	(.38)	(.54)	(.52)	(.53)	(.76)	(.80)
1st Outlier	135	227	247	774	921	914
	(.65)	(.74)	(.81)	(.76)	(.86)	(.92)
2nd Outlier	141	204	252	759	917	920
	(.64)	(.73)	(.78)	(.74)	(.88)	(.88)
AR, 2 IO'S	24	24	33	557	727	682
	(.58)	(.67)	(.55)	(.82)	(.86)	(.84)
1st Outlier	120	137	162	689	835	800
	(.84)	(.80)	(.85)	(.91)	(.94)	(.91)
2nd Outlier	102	136	159	674	831	833
	(.77)	(.88)	(.83)	(.92)	(.91)	(.92)
AR, IO AO	52	41	48	689	781	788
	(.65)	(.63)	(.69)	(.86)	(.87)	(.88)
1st Outlier	136	142	133	718	813	828
	(.90)	(.86)	(.84)	(.98)	(.95)	(.93)
2nd Outlier	184	216	268	842	929	941
	(.72)	(.81)	(.81)	(.88)	(.92)	(.93)

operations, we may estimate the probability of correctly detecting an outlier's location and the probability of correctly identifying the outlier's type given that the location has been correctly detected. Tables 3, 4, and 5 give the results of 1,000 replications of the preceding operations for each of the alternative specifications using critical values $C = 3$, $C = 3.5$, and $C = 4$, respectively. The tables give the frequency with which the location of an outlier was correctly detected. Numbers in parentheses are percentages of correct identification of an outlier's type given that the location has been detected correctly. In the two-outlier cases, rows labeled by 1st and 2nd show the results of detecting and identifying the first outlier and the second outlier, respectively.

In general, the power (probability of correct detection or identification of outliers) of the procedure increases when sample size increases and decreases as the critical value C increases. For large-size outliers, $\omega = 5\sigma_a$, the procedure seems to be fairly powerful. Probabilities of correct detection, using $C = 3.5$, range from 89.6% to 98.8% for one-outlier cases and from 79.2% to 95.2% for two-outlier cases. The percentages of correct identification of outlier types ranges from 76% to 98% except for the case of two AO's with $n = 50$. For medium-size outliers, $\omega = 3\sigma_a$, the performance of the procedure is not as good. Especially when there is more than one outlier, the procedure may very likely miss some of them. Note that the results here are from the first round of outlier-detection procedure. In practice, the parameter-estimation stage will be followed according to the outlier-detection results, and the second-round outlier detection may be conducted if necessary. Hence the actual power of the procedure may be higher than what we report here, especially for the two-outliers case. Finally, note that the estimated standard deviation of the estimated power is $[\hat{p}(1 - \hat{p})/N]^{1/2}$, where N is the number of replications and $\hat{p}$ is the estimated power.

4.2 Estimation in the Presence of Outliers

In this study the estimates of the time series parameters computed from the proposed procedure were compared with the M estimates and the generalized M estimates (GM) proposed by Denby and Martin (1979) and Martin (1980).

Time series were generated from an AR(1) model under several different situations. In addition to the null case, the IO case (1.3), and the AO case (1.4), we

Table 4. Frequency of Correct Detection of Outliers–Location (percentage of correct identification of types): Critical Value C = 3.5; 1,000 Replications

	$\omega = 3\sigma_a$			$\omega = 5\sigma_a$		
n	50	100	150	50	100	150
AR, 1 AO	443	482	481	947	982	975
	(.75)	(.80)	(.82)	(.86)	(.90)	(.92)
AR, 1 IO	319	335	336	901	936	917
	(.85)	(.88)	(.81)	(.94)	(.93)	(.90)
MA, 1 AO	326	477	492	927	975	989
	(.75)	(.83)	(.84)	(.84)	(.91)	(.93)
MA, 1 IO	290	346	347	896	935	924
	(.94)	(.90)	(.89)	(.98)	(.96)	(.96)
AR, 2 AO'S	159	220	229	862	943	952
	(.44)	(.54)	(.62)	(.58)	(.76)	(.80)
1st Outlier	337	428	466	907	968	974
	(.70)	(.75)	(.81)	(.78)	(.86)	(.92)
2nd Outlier	333	400	452	911	973	977
	(.70)	(.76)	(.80)	(.77)	(.88)	(.88)
AR, 2 IO'S	89	100	127	792	881	854
	(.66)	(.65)	(.61)	(.82)	(.85)	(.82)
1st Outlier	246	298	314	877	939	911
	(.83)	(.81)	(.84)	(.91)	(.94)	(.91)
2nd Outlier	241	281	338	870	929	930
	(.82)	(.85)	(.84)	(.91)	(.90)	(.91)
AR, IO AO	157	124	180	861	907	901
	(.64)	(.69)	(.65)	(.86)	(.87)	(.88)
1st Outlier	293	275	316	882	920	925
	(.88)	(.84)	(.84)	(.97)	(.95)	(.93)
2nd Outlier	379	410	506	954	979	975
	(.75)	(.82)	(.79)	(.88)	(.92)	(.93)

also considered the following $100(\gamma_a + \gamma_v - \gamma_a\gamma_v)\%$ contamination model in which the number of outliers is determined by a random mechanism:

$$y_t = \phi y_{t-1} + a_t^*, \qquad z_t = y_t + v_t, \qquad (4.1)$$

where z_t's are the observations, a_t^*'s are iid with a contaminated normal density $(1 - \gamma_a)N(0, \sigma_a^2) + \gamma_a N(0, \kappa_a^2\sigma_a^2)$, v_t's are iid with density $(1 - \gamma_v)\delta(0) + \gamma_v N(0, \kappa_v^2\sigma_a^2)$, $\delta(0)$ represents a degenerate density at 0, and v_t's and a_t^*'s are all independent. This model, considered also by Denby and Martin (1979), may generate IO's or AO's or both. We shall label it as a *mixture model* in the following text.

For each simulated series, the following estimates of the AR(1) parameter ϕ were computed:

LS—The least squares estimate.

M-H—*M* estimate with Huber-type "down-weighting" function for residuals. The tuning constant C_{a_H} is set at 1.5.

M-B—*M* estimate with a bisquare type "down-weighting" function for residuals. The tuning constant C_{a_B} is set at 6.0.

GM-H—Generalized *M* estimate with two Huber-type "down-weighting" functions for residuals and observations, respectively. The tuning constants are set as $C_{a_H} = 1.5$ and $C_{Z_H} = 1.0$.

GM-B—Generalized *M* estimate with two bisquare type "down-weighting" functions for residuals and observations, respectively. The tuning constants are set as $C_{a_B} = 6.0$ and $C_{Z_B} = 3.9$.

P—The estimate computed from the procedure proposed in Section 3 with the critical value $C = 3.0$.

Following Denby and Martin (1979), we computed four iterations of the iterated weighted least squares (IWLS) algorithm (Beaton and Tukey 1974) for the M-H and GM-H estimates using the LS estimate as the starting point. The M-H and GM-H estimates were then used as starting values for another four iterations of IWLS to obtain the M-B and GM-B estimates, respectively. In these IWLS iterations, the scale parameter σ_a was estimated by the median of the absolute values of the residuals divided by .6745, and the scale parameter σ_z was estimated by the median of the absolute deviations of the observations from their sample median divided by .6745.

Since all estimates here, except the LS estimate, were computed from an iterative algorithm, a convergence criterion needs to be set. We considered an

Table 5. Frequency of Correct Detection of Outliers–Location (percentage of correct identification of types): Critical Value C = 3.0; 1,000 Replications

	$\omega = 3\sigma_a$			$\omega = 5\sigma_a$		
n	*50*	*100*	*150*	*50*	*100*	*150*
AR, 1 AO	644	703	692	984	993	990
	(.75)	(.81)	(.81)	(.86)	(.90)	(.92)
AR, 1 IO	504	530	564	959	985	975
	(.83)	(.82)	(.77)	(.93)	(.93)	(.90)
MA, 1 AO	547	672	703	976	993	996
	(.78)	(.83)	(.85)	(.84)	(.91)	(.93)
MA, 1 IO	486	536	552	958	981	973
	(.91)	(.87)	(.88)	(.98)	(.96)	(.97)
AR, 2 AO'S	321	460	453	953	985	986
	(.50)	(.59)	(.63)	(.60)	(.76)	(.80)
1st Outlier	531	652	676	975	991	993
	(.74)	(.77)	(.81)	(.79)	(.86)	(.91)
2nd Outlier	561	636	655	973	994	993
	(.72)	(.77)	(.80)	(.77)	(.88)	(.88)
AR, 2 IO'S	217	283	300	908	962	951
	(.61)	(.63)	(.65)	(.81)	(.85)	(.82)
1st Outlier	444	522	518	956	983	972
	(.82)	(.79)	(.85)	(.91)	(.93)	(.90)
2nd Outlier	449	498	552	946	975	978
	(.78)	(.82)	(.80)	(.90)	(.90)	(.91)
AR, IO AO	347	364	404	934	976	964
	(.65)	(.68)	(.65)	(.86)	(.86)	(.87)
1st Outlier	480	516	537	947	984	973
	(.87)	(.82)	(.84)	(.97)	(.94)	(.93)
2nd Outlier	645	658	712	986	1000	991
	(.76)	(.83)	(.79)	(.89)	(.91)	(.93)

algorithm as having converged here if the estimates given by the most recent two consecutive iterations differed from each other by less than .0001.

For each outlier model, 500 replications of the same length were generated with $\sigma_a^2 = 1$ and, for the mixture model (4.1), $\sigma_v^2 = 1$. The lengths were moderate, $n = 50$ or 75. The means and the standard deviations, SD's as well as the mean squared errors (MSE's) of the estimates of the parameter ϕ, were computed over the 500 replications. Tables 6 and 7 present the results in two different cases, $\phi = .6$ and $\phi = .9$, respectively. The values of the commonly adopted approximation, $((1 - \phi^2)/n)^{1/2}$, to the standard deviation of the LS estimate of ϕ in the null case are also shown at the bottom of both tables for reference.

Note from these results that the bias of the *M* estimates in the AO case is almost as severe as that of the LS estimate. This point was also made earlier by Denby and Martin (1979). On the other hand, the generalized *M* estimates are less efficient in the null and the IO cases. The estimate computed from our proposed procedure, however, gives the smallest MSE's in most of the cases studied here. It compares favorably with the robust procedures M-H, M-B, and GM-H in every case, and only in the AO situation with $\phi = .6$ the proposed procedure is sometimes slightly worse off relative to GM-B.

The MSE's of the estimates from the proposed procedure in most cases are fairly close to those of the LS estimates in the null case, possibly because of the reasonably high success rate of the procedure in detecting outliers (see Table 3). Moreover, note that almost all of the GM-B estimates were obtained here after eight iterations of IWLS and that, throughout this study, our proposed procedure stopped at or before the fourth estimation cycle. As to the precision of this Monte Carlo study, assuming the sampling distribution of $\hat{\phi}$ is normal, the estimated standard deviation of the estimated MSE's is approximately $(2/N)^{1/2}\hat{\sigma}_{\hat{\phi}}^2$, where N is the number of replications and $\hat{\sigma}_{\hat{\phi}}^2$ is the estimated variance of $\hat{\phi}$. The choice that $N = 500$ assures that the estimated SD(MSE) in all of the cases considered is less than .002.

5. ILLUSTRATIVE EXAMPLES

Two examples will be analyzed here to illustrate the application of the proposed procedure described in Section 3.

Table 6. Means, Standard Deviations, and Mean Squared Errors of the Estimates of the AR(1) Parameter (true $\phi = .6$)

Outlier model	LS	M-H	M-B	GM-H	GM-B	p
Null	.5885	.5883	.5885	.5947	.5940	.5933
$n = 50$	(.1143)	(.1157)	(.1147)	(.1223)	(.1225)	(.1165)
	.0132	.0135	.0133	.0150	.0150	.0136
1 IO (fixed)	.5804	.5808	.5814	.5813	.5756	.5932
$T = 24, n = 50,$	(.1110)	(.0989)	(.0978)	(.1125)	(.1251)	(.0968)
$\omega = 5$	.0127	.0101	.0099	.0130	.0162	.0094
1 AO (fixed)	.4456	.4746	.4721	.5441	.5724	.5802
$T = 24, n = 50,$	(.1270)	(.1294)	(.1332)	(.1298)	(.1283)	(.1304)
$\omega = 5$	.0399	.0324	.0341	.0199	.0172	.0174
2 AO'S (fixed)	.4013	.4239	.4202	.5123	.5513	.5496
$T = 17, 34;$	(.1369)	(.1386)	(.1418)	(.1356)	(.1343)	(.1428)
$\omega = -3, 5; n = 50$	.0582	.0502	.0524	.0260	.0204	.0229
2 IO'S (fixed)	.5809	.5847	.5849	.5886	.5854	.5998
$T = 17, 34;$	(.1123)	(.1008)	(.1014)	(.1110)	(.1192)	(.1011)
$\omega = -3, 5; n = 50$	.0130	.0104	.0105	.0124	.0144	.0102
3 AO'S (fixed)	.3265	.3350	.3283	.4505	.5151	.4971
$T = 14, 26, 38;$	(.1354)	(.1300)	(.1319)	(.1393)	(.1524)	(.1686)
$\omega = -4, 5, 4; n = 50$	.0931	.0871	.0912	.0417	.0304	.0390
3 AO'S (fixed)	.3863	.4060	.3990	.5075	.5541	.5649
$T = 20, 42, 61;$	(.1112)	(.1117)	(.1109)	(.1064)	(.1054)	(.1136)
$\omega = -4, 5, 4; n = 75$	.0580	.0501	.0527	.0199	.0132	.0141
Mixture	.4929	.5056	.5094	.5470	.5604	.5606
$\gamma_a = .02, \gamma_v = .02,$	(.1559)	(.1460)	(.1455)	(.1241)	(.1246)	(.1242)
$\kappa_a^2 = 25, \kappa_v^2 = 25, n = 50$	.0357	.0302	.0293	.0182	.0171	.0170
Mixture	.4540	.4720	.4727	.5183	.5360	.5448
$\gamma_a = .05, \gamma_v = .05,$	(.1617)	(.1478)	(.1491)	(.1308)	(.1354)	(.1289)
$\kappa_a^2 = 16, \kappa_v^2 = 16, n = 50$	.0474	.0382	.0384	.0237	.0224	.0196

NOTE: $((1 - \phi^2)/n)^{1/2} \approx .1131$ when $n = 50$. The first entry in each case is the mean; the second entry (in parentheses) is the standard deviation; the third one is the mean squared error.

5.1 Box–Jenkins Series A

The first example is series A from Box and Jenkins (1976). The data are "uncontrolled" concentration readings of a chemical process recorded at every two-hour interval. Two models were suggested by Box and Jenkins (1976, p. 239):

$$x_t = c_0 + \frac{(1 - \theta B)}{(1 - \phi B)} a_t \tag{5.1}$$

and

$$(1 - B)x_t = (1 - \theta B)a_t. \tag{5.2}$$

In our iteration cycle 1, preliminary estimates of model (5.1) and model (5.2) are obtained assuming that there is no outlier, and the results are given in Table 8. Conducting steps 1 and 2 of the procedure in Section 3 using $C = 3.5$ based on the preliminary estimates leads to identification of an IO at $T = 64$ for both models and then the effect of this IO is removed by modifying the residuals accordingly. Following step 3 by using the modified residuals $\check{e}_t$'s and the estimated $\check{\sigma}_a^2$, an AO is found at $T = 43$ in both models. The next iteration of steps 2 and 3 shows that no outlier is identified. Then the time series parameters and the outlier parameters ω_1 and ω_2 are estimated simultaneously as described in the estimation stage. Time series parameters estimates are given under estimation cycle 2 in Table 8. Repeating the outlier-detection stage based on the newly estimated parameters and residuals, no other outlier is found.

For both models, adjusting the effect of two outliers results in changes of parameter estimates and reduction in the estimated σ_a^2. As might be expected, the procedure identifies the same types and locations of outliers for two somewhat different models of the same data.

5.2 The Los Angeles Oxidant Data

The monthly average of hourly readings of ozone in downtown Los Angeles from January 1955 to December 1972 is examined. We adopt the following

Table 7. Means, Standard Deviations, and Mean Squared Errors of the Estimates of the AR(1) Parameter (true $\phi = .9$)

Outlier model	LS	M-H	M-B	GM-H	GM-B	p
Null	.8638	.8639	.8639	.8682	.8657	.8651
$n = 50$	(.0798)	(.0811)	(.0803)	(.0838)	(.0876)	(.0793)
	.0077	.0079	.0077	.0080	.0088	.0075
1 IO (fixed)	.8688	.8752	.8771	.8786	.8783	.8813
$T = 24, n = 50,$	(.0787)	(.0708)	(.0703)	(.0711)	(.0717)	(.0673)
$\omega = 5$	.0071	.0056	.0055	.0055	.0056	.0049
1 AO (fixed)	.7686	.8163	.8264	.8440	.8555	.8658
$T = 24, n = 50,$	(.1155)	(.1073)	(.1137)	(.0938)	(.0894)	(.0819)
$\omega = 5$	.0306	.0185	.0183	.0119	.0100	.0079
2 AO'S (fixed)	.7483	.7927	.7962	.8267	.8353	.8601
$T = 17, 34;$	(.1189)	(.1106)	(.1173)	(.0952)	(.0924)	(.0855)
$\omega = -3, 5; n = 50$	.0371	.0237	.0245	.0144	.0127	.0089
2 IO'S (fixed)	.8678	.8745	.8768	.8768	.8765	.8802
$T = 17, 34;$	(.0693)	(.0631)	(.0629)	(.0645)	(.0698)	(.0635)
$\omega = -3, 5; n = 50$	.0058	.0046	.0045	.0047	.0054	.0044
3 AO'S (fixed)	.6845	.7360	.7340	.7932	.8078	.8554
$T = 14, 26, 38;$	(.1420)	(.1421)	(.1507)	(.1131)	(.1084)	(.1000)
$\omega = -4, 5, 4; n = 50$	.0665	.0470	.0502	.0242	.0202	.0120
3 AO'S (fixed)	.7480	.7956	.7961	.8301	.8393	.8743
$T = 20, 42, 61;$	(.1016)	(.0970)	(.1046)	(.0814)	(.0787)	(.0675)
$\omega = -4, 5, 4; n = 75$	.0334	.0203	.0217	.0115	.0099	.0052
Mixture	.8110	.8420	.8541	.8592	.8621	.8688
$\gamma_a = .02, \gamma_v = .02,$	(.1268)	(.0914)	(.0837)	(.0756)	(.0787)	(.0718)
$\kappa_a^2 = 25, \kappa_v^2 = 25, n = 50$	.0240	.0117	.0091	.0074	.0076	.0061
Mixture	.7760V.8115	.8196	.8344	.8403	.8551	
$\gamma_a = .05, \gamma_v = .05,$	(.1407)	(.1180)	(.1196)	(.0955)	(.0965)	(.0879)
$\kappa_a^2 = 16, \kappa_v^2 = 16, n = 50$	.0351	.0217	.0207	.0134	.0129	.0097

NOTE: $((1 - \phi^2)/n)^{1/2} \approx .0616$ when $n = 50$.

intervention model considered by Box and Tiao (1975):

$$y_t = \omega_1 \eta_{t1} + \omega_2 \frac{\eta_{t2}}{(1 - B^{12})} + \omega_3 \frac{\eta_{t3}}{(1 - B^{12})} + \frac{(1 - \theta_1 B)(1 - \theta_{12} B^{12})}{(1 - B^{12})} a_t, \quad (5.3)$$

where

$$\eta_{t1} = 0, \quad t < 60$$
$$= 1, \quad t \geq 61,$$
$$\eta_{t2} = 1, \quad t = 137 + 12l + k,\ k = 1, \ldots, 6,\ l = 0, 1, \ldots$$
$$= 0, \quad \text{otherwise},$$

Table 8. Box–Jenkins Series A: Outlier Detection and Parameter Estimation

Estimation cycle	Parameters: c_0	$\hat{\phi}$	$\hat{\theta}$	$\hat{\sigma}_a^2 \times 10^3$	Outliers: Time point	Type	$\hat{\omega}$	$\hat{\lambda}$
Model (5.1)								
1	17.1	.91	.59	97.53	64	IO	1.12	3.72
	(.11)	(.04)	(.08)		43	AO	−1.00	−3.68
2	17.1	.89	.47	83.52				
	(.09)	(.04)	(.09)					
Model (5.2)								
1			.70	100.74	64	IO	1.13	3.67
			(.05)		43	AO	−.98	−3.53
2			.63	88.04				
			(.05)					

NOTE: Numbers in parentheses are standard errors of the estimated parameters.

Table 9. Los Angeles Oxidant Data: Outlier Detection and Parameter Estimation

Estimation cycle	Parameters						Outliers			
	$\hat{\theta}_1$	$\hat{\theta}_{12}$	$\hat{\sigma}_a^2 \times 10^3$	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\omega}_3$	Time point	Type	ω	λ
1	−.27 (.07)	.78 (.04)	616.1	−1.34 (.19)	−.24 (.06)	−.10 (.05)	21 54	AO IO	2.40 2.21	3.61 3.14
2	−.30 (.06)	.80 (.04)	552.2	−1.29 (.18)	−.23 (.05)	−.10 (.05)				

and

$$\eta_{t3} = 1, \quad t = 142 + 12l + k, \; k = 1, \ldots, 6, \; l = 0, 1, \ldots$$
$$= 0, \quad \text{otherwise.}$$

The preliminary estimates of the intervention parameters $(\omega_1, \omega_2, \omega_3)$ and the time series parameters are given in Table 9 under estimation cycle 1 assuming no outlier. Let

$$\check{y}_t = y_t + \hat{\omega}_1 \eta_{t1} - \hat{\omega}_2 \eta_{t2}/(1 - B^{12}) - \hat{\omega}_3 \eta_{t3}/(1 - B^{12})$$

be the modified observations. The statistics $\hat{\lambda}_{1t}$ and $\hat{\lambda}_{2t}$ are calculated as described in step 2 of our procedure. From these statistics an AO is identified at $t = 21$. The effect of this AO can be removed by calculating the modified residuals accordingly. Following step 3 of the procedure, a new set of $\hat{\lambda}_{1t}$ and $\hat{\lambda}_{2t}$ are calculated. These indicate a potential IO at time $t = 54$. One more iteration of modifying residuals and recalculating $\hat{\lambda}_{1t}$ and $\hat{\lambda}_{2t}$ does not show evidence of further outliers. Then the time series parameters, intervention parameters as well as outlier parameters, are estimated simultaneously as described in step 4 of the procedure, and the time series and intervention parameter estimates are reported in Table 9 under estimation cycle (2). The next iteration of outlier-detection stage does not indicate the existence of additional outliers. In this example, adjusting the effect of two outliers results in slight changes of the estimated time series and intervention parameters and about a 10% reduction in $\hat{\sigma}_a^2$.

6. CONCLUDING REMARKS

A summary of the iterative outlier detection procedure proposed in Section 3 can be found in a review paper on modeling issues in the seasonal adjustment of economic time series by Hillmer, Bell, and Tiao (1983). They applied the procedure to the logarithms of monthly retail sales of variety stores from January 1967 to September 1979 obtained from the U.S. Bureau of the Census. The model for the series involves the differencing operation $(1 - B)(1 - B^{12})$. It is interesting to note that of the eight AO's and IO's detected, three consecutive IO's occurred beginning at the time point $t_0 = 112$ (April 1976). These three IO's corresponded to a level drop at $t_0 = 112$ and turned out to be associated with the event that a major variety-store chain went out of business at that time. Although the proposed procedure is designed to detect and to distinguish two simple types of outliers, the IO's and the AO's, in this example it actually revealed the effect of a more complex intervention. A more efficient way to detect level shifts at unknown time points was considered by Bell (1983) and Chen (1984).

In this article, the proposed procedure is demonstrated to be useful for estimating time series parameters when there is the possibility of outliers. It can be applied to all invertible ARIMA models. Moreover, it is flexible and easy to interpret, and it can be implemented with very few modifications to existing software packages capable of dealing with ARIMA and transfer function models. In practice, we suggest that this procedure be used in conjunction with other diagnostic tools for time series analysis to produce even better results.

Much further work is needed to investigate the variances and other sampling properties of the resulting estimates of time series parameters from the procedure proposed in Section 3. The simulation results in Section 4 on the AR(1) model seem to suggest that for moderate sample size the variance of such estimates might not be too far from those of the LS estimates obtained in the ideal null case.

Finally, the proposed procedure assumes that the form of the ARIMA model is correctly specified. Depending on the model specification (or identification) tool employed, the existence of AO's and IO's could conceivably lead to misspecification of the model form. As a possible solution to this problem, when an outlier is detected at $t = T$, one may simply adjust the observation z_T from (1.3) or (1.4) by employing the estimated effect of the outlier, and respecify the model accordingly.

ACKNOWLEDGMENTS

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CHAPTER 14

Vector ARMA Models

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14.1. INTRODUCTION

Business, economic, engineering and environmental data are often collected in roughly equally spaced time intervals, such as hour, week, month, or quarter. In many problems, such time series data may be available on several related variables of interest. Two of the reasons for analyzing and modeling such series jointly are:

1. To understand the dynamic relationships among them. They may be contemporaneously related, one series may lead the others or there may be feedback relationships.
2. To improve accuracy of forecasts. When there is information on one series contained in the historical data of another, better forecasts can result when the series are modeled jointly.

In addition, one may be interested in the structure of the relationship among the series. In particular, there may be hidden factors responsible for the dynamic movement of the component series. For examples, there may be combinations that underline the growth of all the components, and there may also be combinations which are more stable than each individual series.

Let

$$\{z_{1t}\}, \ldots, \{z_{kt}\}, \quad t = 0, \pm 1, \pm 2, \ldots \tag{14.1}$$

be k series taken in equally spaced time intervals. Writing

$$\mathbf{z}_t = (z_{1t}, \ldots, z_{kt})' \tag{14.2}$$

188

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we shall refer to the k series as a k-dimensional vector time series. Models that are of possible use in representing such multiple time series, considerations of their properties and methods for relating them to actual data have been extensively discussed in the literature. See in particular, Quenouille (1957), Whittle (1963), Hannan (1970), Zellner and Palm (1974), Brillinger (1975), Dunsmuir and Hannan (1976), Box and Haugh (1977), Parzen (1977), Wallis (1977), Chan and Wallis (1978), Deistler et al. (1978), Hallin (1978), Jenkins (1979), Hsiao (1979), Akaike (1980), Hannan (1980), Hannan et al. (1980), Quinn (1980), and Granger and Newbold (1986). The issue of multivariate linear structure has been given by Hannan and Kavalieris (1984), Hannan and Deistler (1988), Lutkepohl (1991), Reinsel (1993), Reinsel and Velu (1998), and many others.

The principal objective of this chapter is to describe the approach to analyzing multiple time series that have been initiated by Tiao and Box (1981). Our main goal will be on motivating, describing and illustrating the methods used in an iterative model building process with special emphasis on tentative model specification. Much, if not all, of the underlying theory can be found in the references given and, therefore, will not be repeated. Section 14.2 presents a short review of the widely used transfer function models as developed in Box et al. (1994). Section 14.3 discusses a class of vector autoregressive moving-average models. Model building procedures are discussed in Section 14.4 and applied to three actual examples in Section 14.5. We then move on to provide a discussion of issues involved in structural analysis of multivariate time series. The canonical analysis method proposed by Box and Tiao (1977) is discussed in Section 14.6, and finally an introduction to the scalar component model (SCM) approach proposed by Tiao and Tsay (1989) for structural analysis is given in Section 14.7.

14.2. TRANSFER FUNCTION OR UNIDIRECTIONAL MODELS

When $k = 1$ we shall write $\mathbf{z}_t = z_t$. First, recall from equation (3.3) (of Chapter 3) that a widely used linear model for discrete univariate time series is the ARMA(p,q) model

$$\varphi(B)z_t = c + \theta(B)a_t. \tag{14.3}$$

This model can be alternatively written as

$$z_t = c^* + \psi(B)a_t, \quad \psi(B) = \theta(B)/\varphi(B). \tag{14.4}$$

When k series $\{z_{1t}\}, \ldots, \{z_{kt}\}$ are of interest, relationships sometime exist that can be represented by a linear *transfer function* or *unidirectional* model of the form

$$\begin{aligned}
z_{1t} &= L_1(B)\alpha_{1t} \\
z_{2t} &= \nu_{21}(B)z_{1t} + L_2(B)\alpha_{2t} \\
&\vdots \\
z_{1t} &= \nu_{k1}(B)z_{1t} + \cdots + \nu_{k(k-1)}(B)z_{(k-1)t} + L_k(B)\alpha_{kt}
\end{aligned} \tag{14.5}$$

where, for convenience the constant terms are suppressed, the $\nu(B)$'s and $L(B)$'s are polynomials in B, and $\{\alpha_{1t}\}, \ldots, \{\alpha_{kt}\}$ are k independent white noise processes with zero means and variances $\sigma_1^2, \ldots, \sigma_k^2$. Often the polynomials $\nu(B)$'s and $L(B)$'s can be parsimoneously represented as ratios of polynomials of the form

$$\nu(B) = \frac{\omega(B)}{\delta(B)}B^b, \qquad L(B) = \frac{\theta(B)}{\varphi(B)} \tag{14.6}$$

where b is a nonnegative integer, and

$$\omega(B) = \omega_0 - \omega_1 B - \cdots - \omega_s B^s, \qquad \delta(B) = 1 - \delta_1 B - \cdots - \delta_r B^r$$
$$\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q, \qquad \varphi(B) = 1 - \varphi_1 B - \cdots - \varphi_p B^p.$$

It will be understood that the polynomials $\omega(B)$, $\delta(B)$, $\theta(B)$, $\varphi(B)$ and the exponent b will in general be different for the different $\nu(B)$'s and $L(B)$'s.

For $k = 2$, the model in (14.5) reduces to the transfer function model for the output variable $y_t = z_{2t}$ with one stochastic input variable $x_t = z_{1t}$, as discussed in detail in Box et al. (1994). Their model building procedure has been widely adopted, but it becomes cumbersome to apply to situations when there are more than one input variable. Their procedure is compared with an alternative method using the vector ARMA model later in the chapter. It is also worth noting that if the input series is deterministic, the model for the output series is of the same form used in intervention analysis (Box and Tiao, 1975).

Transfer function models of the form (14.5) assume that the k series, when suitably arranged, possess a triangular unidirectional relationship; that is to say, for example, that z_1 depends only on its own past, z_2 depends on its own past and on the present and past of z_1, z_3 on its own past and on the present and past of z_2 and z_1, and so on. A graphical illustration of the model (14.5) for $k = 3$ is given in Figure 14.1. On the other hand, if z_1 depends on the past of z_2, and also z_2 depends on the past of z_1, then we must have a model which allows for this *feedback*.

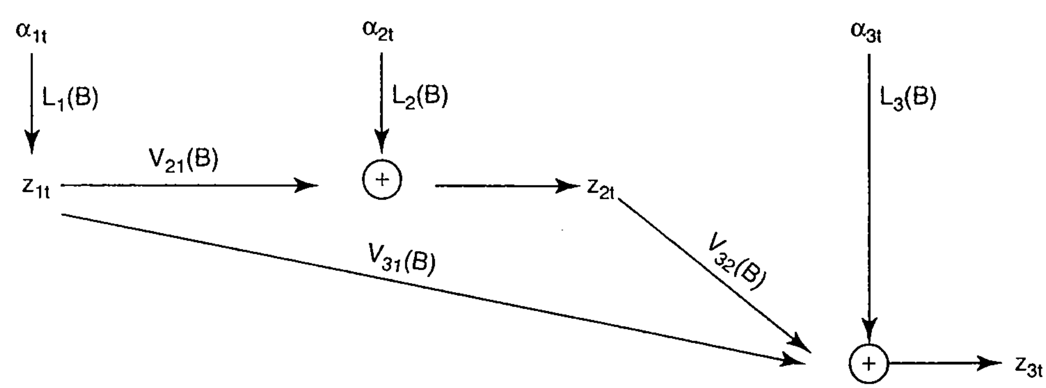

FIGURE 14.1 A trivariate transfer function model.

14.3. THE VECTOR ARMA MODEL

For k series $\{\mathbf{z}_t\}$, the vector autoregressive moving average model takes the form

$$\Phi(B)\mathbf{z}_t = \mathbf{c} + \Theta(B)\mathbf{a}_t \tag{14.7}$$

where $\Phi(B) = \mathbf{I} - \Phi_1 B - \cdots - \Phi_p B^p$, $\Theta(B) = \mathbf{I} - \Theta_1 B - \cdots - \Theta_q B^q$ are matrix polynomials in B, the Φ's and Θ's are $k \times k$ parameter matrices, $\mathbf{c}$ is a $k \times 1$ vector of constants, and $\{\mathbf{a}_t\}$ with $\mathbf{a}_t = (a_{1t}, \ldots, a_{kt})'$ is a series of random shock vectors normally, identically and independently distributed with zero mean vectors and covariance matrix Ω. We shall suppose that the zeros of the determinantal polynomials $|\Phi(B)|$ and $|\Theta(B)|$ are on or outside the unit circle. The series $\{\mathbf{z}_t\}$ will be stationary when the zeros of $|\Phi(B)|$ are all outside the unit circle, and will be invertible when those of $|\Theta(B)|$ are all outside the unit circle. The model in (14.7) will be denoted as ARMA(p,q). The $k \times k$ covariance matrix Ω may either be positive definite or positive semidefinite. In what follows we shall use the notation $N_k(\mathbf{0}, \Omega)$ to denote a k-dimensional multivariate normal distribution with mean vector $\mathbf{0}$ and covariance matrix Ω.

In the literature, the vector ARMA model in (14.7) has been extensively studied by a large number of researchers. Properties of the model have been discussed, for example, in Hannan (1970), Anderson (1971), and Granger and Newbold (1986). The parameter estimation for a given model has been investigated by Tunnicliffe Wilson (1973), Phadke and kedem (1978), Nicholls and Hall (1979), Hillmer and Tiao (1979), and Solo (1984). The problem of model building has been discussed by Akaike (1976), Chan and Wallis (1978), Parzen (1977), Jenkins and Alavi (1981), Cooper and Wood (1982), Tiao and Tsay (1983), and others. Various applications of model (14.7) can be found in Zellner and Palm (1974), Anderson and Moore (1979), and many others.

14.3.1. Some simple examples

To illustrate the behavior of observations from these models, Figure 14.2 shows two series with 250 observations generated from the bivariate $k = 2$ first-order moving-average MA(l) or ARMA(0,1) model

$$\mathbf{z}_t = \mathbf{c} + (\mathbf{I} - \Theta B)\mathbf{a}_t \tag{14.8}$$

with

$$\mathbf{c} = \begin{bmatrix} 17 \\ 25 \end{bmatrix}, \quad \Theta = \begin{bmatrix} .2 & .3 \\ -.6 & 1.1 \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

Figure 14.3 shows two series with 150 observations generated from the bivariate first order autoregressive AR(l) or ARMA(1,0) model,

$$(\mathbf{I} - \Phi B)\mathbf{z}_t = \mathbf{c} + \mathbf{a}_t \tag{14.9}$$

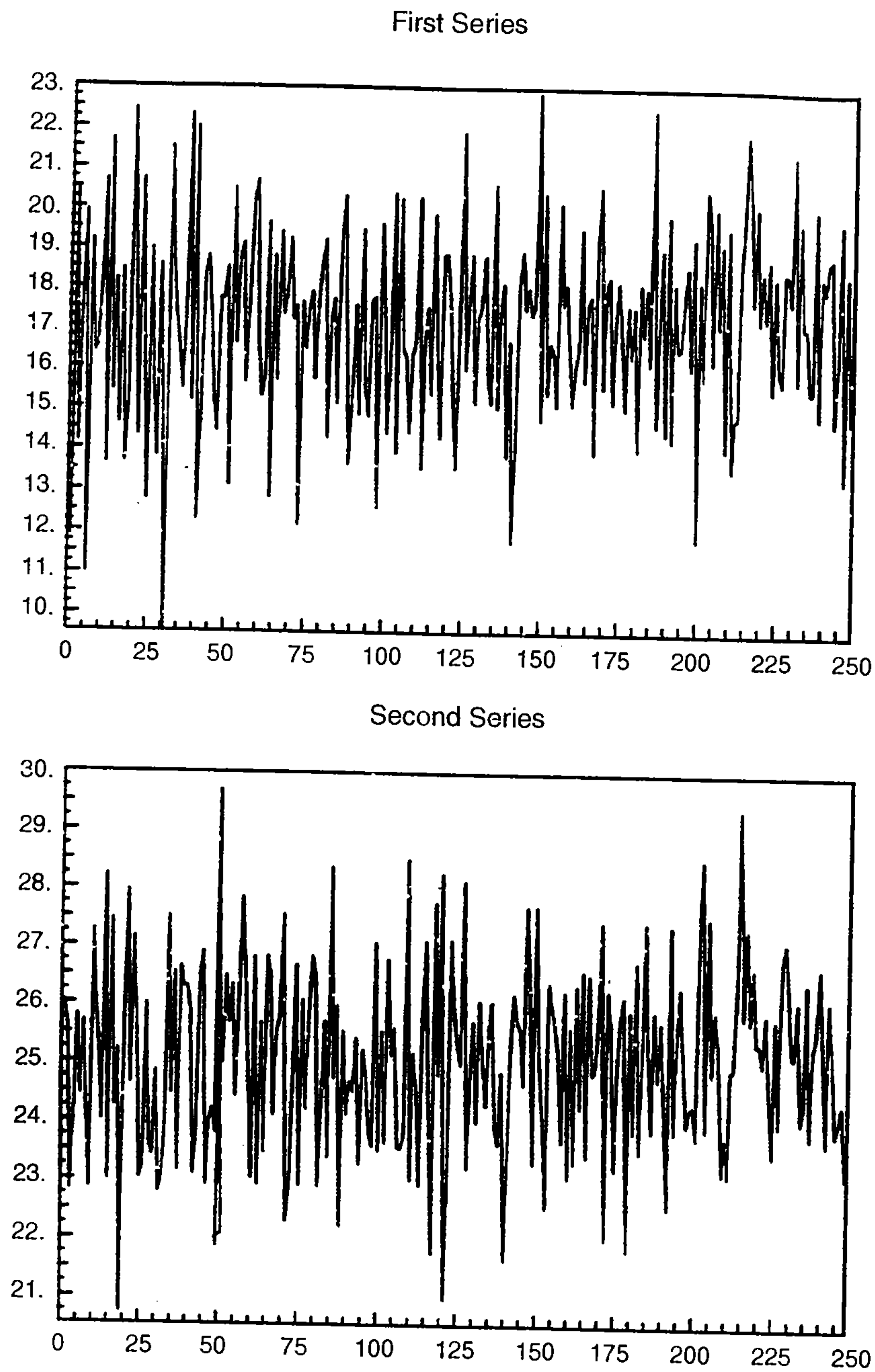

FIGURE 14.2 Data generated from a bivariate MA(1) model with parameter values as in (14.8).

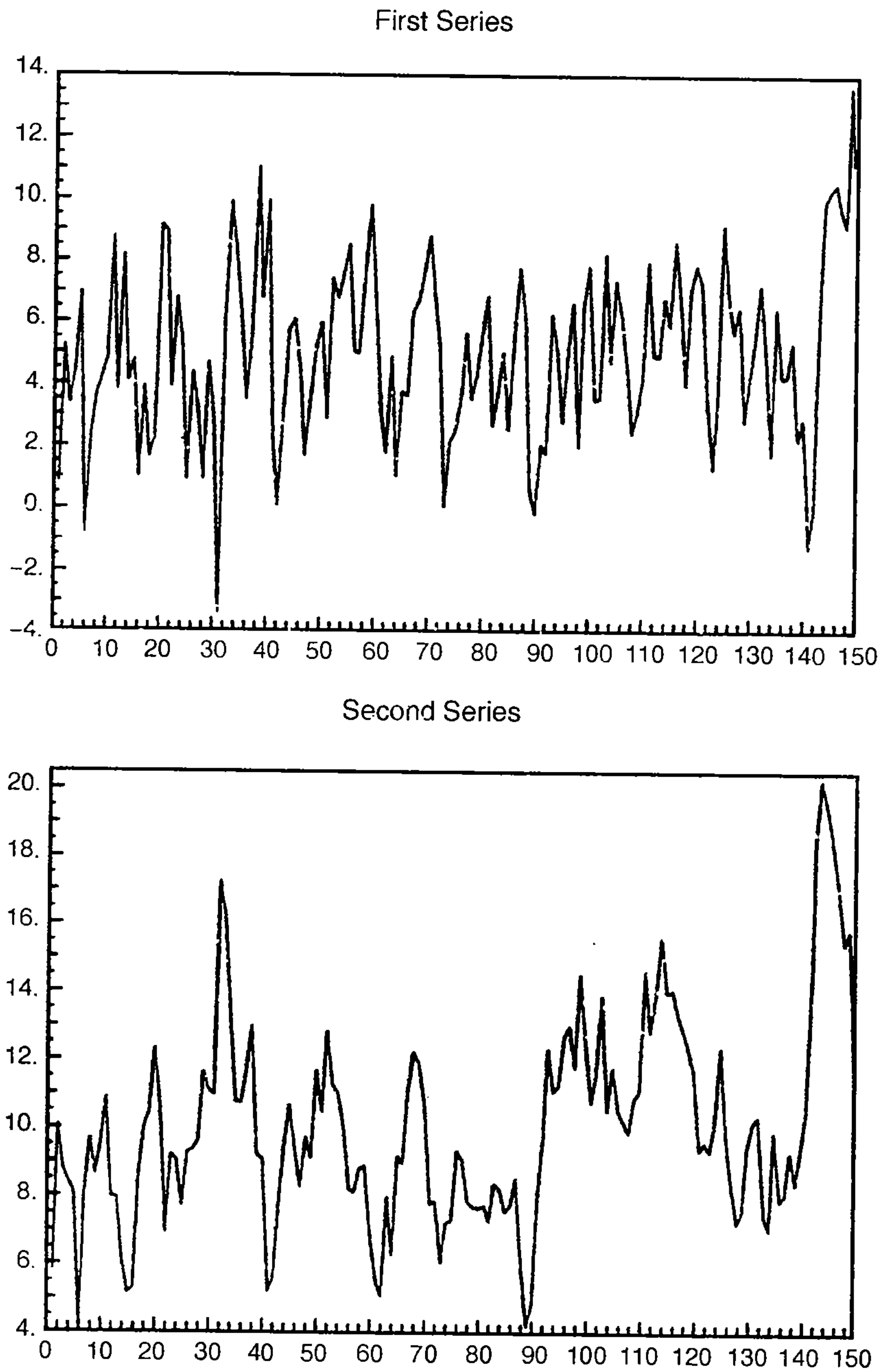

FIGURE 14.3 Data generated from a bivariate AR(1) model with parameter values in (14.9).

with

$$\mathbf{c} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}, \quad \Phi = \begin{bmatrix} .2 & .3 \\ -.6 & 1.1 \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}.$$

Although in both cases the series are seen to be stationary, observations from the autoregressive model are seen to have more "momentum" than those from the moving average model.

In practice, time series often exhibit nonstationary behavior. When several such series are considered jointly, nonstationarity may be modeled by allowing the zeros of $|\Phi(B)|$ in (14.7) to lie on the unit circle. A particular example is the model $(1-B)\mathbf{z}_t = (\mathbf{I} - \Theta B)\mathbf{a}_t$; after differencing each series we obtain a vector MA(1) model. This is a vector analog of the commonly used univariate nonstationary model $(1 - B)z_t = (1 - \theta B)a_t$. However, it should be noted here that for vector time series, linear combinations of the elements of $\mathbf{z}_t$ may often be stationary and simultaneous differencing of all series can lead to unnecessary complications in model fitting, see, for instance the discussions in Box and Tiao (1977) and Hillmer and Tiao (1979).

14.3.2. Relationship to transfer function model

For the vector model in (14.7), in general, all elements of $\mathbf{z}_t$ are related to all elements of $\mathbf{z}_{t-j}$ $(j = 1, 2, \ldots)$ and there can be feedback relationships between all the series. However, if the z_{kt}'s can be arranged so that the coefficient matrices Φ's and Θ's are all lower triangular, then (14.7) can be written as a transfer function model of the form (14.5).

More generally, if the Φ's and Θ's are all lower block triangular, then we obtain a generalization of the transfer function form of (14.5) in which both the input vector series and the output vector series are allowed to have feedback relationships. We note here that relationships between the vector transfer function model and the econometric linear simultaneous equation model have been discussed in Zellner and Palm (1974), and Wallis (1977).

14.3.3. Cross-covariance and correlation matrices

For a stationary vector time series $\{\mathbf{z}_t\}$ with mean vector $\mathbf{0}$, let

$$\Gamma(\ell) = \mathrm{E}(\mathbf{z}_{t-\ell}\mathbf{z}_t') = \{\gamma_{ij}(\ell)\}, \quad i, j = 1, \ldots, k; \quad \ell = 0, \pm 1, \pm 2, \ldots \tag{14.10}$$

be the lag ℓ cross-covariance matrix and $\rho(\ell) = \{\rho_{ij}(\ell)\}$ be the corresponding cross-correlation matrix. When the vector ARMA model in (14.7) is stationary, explicit expressions for the $\Gamma(\ell)$'s can be obtained as follows. First, assuming without loss of generality that $\mathrm{E}(\mathbf{z}_t) = \mathbf{0}$, we can write the model for $\mathbf{z}_t$ in the Ψ form

194

$$\mathbf{z}_t = \Psi(B)\mathbf{a}_t, \quad \Psi(B) = \mathbf{I} + \Psi_1 B + \Psi_2 B^2 + \cdots \tag{14.11}$$

where the Ψ's are obtained by equating coefficients from the relation $\Phi(B)\Psi(B) = \Theta(B)$. Now, from (14.7)

$$\mathbf{z}_{t-\ell}\{(\mathbf{I} - \Phi_1 B - \cdots - \Phi_p B^p)\mathbf{z}_t\}' = \mathbf{z}_{t-\ell}\{(\mathbf{I} - \Theta_1 B - \cdots - \Theta_q B^p)\mathbf{a}_t\}'. \quad (14.12)$$

Taking expectation on both sides of (14.12), and making use of the fact from (14.11) that $\mathrm{E}(\mathbf{z}_t\mathbf{a}'_{t-j}) = \Psi_j\Omega$, where $\Psi_0 = \mathbf{I}$, it is straightforward to verified that

$$\Gamma(\ell) = \begin{cases} \displaystyle\sum_{j=\ell-r}^{\ell-1} \Gamma(j)\Phi'_{\ell-j} - \sum_{j=0}^{r-\ell} \Psi_j\Omega\Theta'_{j+\ell}, & \ell = 0, \ldots, r \\ \displaystyle\sum_{j=1}^{r} \Gamma(\ell - j)\Phi'_j, & \ell > r \end{cases} \quad (14.13)$$

where $\Theta_0 = -\mathbf{I}$, $r = \max(p,q)$ and it is understood that (i) if $p < q$, $\Phi_{p+1} = \cdots = \Phi_r = \mathbf{0}$, and (ii) if $q < p$, $\Theta_{q+1} = \cdots = \Theta_r = \mathbf{0}$. In particular, when $p = 0$, i.e. we have a vector MA(q) model, then

$$\Gamma(\ell) = \begin{cases} \displaystyle\sum_{j=0}^{q-\ell} \Theta_j\Omega\Theta'_{j+\ell}, & \ell = 0, \ldots, q \\ \mathbf{0}, & \ell > q. \end{cases} \quad (14.14)$$

Thus, $\Gamma(\ell) \neq 0$ for $\ell = q$ but all auto- and cross-correlations are zero when $\ell > q$. On the other hand, for a vector autoregressive model, the auto- and cross-correlations in general will decay gradually to zero as $|\ell|$ increases.

14.3.4. The partial autoregression matrices

From the moment equations in (14.13) for a stationary ARMA(p,q) model, we see that, for $m \geq q$, the autocovariance matrices $\Gamma(\ell)$'s and the autoregressive coefficient matrices Φ's are related as follows:

$$\begin{bmatrix} \mathbf{A}(p,m) & \mathbf{b}(p,m) \\ \mathbf{g}'(p,m) & \Gamma(m) \end{bmatrix} \begin{bmatrix} \Lambda(p-1) \\ \Phi'_p \end{bmatrix} = \begin{bmatrix} \mathbf{c}(p,m) \\ \Gamma(p+m) \end{bmatrix}, \quad m = q, q+1, \ldots \quad (14.15)$$

where

$$\mathbf{A}(p,m) = \begin{bmatrix} \Gamma(m) & \Gamma(m-1) & \cdots & & \Gamma(m-p+2) \\ \Gamma(m+1) & \Gamma(m) & & & \\ \vdots & & \ddots & & \vdots \\ & & & \Gamma(m) & \Gamma(m-1) \\ \Gamma(m+p-2) & & \cdots & \Gamma(m+1) & \Gamma(m) \end{bmatrix},$$

$$\mathbf{b}(p,m) = \begin{bmatrix} \Gamma(m-p+1) \\ \vdots \\ \Gamma(m-1) \end{bmatrix}, \qquad \mathbf{c}(p,m) = \begin{bmatrix} \Gamma(m+1) \\ \vdots \\ \Gamma(m+p-1) \end{bmatrix},$$

$$\mathbf{g}(p,m) = \begin{bmatrix} \Gamma'(m+p-1) \\ \vdots \\ \Gamma'(m+1) \end{bmatrix}$$

and $\Lambda'(p-1) = [\Phi_1, \ldots \ldots, \Phi_{p-1}]$.

In the special case $m = q = 0$, (14.15) is a multivariate generalization of the Yule–Walker equations for autoregressive models in univariate time series. In particular, by partitioning inversion, and on writing $\mathbf{A}(p) = \mathbf{A}(p,0)$, $\mathbf{b}(p) = \mathbf{b}(p,0)$, $\mathbf{c}(p) = \mathbf{c}(p,0)$, $\mathbf{g}(p) = \mathbf{g}(p,0)$, we have that

$$\Phi_p' = [\Gamma(0) - \mathbf{g}'(p)\mathbf{A}^{-1}(p)\mathbf{b}(p)]^{-1}[\Gamma(p) - \mathbf{g}'(p)\mathbf{A}^{-1}(p)\mathbf{c}(p)] \tag{14.16}$$

Motivated by this result, we may define a *partial autoregression matrix function* $\wp(\ell)$, which is analogous to the partial autocorrelation function for the univariate case, such that

$$\wp'(\ell) = \begin{cases} \Gamma^{-1}(0)\Gamma(1), & \ell = 1 \\ [\Gamma(0) - \mathbf{g}'(\ell)\mathbf{A}^{-1}(\ell)\mathbf{b}(\ell)]^{-1}[\Gamma(\ell) - \mathbf{g}'(\ell)\mathbf{A}^{-1}(\ell)\mathbf{c}(\ell)], & \ell > 1. \end{cases} \tag{14.17}$$

This function has the property that for a vector AR(p) model

$$\wp'(\ell) = \begin{cases} \Phi_\ell, & \ell = p \\ \mathbf{0}, & \ell > p. \end{cases} \tag{14.18}$$

14.4. MODEL BUILDING STRATEGY FOR MULTIPLE TIME SERIES

In this section we sketch an iterative approach consisting of (1) tentative specification (identification), (2) estimation, and (3) diagnostic checking for the vector ARMA models in (14.7).

14.4.1. Tentative specification

The aim here is to employ statistics that (1) can be readily calculated from the data and (2) facilitate the choice of a subclass of models worthy of further examination.

Sample cross-correlations. The sample cross-correlations

$$\hat{\rho}_{ij}(\ell) = \frac{\sum(z_{it} - \bar{z}_i)(z_{j(t+\ell)} - \bar{z}_j)}{\left\{\sum(z_{it} - \bar{z}_i)^2 \sum(z_{jt} - \bar{z}_j)^2\right\}^{1/2}} \tag{14.19}$$

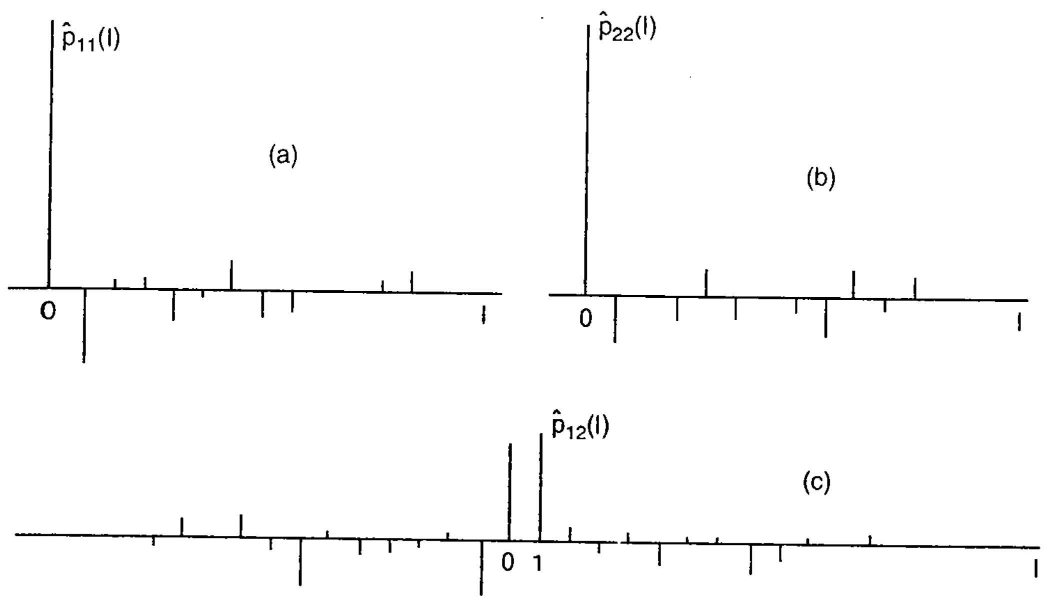

FIGURE 14.4 Sample auto- and cross-correlations for the data in Figure 14.2.

where $\bar{z}_i$ is the sample mean of the ith component series of $\mathbf{z}_t$, are particularly useful in spotting low-order vector moving average models since from (14.14) $\rho_{ij}(\ell) = 0$ for $\ell > q$.

For the data shown in Figure 14.2 generated from a bivariate MA(1) model, Figures 14.4a–14.4c show, respectively, the sample autocorrelations $\hat{\rho}_{11}(\ell)$ and $\hat{\rho}_{22}(\ell)$, and the sample cross-correlations $\hat{\rho}_{12}(\ell)$. The large values occurring at $|\ell| = 1$ would lead to tentative specification of the model as an MA(1). However, graphs of this kind become increasingly cumbersome as the number of series is increased. Furthermore, identification is also not easy from a listing of sample cross-correlation matrices $\hat{\rho}(\ell)$ like that in Table 14.1 (a), particularly when k is greater than 4 or 5.

In this circumstance, we have found the following simple device of great practical value. Instead of the numerical values, a plus sign "+" is used to indicate a value greater than $2n^{-1/2}$, a minus sign "−" for a value less than $-2n^{-1/2}$ and a dot "." to indicate a value in between $-2n^{-1/2}$ and $2n^{-1/2}$. The motivation is that if the series were white noise, the $\hat{\rho}_{ij}(\ell)$'s would, for large n, be normally distributed with mean 0 and variance n^{-1}. The symbols can be arranged either as in Table 14.1 (b) or as in Table 14.1 (c), both of which clearly suggest a vector MA(1) model for the data.

We realize that the variances of the $\hat{\rho}_{ij}(\ell)$'s can be considerably greater than n^{-1} when the series are highly autocorrelated, so that these "indicator symbols," if taken literally, can lead to over parameterization. However, we do not interpret these "indicator symbols" in the sense of formal significance tests but as a rather crude "signal-to-noise ratio" guide. Taken together they can give useful and assimilable indicators of the general correlation pattern.

Table 14.2 shows sample cross correlation matrices in terms of these "indicator symbols" for the series (Fig. 14.3) generated from a bivariate AR(1) model. The

TABLE 14.1. Sample Cross-Correlation Matrices for Data in Figure 14.2

(a) Sample Cross-Correlation Matrices $\hat{\rho}(\ell)$

Lags 1–6

$$\begin{bmatrix} -.28 & .37 \\ -.21 & -.19 \end{bmatrix} \begin{bmatrix} .03 & .08 \\ .02 & .01 \end{bmatrix} \begin{bmatrix} .04 & -.03 \\ -.01 & -.08 \end{bmatrix} \begin{bmatrix} -.11 & .04 \\ -.03 & -.09 \end{bmatrix} \begin{bmatrix} -.02 & -.09 \\ -.02 & -.08 \end{bmatrix} \begin{bmatrix} .10 & .01 \\ .01 & -.00 \end{bmatrix}$$

Lags 7–12

$$\begin{bmatrix} -.11 & .01 \\ -.17 & -.06 \end{bmatrix} \begin{bmatrix} -.09 & -.12 \\ -.03 & -.16 \end{bmatrix} \begin{bmatrix} .01 & -.06 \\ .08 & .10 \end{bmatrix} \begin{bmatrix} -.00 & .02 \\ .01 & -.04 \end{bmatrix} \begin{bmatrix} .03 & .00 \\ .08 & .08 \end{bmatrix} \begin{bmatrix} .06 & .04 \\ -.01 & .01 \end{bmatrix}$$

(b) $\hat{\rho}(\ell)$ *in Terms of Indicator Symbols*

Lags 1–6

$$\begin{bmatrix} - & + \\ - & - \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

Lags 7–12

$$\begin{bmatrix} \cdot & \cdot \\ - & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & - \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

(c) Pattern of Correlations for Each Element in Matrix Over All Lags

	z_1	z_2
z_1	- · · · · · · · · · · ·	+ · · · · · · · · · · ·
z_2	- · · · · · - · · · · ·	- · · · · · · - · · · ·

persistence of large sample auto- and cross-correlations indicates that the data are not likely to have come from a low-order MA model, and suggests the possibility of autoregressive behavior. In general, the pattern of indicator symbols for the cross correlation matrices makes it very easy to identify a low-order moving-average model.

TABLE 14.2. Sample Cross-Correlation Matrices $\hat{\rho}(\ell)$ **for Data in Figure 14.3 in Terms of Indicator Symbols**

Lags 1–6

$$\begin{bmatrix} + & \cdot \\ + & + \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ + & + \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ + & + \end{bmatrix} \begin{bmatrix} \cdot & - \\ + & + \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ + & + \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix}$$

Lags 7–12

$$\begin{bmatrix} - & - \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} - & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

Sample partial autoregression and related summary statistics

For an AR(p) process the partial autoregression matrices $\wp(\ell)$ in (14.17) are zero for $\ell > p$. They are particularly useful, therefore, for identifying an autoregressive model. Estimates of $\wp(\ell)$ and their standard errors can be obtained by fitting autoregressive models of successively high-order 1,2... by standard multivariate least squares.

It is well known (see, e.g., Anderson 1971) that for a stationary AR(p) model, asymptotically the estimates $\hat{\Phi}_1, \ldots, \hat{\Phi}_p$ are jointly normally distributed. A useful summary of the pattern of the partials is obtained by listing indicator symbols, assigning a plus (minus) sign when a coefficient in $\hat{\wp}(\ell)$ is greater (resp. less) than 2 (resp. -2) times its estimated standard errors, and a dot for in between values.

To help determine tentatively the order of an autoregressive model, we may also employ the likelihood ratio statistics corresponding to testing the null hypothesis $\Phi_\ell = \mathbf{0}$ against the alternative $\Phi_\ell \neq \mathbf{0}$ when an AR(ℓ) model is fitted. Let

$$\mathrm{S}(\ell) = \sum_{t=\ell+1}^{n} (\mathbf{z}_t - \hat{\Phi}_1 \mathbf{z}_{t-1} - \cdots - \hat{\Phi}_\ell \mathbf{z}_{t-\ell})(\mathbf{z}_t - \hat{\Phi}_1 \mathbf{z}_{t-1} - \cdots - \hat{\Phi}_\ell \mathbf{z}_{t-\ell})' \qquad (14.20)$$

be the matrix of residual sum of squares and cross-products after fitting an AR(ℓ). The likelihood ratio statistic is the ratio of the determinants

$$U = |\mathrm{S}(\ell)|/|\mathrm{S}(\ell - 1)|. \qquad (14.21)$$

Using Bartlett's (1938) approximation, the statistic

$$M(\ell) = -(N - .5 - \ell k)\log_e U \qquad (14.22)$$

is, on the null hypothesis, asymptotically distribute.d as χ^2 with k^2 degrees of freedom where $N = n - p - 1$ is the effective number of observations, assuming that a constant term is included in the model.

Finally, a measure of the extent to which the fit is improved as the order is increased is provided by the diagonal elements of the estimated residual covariance matrices $\hat{\Omega}$ corresponding to the successive AR models.

For illustration, for the series in Figure 14.3 generated from a bivariate AR(1) model, the matrices of summary symbols, the $M(\ell)$ statistics and the diagonal elements of the residual covariance matrices are shown in Table 14.3 for $\ell = 1, \ldots, 5$. They indicate that a bivariate AR(1) or at most an AR(2) would be adequate for the data.

For the series shown in Figure 14.2, the pattern of the partials and related statistics are given in Table 14.4. Notice here that if we had confined attention to autoregressive models as is advocated in many econometric textbooks, we would have needed p to be as high as 7. This is not surprising since with the bivariate MA(1) model of (14.8) written in the autoregressive form (ignoring the constant term $\mathbf{c}$)

$$(\mathbf{I} - \Theta B)^{-1}\mathbf{z}_t = \mathbf{a}_t, \quad \text{or} \quad \mathbf{z}_t = \Pi_1 \mathbf{z}_{t-1} + \Pi_2 \mathbf{z}_{t-2} + \cdots + \mathbf{a}_t$$

TABLE 14.3. Indicator Symbols for Partial Autoregression and Related Statistics for Data in Figure 14.3

Lag ℓ	Indicator Symbols	$M(\ell)^a \mathrel{\dot\sim} \chi_4^2$	Diagonal Elements of $\hat{\Omega}$
1	$\begin{bmatrix} \cdot & + \\ - & + \end{bmatrix}$	356.96	5.30 1.08
2	$\begin{bmatrix} \cdot & \cdot \\ \cdot & + \end{bmatrix}$	7.04	5.16 1.03
3	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	2.63	5.07 1.03
4	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	4.38	5.01 1.02
5	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	2.42	4.95 1.01

[a] $\mathrel{\dot\sim}$ means approximately distributed as.

TABLE 14.4. Pattern of Partial Autoregression and Related Statistics for Data in Figure 14.2

Lag	Pattern of $\hat{\rho}(\ell)$	$M(\ell) \mathrel{\dot\sim} \chi_4^2$	Diagonal Elements of $\hat{\Omega}$
1	$\begin{bmatrix} - & - \\ + & - \end{bmatrix}$	123.2	4.78 1.88
2	$\begin{bmatrix} \cdot & \cdot \\ + & - \end{bmatrix}$	75.9	4.75 1.43
3	$\begin{bmatrix} + & \cdot \\ + & - \end{bmatrix}$	35.2	4.63 1.23
4	$\begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix}$	27.5	4.63 1.08
5	$\begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix}$	16.6	4.61 1.04
6	$\begin{bmatrix} \cdot & \cdot \\ + & \cdot \end{bmatrix}$	13.5	4.53 .98
7	$\begin{bmatrix} \cdot & - \\ + & \cdot \end{bmatrix}$	16.5	4.38 .94
8	$\begin{bmatrix} \cdot & \cdot \\ \cdot & - \end{bmatrix}$	8.1	4.31 .91

we find

$$\Pi_1 = \begin{bmatrix} -.2 & -.3 \\ .6 & -1.1 \end{bmatrix}, \Pi_2 = \begin{bmatrix} .14 & -.39 \\ .78 & -1.03 \end{bmatrix}, \ldots, \Pi_6 = \begin{bmatrix} .23 & -.25 \\ .49 & -.51 \end{bmatrix} \tag{14.23}$$

$$|\Pi_1| = .4, |\Pi_2| = .16, \ldots, |\Pi_6| = .0041.$$

Thus, although the determinants $|\Pi_j|$ decrease rapidly towards zero as j increases, the elements of Π_j converge to zero very slowly so that many autoregressive terms would be needed to provide an adequate approximation.

In general, the pattern of the partial autoregression matrices, the $M(\ell)$ statistic, and the diagonal elements of the residual covariance matrix are useful to distinguish between moving-average and low-order autoregressive models and, for the latter to tentatively select the appropriate order.

14.4.2. Estimation

Once the order of the model in (14.7) has been tentatively selected, efficient estimates of the associated parameter matrices $\Phi = (\Phi_1, \ldots, \Phi_p)$ and $\Theta = (\Theta_1, \ldots, \Theta_q)$ are then determined by maximizing the likelihood function. Approximate standard errors and correlation matrix of the estimates of elements of the Φ_j's and Θ_j's can also be obtained.

Conditional likelihood

For the ARMA(p,q) model in (14.7), we can write

$$\mathbf{a}_t = \mathbf{z}_t - \mathbf{c} - \Phi_1\mathbf{z}_{t-1} - \cdots - \Phi_p\mathbf{z}_{t-p} + \Theta_1\mathbf{a}_{t-1} + \cdots + \Theta_q\mathbf{a}_{t-q}. \tag{14.24}$$

As in the univariate case discussed in Box et al. (1994), the likelihood function can be approximated by a "conditional" likelihood function as follows. The series is regarded as consisting of the n-p vector observations $\mathbf{z}_{p+1}, \ldots, \mathbf{z}_n$. The likelihood function is then determined from $\mathbf{a}_{p+1}, \ldots, \mathbf{a}_n$ using $\mathbf{z}_1, \ldots, \mathbf{z}_p$ as preliminary values and conditional on zero values for $\mathbf{a}_p, \ldots, \mathbf{a}_{p-q+1}$. Thus, as shown in Tunnicliffe-Wilson (1973)

$$\ell_{\text{con}}(\mathbf{c}, \Phi, \Theta, \Omega \mid \mathbf{z}) \propto |\Omega|^{-\frac{n-p}{2}} \exp\left\{-\frac{1}{2}tr[\Omega^{-1}S(\mathbf{c}, \Phi, \Theta)]\right\} \tag{14.25}$$

where

$$S(\mathbf{c}, \Phi, \Theta) = \sum_{t=p+1}^{n} \mathbf{a}_t\mathbf{a}_t'.$$

Properties of the maximum likelihood estimates obtained from (14.25) have been discussed in Nicholls (1976, 1977) and Anderson (1980).

It has been shown in Hillmer and Tiao (1979) that this approximation can be seriously inadequate if n is not sufficiently large and if one or more zeros of $|\Theta(B)|$ lie on or close to the unit circle. Specifically, this would lead to estimates of the moving-average parameters with large bias.

Exact likelihood function

For univariate ARMA models, the exact likelihood function has been considered by Tiao and Ali (1971), Newbold (1974), Dent (1977), Ansley (1979), and others. For vector models, this function has been studied by Osborn (1977) for the pure moving-average case, and by Phadke and Kedem (1978), Nicholls and Hall (1979), and Hillmer and Tiao (1979). It takes the form

$$\ell(\mathbf{c}, \Phi, \Theta, \Omega \mid \mathbf{z}) \propto \ell_{\text{con}}(\mathbf{c}, \Phi, \Theta, \Omega \mid \mathbf{z})\ell_1(\mathbf{c}, \Phi, \Theta, \Omega \mid \mathbf{z}) \qquad (14.26)$$

where $\ell_1(\cdot)$ depends (1) only on $\mathbf{z}_1, \ldots, \mathbf{z}_p$ if $q = 0$, and (2) on all the data vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ if $q \neq 0$. For the general vector ARMA(p,q) model, it has been shown that a close approximation to the exact likelihood can be obtained by considering the transformation $\mathbf{w}_t = \Phi(B)\mathbf{z}_t$ so that

$$\mathbf{w}_t = \Theta(B)\mathbf{a}_t \qquad (14.27)$$

and then apply the exact likelihood results for vector MA(q) model to $\mathbf{w}_t$, $t = p + 1, \ldots n$.

Because estimation of moving-average parameters using the exact likelihood is computationally rather complex, we propose to employ the conditional method in the preliminary stages of iterative model building and switch to the exact method toward the end.

14.4.3. Diagnostic checking

To guard against model misspecification and to search for directions of improvement, a detailed diagnostic analysis of the residual series $\{\hat{\mathbf{a}}_t\}$ where

$$\hat{\mathbf{a}}_t = \mathbf{z}_t - \hat{\mathbf{c}} - \hat{\Phi}_1\mathbf{z}_{t-1} - \cdots - \hat{\Phi}_p\mathbf{z}_{t-p} + \hat{\Theta}_1\hat{\mathbf{a}}_{t-1} + \cdots + \hat{\Theta}_q\hat{\mathbf{a}}_{t-q}. \qquad (14.28)$$

is performed. Useful diagnostic checks include (1) plots of standardized residual series against time and/or other variables and (2) cross-correlation matrices of the residuals $\hat{\mathbf{a}}_t$'s. As before, the structure of the correlations is summarized by indicator symbols. Hosking (1980) and Li and McLeod (1981) have proposed overall χ^2 tests based on the sample cross-correlations of the residuals. However, as noted in Box et al. (1994), such overall tests are not substitutes for more detailed study of the correlation
202 structure.

14.5. ANALYSES OF THREE EXAMPLES

We now apply the model building approach introduced in the preceding section to the following three sets of data:

1. The *Financial Times* ordinary share index, U.K. car production and the *Financial Times* commodity price index: quarterly data 3/1952–4/1967, obtained from Coen et al. (1969). This will be referred to as the SCC data. (See Fig. 1.11.)
2. The gas furnace data given in Box et al. (1994). (See Fig. 1.10.)
3. The census housing data (Hillmer and Tiao 1979).

14.5.1. The SCC data

Let

z_{1t} = *Financial Times* ordinary share index
z_{2t} = U.K. car production index
z_{3t} = *Financial Times* commodity price index

The authors of the original study were interested in the possibility of predicting z_{1t} from lagged values of z_{2t} and z_{3t} using a standard regression analysis in which z_{1t} was treated as a dependent variable and $z_{2(t-6)}$ and $z_{3(t-7)}$ as regressors or independent variables. For a critical evaluation of this approach, see Box and Newbold (1970). Here we consider what structure is revealed by the present multiple time series analysis, in which the three series are jointly modeled.

Tentative specification
We see in Table 14.5 that the original series show high and persistent auto- and cross-correlations. Examination of the partials and related statistics in Table 14.6 shows that for $\ell > 2$, most of the elements of $\wp(\ell)$ are small compared with their estimated standard errors, and the $M(\ell)$ statistic fails to show significant improvement. Table 14.7 shows that the pattern of the cross correlations of the residuals after AR(2) is consonant with estimated white noise. However, note that there is one large residual correlation at lag 1 after the AR(1) fit, suggesting also the possibility of an ARMA(1,1) model.

Estimation
Both an AR(2) model and an ARMA(1,1) model were fitted using the exact likelihood method but results are given only for the ARMA(1,1) model which produced a marginally better representation. For this model,

$$(\mathbf{I} - \Phi)\mathbf{z}_t = \mathbf{c} + (\mathbf{I} - \Theta B)\mathbf{a}_t \tag{14.29}$$

TABLE 14.5. Pattern of Sample Cross-Correlations for the SCC Data

	Z_1 Stocks	Z_2 Cars	Z_3 Commodities
Z_1 Stocks	++++++++···· ······	------------- ---------	------------- ---------
Z_2 Cars	············· ·····+++	++++++++++++ +++·····	+++++++++++ ++·····
Z_3 Commodities	-------····· ···+++++	++++++++++++ ++++++··	++++++++++++ +++++···

TABLE 14.6. Partial Autoregression and Related Statistics: SCC Data

Lag	Indicator Symbols for Partials	$M(\ell)$ Statistic $\approx \chi_9^2$	Diagonal Elements of $\hat{\Omega} \times 10$
1	$\begin{bmatrix} + & \cdot & \cdot \\ \cdot & + & \cdot \\ \cdot & \cdot & + \end{bmatrix}$	301.3	.44 .89 1.62
2	$\begin{bmatrix} - & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ - & + & - \end{bmatrix}$	18.6	.40 .84 1.23
3	$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$	9.6	.37 .81 1.21
4	$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$	3.6	.36 .79 1.19
5	$\begin{bmatrix} \cdot & + & \cdot \\ \cdot & + & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$	11.9	.32 .70 1.11

TABLE 14.7. Pattern of Cross-Correlation Matrices of Residuals: SCC Data

Lag	1	2	3	4	5	6	7	8
				(a) AR(1) Model				
	··+ ··· ···	··· ··· ···	··· ··· ···	··· ··· ···	··· ··· ···	··· ··· ···	··· ··· ··+	··· ··· ···
				(b) AR(2) Model				
	··· ··· ···	··· ··· ···	··· ··· ···	··· ··· ···	··· ··· ···	··· ··· ···	··· ··· ··+	··· ··· ···

TABLE 14.8. Estimation Results for the Model (14.29): SCC Data (Exact Likelihood)[a]

$\hat{c}$	$\hat{\Phi}$	$\hat{\Theta}$	$\hat{\Omega}$

(a) Full Model

$$\begin{bmatrix} 1.11 \\ (.64) \\ 1.74 \\ (.82) \\ 4.08 \\ (1.47) \end{bmatrix} \quad \begin{bmatrix} .81 & .15 & -.06 \\ (.08) & (.07) & (.04) \\ -.07 & .98 & -.09 \\ (.10) & (.10) & (.05) \\ -.32 & .30 & .76 \\ (.18) & (.17) & (.08) \end{bmatrix} \quad \begin{bmatrix} -.29 & .23 & .06 \\ (.15) & (.11) & (.07) \\ -.45 & .20 & -.15 \\ (.22) & (.17) & (.11) \\ -.79 & .57 & -.44 \\ (.28) & (.21) & (.13) \end{bmatrix} \quad \begin{bmatrix} .037 & & \\ & & \\ .022 & .078 & \\ & & \\ .013 & .022 & .129 \\ & & \end{bmatrix}$$

(b) Restricted Model (Intermediate)

$$\begin{bmatrix} .13 \\ (.09) \\ .59 \\ (.05) \\ 2.48 \\ (1.10) \end{bmatrix} \quad \begin{bmatrix} .90 & .08 & \\ (.06) & (.06) & . \\ . & .92 & -.02 \\ . & (.04) & (.04) \\ & & .85 \\ . & . & (.07) \end{bmatrix} \quad \begin{bmatrix} -.22 & .15 & \\ (.14) & (.10) & . \\ -.31 & & \\ (.17) & . & . \\ -.55 & .22 & -.44 \\ (.23) & (.15) & (.12) \end{bmatrix} \quad \begin{bmatrix} .042 & & \\ & & \\ .022 & .079 & \\ & & \\ .017 & .021 & .131 \\ & & \end{bmatrix}$$

(c) Restricted Model (Final)

$$\begin{bmatrix} .12 \\ (.08) \\ .24 \\ (.10) \\ 2.76 \\ (1.07) \end{bmatrix} \quad \begin{bmatrix} .98 & & \\ (.03) & & . \\ . & .93 & \\ & (.04) & . \\ & & .83 \\ . & . & (.06) \end{bmatrix} \quad \begin{bmatrix} & & \\ . & . & . \\ & & \\ . & . & . \\ -.40 & & -.41 \\ (.23) & . & (.12) \end{bmatrix} \quad \begin{bmatrix} .045 & & \\ & & \\ .024 & .085 & \\ & & \\ .019 & .023 & .134 \\ & & \end{bmatrix}$$

[a] For this example, estimates from the conditional likelihood for the ARMA(1,1) case are very close to the exact results.

Table 14.8 shows the initial unrestricted fit and also the fits for two simpler versions obtained by setting to zero those coefficients whose estimates were small compared to their standard errors.

Diagnostic checking

Table 14.9 suggests that the restricted ARMA(1,1) model provides an adequate representation of the data.

Implication of the model

The final model implies that the system is approximately

$$\begin{aligned} (1 - .98B)z_{1t} &= a_{1t} \\ (1 - .93B)z_{2t} &= .2 + a_{2t} \qquad (14.30) \\ (1 - .83B)z_{3t} &= 2.8 + .40a_{1(t-1)} + (1 + .41B)a_{3t}. \end{aligned}$$

TABLE 14.9. Pattern of Residual Cross-Correlations after Final Restricted ARMA(1,1) Model Fit: SCC Data

	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$
$\hat{a}_1$	....	....	....
$\hat{a}_2$	....	....	....
$\hat{a}_3$	....	....	....

Thus, the ordinary share $\{z_{1t}\}$ series behaves like a random walk and does not depend on the lagged values of the other two series. The same can roughly be said about the car production index $\{z_{2t}\}$ series. From (14.30) we can make use of the first equation to write the model for z_{3t} as

$$(1 - .83B)z_{3t} = 2.8 + .40(1 - .98B)z_{1(t-1)} + (1 + .41B)a_{3t} \tag{14.31}$$

which implies that the ordinary share z_{1t} is a *leading indicator* at lag 1 of the commodity index z_{3t}. The impact of z_{1t} is, however, small, as can be seen, for example, by the improvement achieved over the corresponding best fitting univariate model for z_{3t}, which was

$$(1 - .78B)z_{3t} = 3.6 + (1 + .53B)a_t. \tag{14.32}$$

The estimated residual variance $\hat{\sigma}_a^2 = .151$ from the univariate model is not much larger than the value .134 for the estimated variance of a_{3t} obtained from the final vector model.

In conclusion, we see that the three series z_{1t}, z_{2t}, and z_{3t} are essentially *not* dynamically related. Although the multiple time series analysis fails to reveal anything very surprising for this example, it shows what is there and does not mislead.

14.5.2. The gas furnace data

The two series shown in Figure 1.10 consist of (1) input gas rate and (2) output CO_2 concentration at 9-s intervals from a gas furnace. We shall let z_{1t} = gas rate + .057 and $z_{2t} = CO_2 - 5.35$. This set of data was employed in Box et al. (1994) to illustrate a procedure of tentative specification, fitting, and checking transfer function model of the form (14.5) for $k = 2$ relating two time series, one of which is *known* to be input for the other. Using this approach, the following models were found for the input z_{1t} and the output z_{2t}

$$\begin{aligned} &(1 - 1.97B + 1.37B^2 - .34B^3)z_{1t} = a_{1t}, && \hat{\sigma}_{a_1}^2 = .0353 \\ &z_{2t} = \frac{\omega(B)}{\delta(B)}B^3 z_{1t} + \varphi^{-1}(B)a_{2t}, && \hat{\sigma}_{a_2}^2 = .0561 \end{aligned} \tag{14.33}$$

TABLE 14.10. Tentative Identification for the Gas Furnace Data

(a) Pattern of Cross-Correlations of Original Data

	Z_{1t}	Z_{2t}
Z_{1t}	++++++++++++	------------
Z_{2t}	------------	++++++++++++

(b) M Statistic for Partial Autoregression

Lag ℓ	1	2	3	4	5	6	7	8	9	10	11
$M(\ell)$	1650	665	31.7	22.5	5.6	12.9	1.8	8.0	3.5	0	2.0

(c) Pattern of Cross-Correlations of Residuals after AR(6) Fit

	$\hat{a}_{1t}$	$\hat{a}_{2t}$
$\hat{a}_{1t}$	....	....
$\hat{a}_{2t}$	....	....

where $\omega(B) = -(.53 + .37B + .51B^2)$, $\delta(B) = 1 - .57B$, $\varphi(B) = 1 - 1.53B + .63B^2$, and the $\{a_{1t}\}$ and $\{a_{2t}\}$ series are assumed independent.

Particularly when we are dealing with econometric rather than engineering models, feedback relationships may not be known a priori, and it is of interest, therefore, to analyze the data using the present multivariate approach where no distinction is made between an input and output variable, and the fact that no feedback could occur in the system is not used.

Tentative specification

In Table 14.10, we see that the auto- and cross-correlations of the original data in part (a) are persistently large in magnitude, ruling out low-order moving-average models, the $M(\ell)$ statistic in part (b) suggests that an AR(6) model might be appropriate, and the residual cross-correlation pattern after an AR(6) fit in part (c) seems to verify the appropriateness of this model.

Estimation results

Estimation results corresponding to an unrestricted AR(6) model

$$(\mathbf{I} - \Phi_1 B - \cdots - \Phi_6 B^6)\mathbf{z}_t = \mathbf{a}_t \tag{14.34}$$

are as follows:

$$\hat{\Phi}_1 = \begin{bmatrix} 1.93 & -.05 \\ (.06) & (.05) \\ .06 & 1.55 \\ (.08) & (.06) \end{bmatrix}, \quad \hat{\Phi}_2 = \begin{bmatrix} -1.20 & .10 \\ (.13) & (.08) \\ -.14 & -.59 \\ (.16) & (.11) \end{bmatrix}, \quad \hat{\Phi}_3 = \begin{bmatrix} .17 & -.08 \\ (.15) & (.09) \\ -.44 & -.17 \\ (.19) & (.11) \end{bmatrix},$$

$$\hat{\Phi}_4 = \begin{bmatrix} -.16 & .03 \\ (.15) & (.09) \\ .15 & .13 \\ (.19) & (.11) \end{bmatrix}, \quad \hat{\Phi}_5 = \begin{bmatrix} .38 & -.04 \\ (.14) & (.08) \\ -.12 & .06 \\ (.18) & (.10) \end{bmatrix}, \quad \hat{\Phi}_6 = \begin{bmatrix} -.22 & .03 \\ (.08) & (.03) \\ .25 & -.04 \\ (.11) & (.04) \end{bmatrix},$$

$$\hat{\Omega} = \begin{bmatrix} .0345 & \\ & \\ -.0023 & .0566 \end{bmatrix}, \quad \text{and} \quad \hat{\rho}_{a_1,a_2}(0) = .045.$$

If we let

$$\hat{\Phi}_\ell = \begin{bmatrix} \hat{\phi}_{11}^{(\ell)} & \hat{\phi}_{12}^{(\ell)} \\ \hat{\phi}_{21}^{(\ell)} & \hat{\phi}_{22}^{(\ell)} \end{bmatrix} \tag{14.35}$$

then we see that $\hat{\phi}_{12}^{(\ell)}$ are small compared with their standard errors over all lags ℓ, confirming (as in this case is known from the physical nature of the apparatus generating the data) that there is a unidirectional relationship between z_{1t} and z_{2t} involving no feedback. Also, $\hat{\phi}_{21}^{(\ell)}$ is small for $\ell = 1,2$, and the residuals $\hat{a}_{1t}$ and $\hat{a}_{2t}$ are essentially uncorrelated, implying a delay of three periods. It should be noted in addition, that the estimated variances for a_{1t} and a_{2t} are very close to those for a_{1t} and a_{2t} in (14.33), and their correlation is negligible.

To facilitate comparison with (14.33), we set

$$\phi_{11}^{(\ell)} = 0,\ \ell > 3; \quad \phi_{12}^{(\ell)} = 0,\ \text{all } \ell; \quad \phi_{21}^{(\ell)} = 0,\ \ell = 1,2; \quad \phi_{22}^{(\ell)} = 0,\ \ell = 5,6.$$

Estimation results for this restricted AR(6) model are then

$$\hat{\phi}_1 = \begin{bmatrix} 1.98 & \\ (.06) & \\ & 1.53 \\ & (.06) \end{bmatrix}, \quad \hat{\phi}_2 = \begin{bmatrix} -1.38 & \\ (.10) & \\ & -.58 \\ & (.11) \end{bmatrix}, \quad \hat{\phi}_3 = \begin{bmatrix} .35 & \\ (.06) & \\ -.53 & -.14 \\ (.07) & (.10) \end{bmatrix}, \tag{14.36}$$

$$\hat{\phi}_4 = \begin{bmatrix} & \\ & \\ .11 & .12 \\ (.16) & (.04) \end{bmatrix}, \quad \hat{\phi}_5 = \begin{bmatrix} & \\ & \\ -.04 & \\ (.17) & \end{bmatrix}, \quad \hat{\phi}_6 = \begin{bmatrix} & \\ & \\ .21 & \\ (.11) & \end{bmatrix},$$

$$\hat{\Omega} = \begin{bmatrix} .0359 & \\ & \\ -.0029 & .0561 \end{bmatrix}, \quad \hat{\rho}_{a_1,a_2}(0) \cong 0.$$

Examination of the pattern of the cross-correlations of the residuals suggests that the model is adequate.

Implication of the bivariate model
The final AR(6) model (14.36) can be written

$$\begin{bmatrix} \phi_{11}(B) & \\ \phi_{21}(B) & \phi_{22}(B) \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix} \tag{14.37}$$

where $\phi_{11}(B) = (1 - 1.98B + 1.38B^2 - .35B^3)$, $\phi_{21}(B) = (.53 - .11B - .21B^3)B^3$, and $\phi_{22}(B) = (1 - 1.53B + .58B^2 + .14B^3 - .12B^4)$. Assuming that a_{1t} and a_{2t} are uncorrelated, the input model $\phi_{11}(B)z_{1t} = a_{1t}$ with $\mathrm{Var}(a_{1t}) = .0359$ is essentially the same as that in (14.33). Now the model relating the output z_{2t} to the input z_{1t} is

$$z_{2t} = -\frac{\phi_{21}(B)}{\phi_{22}(B)} z_{1t} + \frac{1}{\phi_{22}(B)} a_{2t} \tag{14.38}$$

with $\mathrm{Var}(a_{2t}) = .0561$. The noise model $\phi_{22}^{-1}(B)a_{2t}$ is not very different from the corresponding one $\varphi^{-1}(B)a_{2t}$ in (14.33), but the dynamic model $-\phi_{22}^{-1}(B)\phi_{21}(B)z_{1t}$ at first sight appears markedly different from the first term on the RHS of the output model in (14.33). The reason is that in the form (14.38) the denominators of the dynamic model and that of the noise model are constrained to be identical. This restriction is not present in the transfer function model (14.33). This less restrictive form can however be written in the form of (14.38) if we set

$$\phi_{22}(B) = \varphi(B); \qquad -\phi_{21}(B) = \omega(B)B^3\{\varphi(B)\delta^{-1}(B)\}.$$

For this example, the factor $\varphi(B)\delta^{-1}(B) \cong 1 - .96B$ and it is then seen that the models are in fact very similar. This may be confirmed by comparing the impulse response weights in Table 14.11, where

$$\omega(B)B^3\delta^{-1}(B) = \sum_{j=0}^{\infty} \nu_j B^j; \qquad -\phi_{21}(B)\phi_{22}^{-1}(B) = \sum_{j=0}^{\infty} \nu_j^* B^j.$$

Implications on general time series model building
The relative merit of the vector ARMA procedure and the transfer function modeling of the system will depend on how much is known or we are prepared to assume. In some applications, particularly in engineering and most examples of intervention analysis,

TABLE 14.11. Impulse Response Weights for the Gas Furnace Data

j	0	1	2	3	4	5	6	7	8	9	10	11	12
ν_j				−.53	−.67	−.89	−.51	−.29	−.17	−.09	−.05	−.03	−.02
ν_j^*				−.53	−.70	−.77	−.48	−.26	−.09	−.01	.01	.00	−.01

an adequate initial specification may be possible from knowledge of the nature of the problem. This may allow a flow diagram showing the feedback structure to be drawn and likely orders to be guessed for the various dynamic components. The resulting models can then be directly fitted in the manner described and illustrated in Box and MacGregor (1974, 1976) and Box and Tiao (1975). For a single input with feedback known to be absent, a prewhitening method is given in Box et al. (1994) for *identifying* an unknown dynamic system but extension of this identification method to multiple inputs is rather complex.

Particularly for economic and business examples, however, the feedback structure and orders of the multiple system are seldom known. The present multiple time series procedure has the great advantage that it allows *identification* of the feedback and dynamic structure. Furthermore

1. A one-sided causal relationship, if it exists, will emerge in the identification process, and the stochastic structures of the input as well as the transfer function relationship between input and output will be modeled simultaneously.
2. Stochastic multiple input and multiple output situations are readily handled.
3. A useful method is provided for seeking leading indicators in economic and business applications. In this context it should be noted that a unidirectional dynamic relationship may not exist between two time series even when one variable is known to be the input for the other. One reason for this phenomenon is the effect of temporal aggregation. As shown in Tiao and Wei (1976), pseudofeedback relationships could occur because of this temporal aggregation effect, and it would be a mistake to impose a transfer function model in such a situation;
4. However, when a simple transfer function structure of the form (14.5) *is* appropriate, the present multiple time series approach could rarely reproduce it directly [see, e.g., (14.33) and (14.38)], and some analysis of the fitted form might be necessary to reveal a more parsimonious and more easily understood structure.

14.5.3. The census housing data

As an further example, we illustrate the modeling and analysis of seasonal time series by considering the monthly single-family housing starts z_{1t} and houses sold z_{2t} for the period January 1965 through May 1975. The data were obtained from the "survey of current business," and are plotted in Figure 14.5. Because of the strong seasonal behavior in these series, one might well be led to consider the seasonally difference data $u_{it} = (1 - B^{12})z_{it}$, $i = 1,2$. The seasonally differenced series are shown in Figure 14.6.

Models for individual series

In Hillmer and Tiao (1979), it is found that each series individually can be adequately represented by a univariate multiplicative model of the form

210

$$(1 - B)(1 - B^{12})z_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12})a_t \qquad (14.39)$$

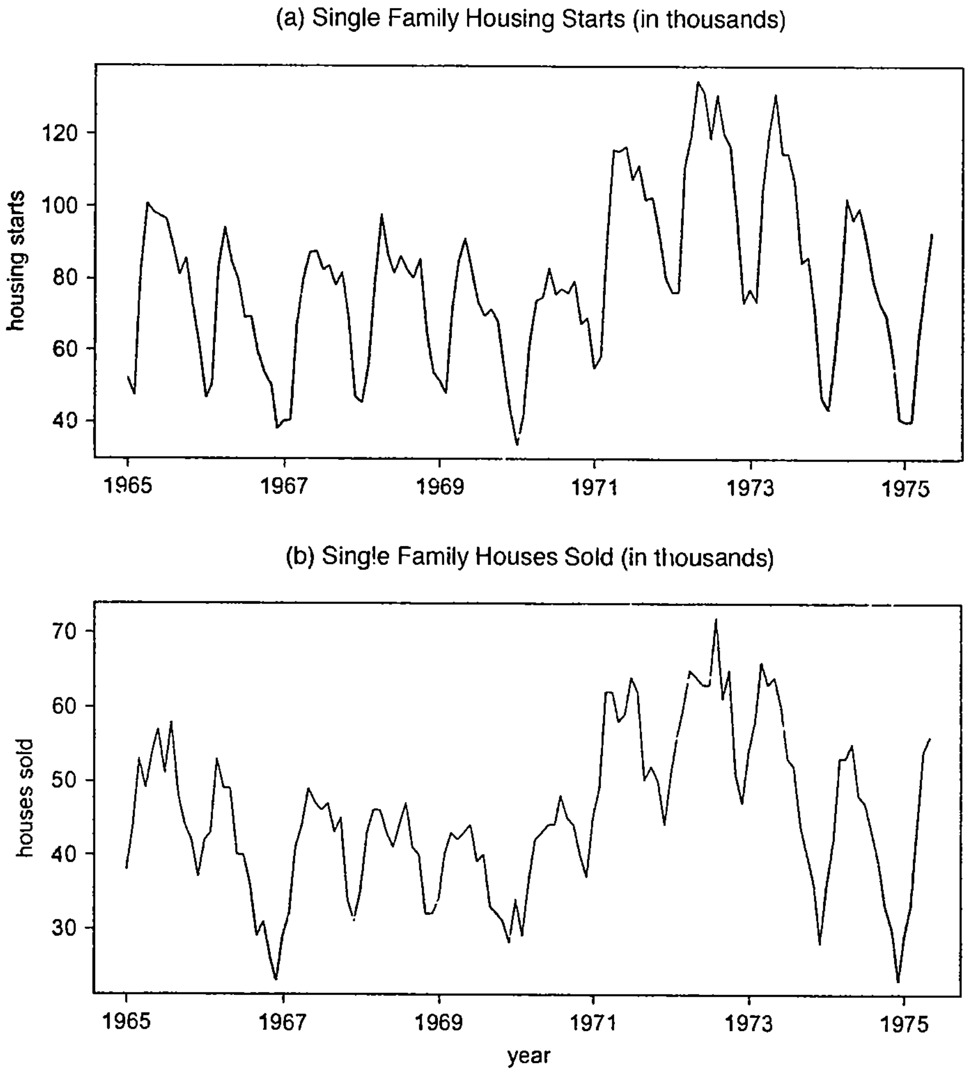

FIGURE 14.5 The census housing data for January 1965–May 1975.

Table 14.12 compares the parameter estimates obtained by employing the exact (E) likelihood (14.26) with those from the conditional (C) likelihood (14.25), where the numbers in the parentheses are the associated estimated standard errors.

We observe that for both series the estimates $\hat{\theta}_{12}$ of the seasonal moving average parameter are appreciably smaller and the corresponding estimates $\hat{\sigma}_a^2$ larger for C

TABLE 14.12. Parameter Estimates for Individual Models: Census Housing Series

		$\hat{\theta}_1$	$\hat{\theta}_{12}$	$\hat{\sigma}_a^2$
Housing starts z_{1t}	E	.28 (.09)	.91 (.06)	41.61
	C	.30 (.09)	.75 (.07)	50.49
Houses sold z_{2t}	E	.16 (.10)	1.00 (.06)	11.93
	C	.24 (.10)	.72 (.08)	16.46

FIGURE 14.6 The seasonally differenced series: January 1966–May 1975.

than for E. It appears that θ_{12} is close to unity, especially for the "houses sold" series z_{2t}, implying a possibly deterministic seasonal structure:

$$(1 - B)z_t = S_t + (1 - \theta B)a_t \tag{14.40}$$

where $(1 - B^{12})S_t = 0$. Such a structure, however, would not be detected if the conditional likelihood method were employed.

Bivariate model

Part (a) of Table 14.13 shows the pattern of the sample cross correlations of $\mathbf{u}_t = (u_{1t}, u_{2t})'$ indicating that low order vector MA model would not be appropriate. Part (b) of the same table gives the $M(\ell)$ statistics for $\ell = 1, \ldots, 5$, and part (c) shows the pattern of the cross correlations of the residuals after a bivariate AR(1) fit. These

TABLE 14.13. Tentative Identification for the Seasonally Differenced Housing Data U_t

(a) Pattern of Cross-Correlations

	U_{1t}	U_{2t}
U_{1t}	++++++++++..	++++++++....
	.....-------	.........---
	------------	-----------.
U_{2t}	++++++++++..	++++++++....
	..........--	....
	-----------.	.----------.

(b) $M(\ell)$ Statistics Corresponding to Partial Autoregression Matrices

Lag ℓ	1	2	3	4	5
$M(\ell)$	218.6	3.5	2.3	4.7	5.4

(c) Pattern of Residual Cross-Correlations after AR(1) Fit

	a_{1t}	a_{2t}
a_{1t}	....	-...........
	....	....
	....	..........+.
a_{2t}	....	...........-
	..+..+......	....
	..-..-......	....

summaries suggest the tentative model

$$(\mathbf{I} - \Phi B)(1 - B^{12})\mathbf{z}_t = (\mathbf{I} - \Theta B^{12})\mathbf{a}_t \tag{14.41}$$

Estimation and checking
Table 14.14 summaries the estimation results corresponding to

1. The full bivariate model in (14.41) using the conditional likelihood method
2. The full bivariate model using the exact likelihood method
3. The restricted model by setting "small" parameter estimates to zero

Table 14.15 shows the pattern of the sample cross-correlations of the residuals corresponding to the restricted case, showing that the model gives an adequate representation of the series.

Interpretation
For the full bivariate model, comparing the results of the conditional likelihood with those from the exact likelihood, we see that

TABLE 14.14. Estimation Results for the Model (16.41): Census Housing Data

	(1) Full Model Conditional Likelihood	(2) Full Model Exact Likelihood	(3) Restricted Model Exact Likelihood
$\hat{\Phi} =$	$\begin{bmatrix} .47 & .89 \\ (.07) & (.13) \\ .14 & .69 \\ (.05) & (.08) \end{bmatrix}$	$\begin{bmatrix} .46 & .95 \\ (.07) & (.14) \\ .10 & .76 \\ (.05) & (.09) \end{bmatrix}$	$\begin{bmatrix} .42 & 1.03 \\ (.07) & (.13) \\ & .93 \\ & (.04) \end{bmatrix}$
$\hat{\Theta} =$	$\begin{bmatrix} .75 & .06 \\ (.07) & (.11) \\ .07 & .69 \\ (.05) & (.08) \end{bmatrix}$	$\begin{bmatrix} 1.01 & -.04 \\ (.07) & (.12) \\ .05 & .99 \\ (.05) & (.07) \end{bmatrix}$	$\begin{bmatrix} .94 & \\ (.06) & \\ & 1.00 \\ & (.06) \end{bmatrix}$
$\hat{\Omega} =$	$\begin{bmatrix} 37.51 & \\ 6.29 & 15.15 \end{bmatrix}$	$\begin{bmatrix} 28.09 & \\ 4.98 & 11.13 \end{bmatrix}$	$\begin{bmatrix} 29.75 & \\ 5.89 & 11.83 \end{bmatrix}$
			$\hat{\rho}(a_1, a_2) \doteq .31$

1. There is a substantial increase in the estimated values of the diagonal elements of Θ to near unity, and a corresponding decrease in the estimated variances of the residuals, when the exact method is used
2. In contrast, little change occurs in the estimates of the elements of Φ

Now from the restricted model, we can write, for the "houses sold" series z_{2t}

$$(1 - .93B)(1 - B^{12})z_{2t} = (1 - B^{12})a_{2t} \tag{14.42}$$

which means that

$$(1 - .93B)z_{2t} = S_{2t} + a_{2t}, \quad \hat{\sigma}_{22} = 11.8 \tag{14.43}$$

where S_{2t} satisfies the relation $S_{2t} = S_{2(t-12)}$. Thus, z_{2t} behaves nearly like a random walk with a deterministic seasonal component and does not depend on the past of z_{1t}. We note that this result is essentially the same as the individual model, shown in Table 14.12, for the "house sold" series fitted using the exact likelihood.

TABLE 14.15. Residual Cross-Correlations for the Restricted Model: Census Housing Data

	a_{1t}	a_{2t}
a_{1t}	....	......+.....
	....	....
a_{2t}	....	....
	....+......	......+.....

On the other hand, for the "housing starts" series z_{1t}, we have that

$$(1 - .42B)(1 - B^{12})z_{1t} = 1.03(1 - B^{12})z_{2(t-1)} + (1 - .94B^{12})a_{1t} \quad (14.44)$$

so that the seasonal differencing operator $(1 - B^{12})$ again nearly cancels yielding approximately

$$(1 - .42B)z_{1t} = S_{1t} + 1.03z_{2(t-1)} + a_{1t}, \quad \hat{\sigma}_{11} = 29.8 \quad (14.45)$$

where $S_{1t} = S_{1(t-12)}$. Thus, housing starts z_{1t} depends not only on its own past, $z_{1(t-1)}$ but also on the past of houses sold, $z_{2(t-1)}$. From Table 14.12 the appropriate individual model for z_{1t} is

$$(1 - B)(1 - B^{12})z_{1t} = (1 - .28B)(1 - .91B^{12})a_t \quad (14.46)$$

or approximately

$$(1 - B)z_{1t} = S_{1t} + (1 - .28B)a_t, \quad \hat{\sigma}_a^2 = 41.61 \quad (14.47)$$

We see from (14.45) that the difference operator $(1 - B)$ in (14.47), indicating that z_{1t} is nonstationary, arises because of the dependence of z_{1t} on $z_{2(t-1)}$. Also, by comparing the estimated variance $\hat{\sigma}_a^2$ in (14.47) with the corresponding estimated variance $\hat{\sigma}_{11}$ in (14.45), a substantial reduction occurs when the information $z_{2(t-1)}$ is utilized.

In summary, this example shows that (1) the existence of a deterministic seasonal component can be detected when the exact likelihood method is employed, (2) an appreciable reduction in the one step ahead forecast variance can occur by modeling several series jointly, and (3) although individually both z_{1t} and z_{2t} are nonstationary, there is really only one nonstationary component z_{2t} since nonstationarity of z_{1t} stems from its dependence on $z_{2(t-1)}$. The last point is closely related to the concept of cointegration discussed below.

14.6. STRUCTURAL ANALYSIS OF MULTIVARIATE TIME SERIES

In modeling and analysis of multivariate time series, it is of interest, and often critical, to examine the structure of the relations among the component series. The vector ARMA model (14.7) is a straightforward generalization of the univariate ARMA model (14.3). This direct generalization, however, creates a number of difficulties. In particular, each AR or MA term in the vector model contains k^2 number of parameters so that even for a moderate k, there will be an overflow of parameters whose estimates are often highly correlated. This will make it difficult, if not impossible, to comprehend the fitting result. For univariate ARIMA models, parsimony is achieved by making the orders p and q as small as possible, but this is not sufficient for the multivariate case. Here we need to simplify the structure of the parameter matrices Φ's and Θ's, and a possible way to achieve this is by considering *linear transformations* of the original component series. In addition, in many problems it is also of interest to explore the

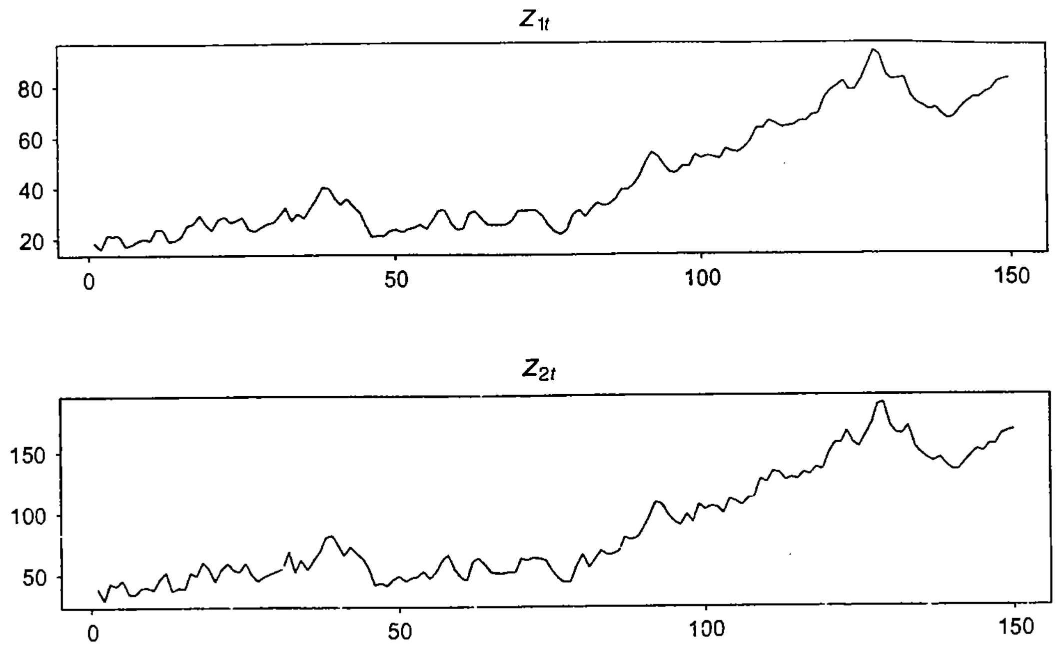

FIGURE 14.7 Generated bivariate ARMA(1,1) series.

existence of possibly simpler underlying structures which explain important features of the observed series. For example, there may exist a linear combination of the component series which is responsible for most of the variation of the data, or another combination responsible for the dynamic growth of a number of the components. In this and the next sections, we discuss some useful types of linear transformations for structural analysis.

As an illustration of the usefulness of linear transformations, consider the bivariate time series $\mathbf{z}_t = (z_{1t}, z_{2t})'$ of 150 observations shown in Figure 14.7, generated from the following ARMA(1,1) model

$$(\mathbf{I} - \Phi B)\mathbf{z}_t = (\mathbf{I} - \Theta B)\mathbf{a}_t \tag{14.48}$$

where

$$\Phi = \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}, \quad \Theta = \begin{bmatrix} -.5 & .5 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} 1 & .3 \\ .3 & 1 \end{bmatrix}.$$

The matrices Φ and Θ contain a total of eight non-zero parameters. Both series appear to be nonstationary and move in tandem. It is easily shown that individually each component in $\mathbf{z}_t = (z_{1t}, z_{2t})'$ follows a nonstationary ARIMA(0, 1, 1) model. Consider, however, the linear transformation

$$\mathbf{y}_t = \mathbf{T}\mathbf{z}_t \quad \text{where} \quad \mathbf{T}' = \left[\boldsymbol{\nu}_0^{(1)}, \quad \boldsymbol{\nu}_0^{(2)}\right] = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}. \tag{14.49}$$

The transformed bivariate series $\mathbf{y}_t = (y_{1t}, y_{2t})'$ is shown in Figure 14.8, where y_{1t}

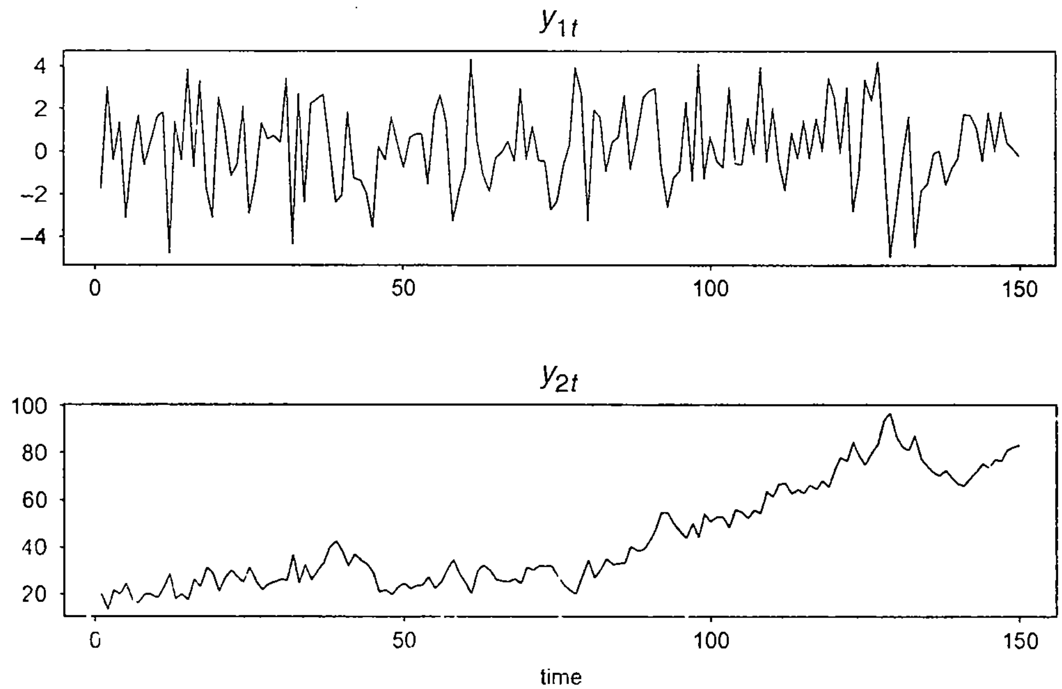

FIGURE 14.8 Transformed bivariate ARMA(1,1) series.

is clearly seen to be stationary and y_{2t}, nonstationary. It is easily shown that the transformed process $\mathbf{y}_t$ has the bivariate ARMA(1,1) model

$$(\mathbf{I} - \Phi^* B)\mathbf{y}_t = (\mathbf{I} - \Theta^* B)\mathbf{b}_t \tag{14.50}$$

where

$$\Phi^* = \mathbf{T}\Phi\mathbf{T}^{-1} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad \Theta^* = \mathbf{T}\Theta\mathbf{T}^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & .5 \end{bmatrix}, \quad \text{and}$$

$$\mathbf{b}_t = \mathbf{T}\mathbf{a}_t = \begin{bmatrix} b_{1t} \\ b_{2t} \end{bmatrix}.$$

From the model for y_t in (14.50), we see that

1. Φ^* and Θ^* together contain only three nonzero parameters.
2. The first component $y_{1t} = b_{1t}$ is simply a white-noise process.
3. There is only one nonstationary component y_{2t}.
4. The nonstationarity of the original components z_{1t} and z_{2t}, is due to this common component y_{2t}.

For this example, the linear transformation $\mathbf{T}\mathbf{z}_t$ produces not only parsimony in parameterization but also components whose structure can be substantively meaningful. In the analysis of economic time series, it can happen that a linear combination of several individually nonstationary time series becomes stationary, implying a stable relationship among the component series. The basic idea is discussed in Box and Tiao (1977) and was later popularized by Engle and Granger (1987) in the econometric literature in what has since been called *cointegration*.

14.6.1. A canonical analysis of multiple time series

Here we discuss the linear transformation method proposed by Box and Tiao in their 1977 paper. The basic objective is to assess linear combinations of the observed series $\mathbf{z}_t$ according to their dynamic dependence on the past history of the series. The vector ARMA(p,q) model in (14.7) can be written as

$$\mathbf{z}_t = \hat{\mathbf{z}}_{t-1}(1) + \mathbf{a}_t \tag{14.51}$$

where $\hat{\mathbf{z}}_{t-1}(1)$ is the one-step-ahead forecast of $\mathbf{z}_t$ made at time $t-1$. Now, $\hat{\mathbf{z}}_{t-1}(1)$ is a linear function of the past observations $(\mathbf{z}_{t-1}, \mathbf{z}_{t-2}, \ldots)$ and is independent of $\mathbf{a}_t$. When the model is invertible, we have that

$$\hat{\mathbf{z}}_{t-1}(1) = [\mathbf{I} - \Pi(B)]\mathbf{z}_t. \tag{14.52}$$

where $\Pi(B) = \mathbf{I} - \Pi_1 B - \Pi_2 B^2 - \cdots$ and $\Pi(B)$ satisfies the relation $\Theta(B)\Pi(B) = \Phi(B)$.

For stationary series with zero mean vector, let

$$\Gamma_{\mathbf{z}}(0) = \mathrm{E}(\mathbf{z}_t\mathbf{z}_t') \quad \text{and} \quad \Gamma_{\hat{\mathbf{z}}}(0) = \mathrm{E}(\hat{\mathbf{z}}_{t-1}(1)\hat{\mathbf{z}}_{t-1}(1)'), \tag{14.53}$$

Consider the linear transformation $y_t = \mathbf{h}'\mathbf{z}_t$ where $\mathbf{h}$ is a $k \times 1$ vector of constants. The variance of y_t is $\mathbf{h}'\Gamma_{\mathbf{z}}(0)\mathbf{h}$ where

$$\mathbf{h}'\Gamma_{\mathbf{z}}(0)\mathbf{h} = \mathbf{h}'\Gamma_{\hat{\mathbf{z}}}(0)\mathbf{h} + \mathbf{h}'\Omega\mathbf{h} \tag{14.54}$$

Thus, an appropriate measure of the dynamic dependence of y_t on the past history $(\mathbf{z}_{t-1}, \mathbf{z}_{t-2}, \ldots)$ is the variance ratio

$$\lambda = \frac{\mathbf{h}'\Gamma_{\hat{\mathbf{z}}}(0)\mathbf{h}}{\mathbf{h}'\Gamma_{\mathbf{z}}(0)\mathbf{h}}. \tag{14.55}$$

It follows from standard canonical correlation analysis that the combination y_t which maximizes this ratio is such that λ is the largest eigenvalue and $\mathbf{h}$ the corresponding eigenvector of

$$\Gamma_{\mathbf{z}}^{-1}(0)\Gamma_{\hat{\mathbf{z}}}(0) \tag{14.56}$$

After the largest eigenvalue and the corresponding eigenvector, it may be of interest to consider the next largest, and so on. In general, let

$$0 \le \lambda_1 \le \cdots \le \lambda_k \le 1 \quad \text{and} \quad \mathbf{M}' = [\mathbf{m}_1, \ldots, \mathbf{m}_k] \tag{14.57}$$

be the k (ordered) eigenvalues and the corresponding matrix of eigenvectors. Consider the linear transformation $\mathbf{y}_t = \mathbf{M}\mathbf{z}_t$. Then

$$\mathbf{y}_t = \hat{\mathbf{y}}_{t-1}(1) + \mathbf{b}_t \tag{14.58}$$

where $\hat{\mathbf{y}}_{t-1}(1) = \mathbf{M}\hat{\mathbf{z}}_{t-1}(1)$, and $\mathbf{b}_t = \mathbf{M}\mathbf{a}_t$, $\hat{\mathbf{y}}_{t-1}(1)$, and $\mathbf{b}_t$ are independent, and it can be readily shown that the covariance matrices of $\mathbf{y}_t$, $\hat{\mathbf{y}}_{t-1}(1)$ and $\mathbf{b}_t$ are all diagonal. In particular, if the eigenvectors are nomalized so that all the components of $\mathbf{y}_t = (y_{1t}, \ldots, y_{kt})'$ have unit variance, then the covariance matrix of $\hat{\mathbf{y}}_{t-1}(1)$ will be $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_k\}$ and that of $\mathbf{b}_t$, $\mathbf{I} - \Lambda$.

These results highlight the characteristics of the canonical transformation (14.58). First, the transformed component y_{1t} is the least dynamically dependent linear combination of $\mathbf{z}_t$ and y_{kt} is the most dynamically dependent linear combination. The dynamical dependency of the y_{it} series ranges from λ_1 to λ_k. If $\lambda_1 = 0$, $y_{1t} = b_{1t}$ so that y_{1t} is a white noise process with a constant variance. Since y_{1t} is a linear combination of $\mathbf{z}_t$, this implies that there exists a very stationary contemporaneous relationship among the k original series. On the other extreme, when $\lambda_k \to 1$, it can be shown that y_{kt} will approach a series with a root on the unit circle in its model so that $\mathbf{z}_t$ will contain a nonstationary component. Thus, one may also regard the λ's as measures of the stationarity of the transformed components, with small values signifying the existence of very stationary components and values close to unity, nonstationart components. The idea here is, of course, closely related to cointegration.

For illustration, Box and Tiao have applied the canonical transformation method to the hog data given in Quenouille (1957) consisting of annual observations of hog supply, hog prices, corn supply, corn prices, and farm wages for the period 1867–1948. Individually, all five series are apparently nonstationary. (See Fig. 1.12 of Chapter 1.) By fitting a vector AR(1) model to the data, they have found that

$$\hat{\lambda}_1 = .0232, \quad \hat{\lambda}_2 = .1421, \quad \hat{\lambda}_3 = .5061, \quad \hat{\lambda}_4 = .6901, \quad \hat{\lambda}_5 = .8868.$$

The five transformed series are shown in Figure 14.9. The first two transformed series corresponding to the two smallest eignvalues are seen to be very stable over time, and the last series corresponding to the largest eigenvalue underlines much of the growth in the data. Out of the originally five nearly nonstationary series, there seem to exist two linear combinations that are very stationary. For further details of the analysis and substantive interpretation, see the 1977 paper.

14.7. SCALAR COMPONENT MODELS IN MULTIPLE TIME SERIES

This section introduces an approach for modeling vector time series proposed by Tiao and Tsay (1989). The goal is to find

1. An overall parsimonious model
2. Possibly simplifying structures that may not be obvious by direct modeling of the observed data

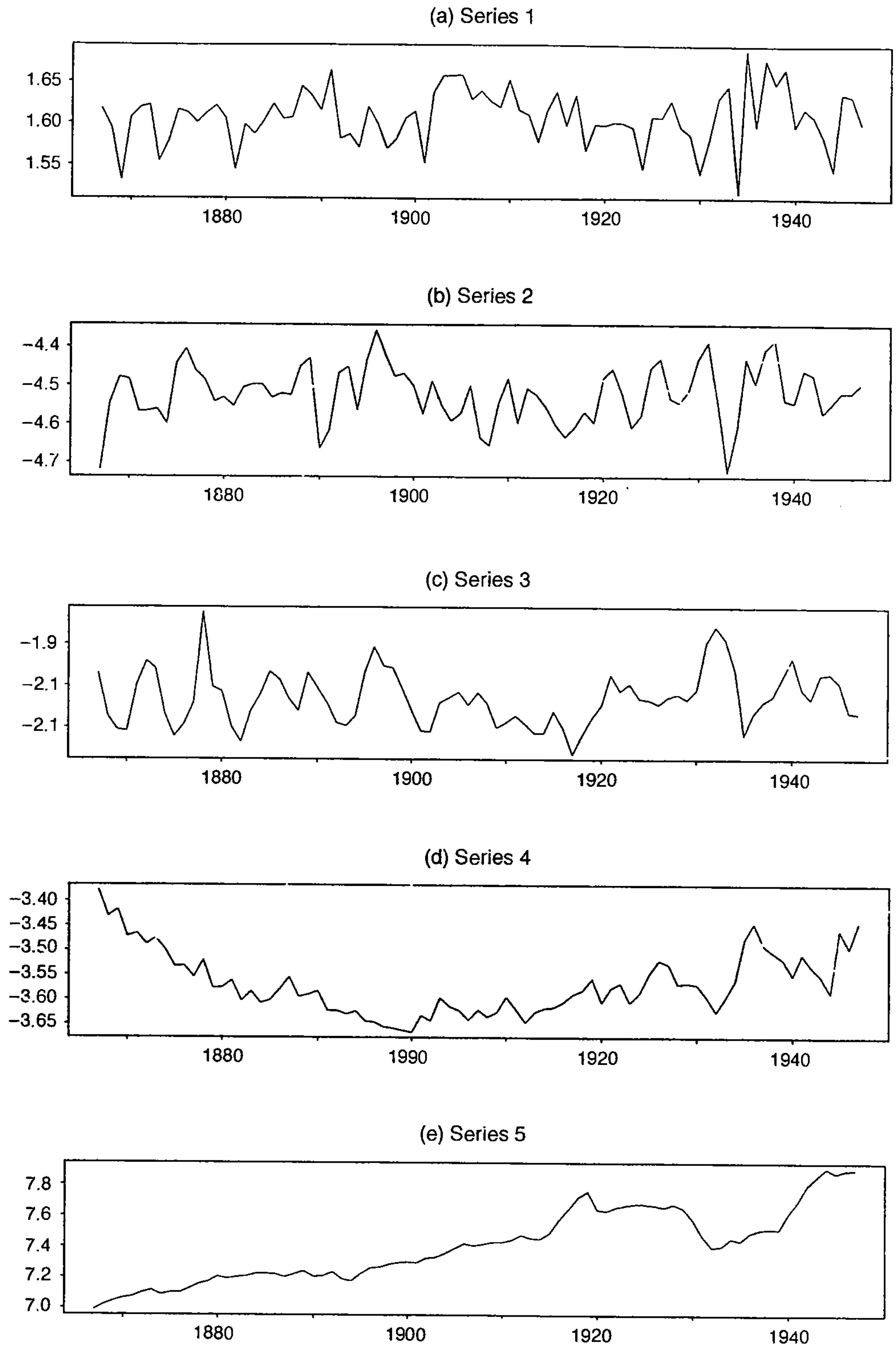

FIGURE 14.9 Transformed hog data.

This is achieved at the model specification stage by investigating linear combinations of the observed series.

As mentioned earlier, while the vector ARMA(p,q) model (14.7) is a straight forward generalization of the univariate ARIMA model (14.3), this direct generalization creates a major difficulty that, for moderate or large k, there will be an overflow of parameters whose estimates are often highly correlated. Another major difficulty in this generalization is the lack of identifiable models. To illustrate, consider again the bivariate ARMA(1,1) model in (14.48) and its transform in (14.50). Since $y_{1(t-1)} = b_{1(t-1)}$, the model for $\mathbf{y}_t$ in (14.50) can alternatively be written as

$$(\mathbf{I} - \Phi^{**}B)\mathbf{y}_t = (\mathbf{I} - \Theta^{**}B)\mathbf{b}_t \tag{14.59}$$

where

$$\Phi^{**} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad \Theta^{**} = \begin{bmatrix} 0 & 0 \\ -2 & .5 \end{bmatrix}.$$

The two bivariate ARMA(1,1) models, (14.50) and (14.59), are exchangeable in the sense that they have different parameter values for the autoregressive (AR) and moving-average (MA) matrices but yield the same probabilistic structure of the process $\mathbf{y}_t$. In general, the vector ARMA model representation allows for exchangeable models because it opens the possibility of parameterization in both the AR and the MA parts. The possibility of multiple-model representations gives rise to the problem of identifiability of the vector ARMA model. This identifiability problem has been discussed extensively in the literature. See Hannan (1969), Akaike (1976), Solo (1986), and the references cited therein. Here, we shall not discuss the theory of identifiability or the sufficient conditions for a unique model representation, but consider methods that can recognize exchangeable models when they exist.

In what follows we first introduce the idea of *scalar component models* (SCMs) to describe a components structure in a multivariate framework. Some issues concerned with exchangeable models are then discussed. Finally we present an example illustrating the use of canonical correlation analysis to specify scalar component models. For details, see the paper by Tiao and Tsay.

14.7.1. Scalar component models

To introduce the concept of SCMs, we begin with another simple example. Suppose that $\mathbf{z}_t$ follows the bivariate ARMA(1,1) model

$$(\mathbf{I} - \Phi B)\mathbf{z}_t = (\mathbf{I} - \Theta B)\mathbf{a}_t \tag{14.60}$$

where

$$\Phi = \begin{bmatrix} .3 & .2 \\ .9 & .6 \end{bmatrix}, \qquad \Theta = \begin{bmatrix} .25 & .175 \\ .5 & .35 \end{bmatrix}$$

containing eight nonzero parameters. Consider the linear transformation

$$\mathbf{y}_t = \mathbf{T}\mathbf{z}_t \qquad \text{where} \qquad \mathbf{T}' = \left[\boldsymbol{\nu}_0^{(1)}, \quad \boldsymbol{\nu}_0^{(2)}\right] = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}.$$

Premultiplying expression (14.60) by $\mathbf{T}$ we have that

$$\mathbf{T}\mathbf{z}_t + \mathbf{G}\mathbf{z}_{t-1} = \mathbf{T}\mathbf{a}_t + \mathbf{H}\mathbf{a}_{t-1} \tag{14.61}$$

where

$$\mathbf{G} = \begin{bmatrix} .3 & .2 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 & 0 \\ .25 & .175 \end{bmatrix}$$

and it is seen that $\mathbf{G}$ and $\mathbf{H}$ each contains a row of zero values. By inserting $\mathbf{T}^{-1}\mathbf{T}$ in front of $\mathbf{z}_{t-1}$ and $\mathbf{a}_{t-1}$, the model for $\mathbf{y}_t$ is

$$(\mathbf{I} - \Phi^* B)\mathbf{y}_t = (\mathbf{I} - \Theta^* B)\mathbf{b}_t \tag{14.62}$$

where

$$\Phi^* = \begin{bmatrix} .9 & .7 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Theta^* = \begin{bmatrix} 0 & 0 \\ .775 & .6 \end{bmatrix}.$$

Now for the transformed $\mathbf{y}_t$ process, the component y_{1t} does not need any non-zero MA coefficient and the component y_{2t} has no AR coefficient. In other words, corresponding to the zero rows of $\mathbf{G}$ and $\mathbf{H}$, Φ^* and Θ^* each contains a row of zero values. This is the type of simplification that we intend to capture in this approach.

To describe the structure of y_{1t}, we say that it follows an SCM of order (1,0). The order (1,0) signifies that, within the bivariate ARMA(1,1) framework, y_{1t} needs AR parameters at lag 1, but it does not require any nonzero MA coefficient. This is because the vector $\boldsymbol{\nu}_0^{(1)}$ has the property that $\boldsymbol{\nu}_0^{(1)\prime}\Phi = -(.3, .2) \neq \mathbf{0}'$ but $\boldsymbol{\nu}_0^{(1)\prime}\Theta = \mathbf{0}'$. More generally, given the vector ARMA(p,q) model (14.7) we say that a non-zero linear combination $y_t = \boldsymbol{\nu}_0'z_t$ follows an SCM of order (p_1,q_1) if $\boldsymbol{\nu}_0$ has the properties

$$\begin{aligned} \boldsymbol{\nu}_0'\Phi_{p_1} &\neq \mathbf{0}', & 0 &\le p_1 \le p \\ \boldsymbol{\nu}_0'\Phi_\ell &= \mathbf{0}', & \ell &= p_1 + 1, \ldots, p \\ \boldsymbol{\nu}_0'\Theta_{q_1} &\neq \mathbf{0}', & 0 &\le q_1 \le q \\ \boldsymbol{\nu}_0'\Theta_\ell &= \mathbf{0}', & \ell &= q_1 + 1, \ldots, q. \end{aligned} \tag{14.63}$$

Since $\boldsymbol{\nu}_0'\Phi(B)\mathbf{z}_t = \boldsymbol{\nu}_0'\Theta(B)\mathbf{a}_t$, the structure of y_t can be written as

$$y_t + \sum_{\ell=1}^{p_1} \mathbf{g}_\ell'\mathbf{z}_{t-\ell} = \boldsymbol{\nu}_0'\mathbf{a}_t + \sum_{\ell=1}^{q_1} \mathbf{h}_\ell'\mathbf{a}_{t-\ell} \tag{14.64}$$

where $\mathbf{g}_\ell' = -\boldsymbol{\nu}_0'\Phi_\ell$ and $\mathbf{h}_\ell' = -\boldsymbol{\nu}_0'\Theta_\ell$. We shall denote the model structure of y_t

as SCM(p_1, q_1). Thus, by allowing $\boldsymbol{\nu}_0$ to be an arbitrary nonzero vector, the SCM is a direct generalization of the model of each component z_{jt} in the vector ARMA framework so that the model structure can be simplified.

The SCM is a device designed to capture the structure of a component within a vector model. It is *not* a univariate ARMA model. Thus, the fact that y_t has an SCM(p_1, q_1) structure in a vector model does not necessarily imply that y_t follows an univariate ARMA(p_1, q_1) process. For example, y_{1t} in model (14.48) follows an SCM(1, 0) structure but it is not a univariate ARMA(1, 0) process, because the SCM structure of y_{1t} involves nonzero coefficients of $y_{2(1-t)}$.

Given a k-dimensional vector ARMA process $\mathbf{z}_t$, let $\mathbf{y}_t = \mathbf{T}\mathbf{z}_t$, where $\mathbf{T} = [\boldsymbol{\nu}_0^{(1)}, \ldots, \boldsymbol{\nu}_0^{(k)}]'$ is a $k \times k$ nonsingular matrix, be the transformed process associated with k SCMs of orders (p_i, q_i), $i = 1, \ldots, k$. Such a transformation can lead to considerable parsimony in parameterization of the transformed model. As a further illustration, suppose that the observed process $\mathbf{z}_t$ follows a four dimensional ARMA(2,1) model and that the transformation $\mathbf{T}\mathbf{z}_t$ produces four scalar component models of orders (0,0), (0,1), (1,0), and (2,1). We then have, for the transformed process $\mathbf{y}_t$, the ARMA(2,1) model

$$(\mathbf{I} - \Phi_1 B - \Phi_2 B^2)\mathbf{y}_t = (\mathbf{I} - \Theta_1 B)\mathbf{b}_t \tag{14.65}$$

where (suppressing the superscript *)

$$\Phi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \times & \times & \times & \times \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \times & \times & \times & \times \\ 0 & 0 & 0 & 0 \\ \times & \times & \times & \times \end{bmatrix},$$

with 0 and $\times$ denoting zero and nonzero parameters, respectively. In this particular instance then, modeling the transformed series $\mathbf{y}_t$ would involve 20 parameters in the coefficient matrices instead of 48, a saving of 28 parameters in estimation. As will be shown in the next section, because of the structure of the first and the third SCMs, we may further set the (4,1)th and the (4,3)th elements of Θ_1 to zero, achieving a total reduction of 30 parameters. The reduction could be even more substantial when the dimension k is relatively large. In view of the possibility of high correlations among the unconstrained 48 parameter estimates, the reduction could drastically simplify the complexity in estimation.

14.7.2. Exchangeable models and overparameterization

Exchangeable models are a special feature of vector time series that does not occur in the univariate case. In what follows, we discuss two issues:

1. Models with exchangeable structures
2. Elimination of redundant parameters

Models with exchangeable structures

Two vector ARMA models are exchangeable if they are of finite order and give the same probability distribution of $\mathbf{z}_t$. This can happen, for instance, when $|\Phi(B)|$ or $|\Theta(B)|$ is a constant. A simple example is that a bivariate AR(1) model $\mathbf{z}_t = \Phi\mathbf{z}_{t-1} + \mathbf{a}_t$ can be written exactly as a bivariate MA(l) model $\mathbf{z}_t = -\Theta\mathbf{a}_{t-1} + \mathbf{a}_t$ if

$$\Phi = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \qquad \Theta = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}. \tag{14.66}$$

Here the lack of identifiability exists because $z_{2t} = a_{2t}$ so that $z_{2(t-1)}$ and $a_{2(t-1)}$ can appear as alternatives in the model structure of $z_{1t} : z_{1t} = 2z_{2(t-1)} + a_{1t} = 2a_{2(t-1)} + a_{1t}$ and generally $z_{1t} = (2-\delta)z_{2(t-1)} + \delta a_{2(t-1)} + a_{1t}$ so that we can also write the model structure for $\mathbf{z}_t$ in the form of a bivariate ARMA(1,1) model. In general, when the AR matrix polynomial $\Phi(B)$ and the MA matrix polynomial $\Theta(B)$ of model (4.7) are left coprime and if $\Phi(B)$ can be written as the product $\Phi(B) = \Xi(B)\Lambda(B)$ where $\Xi(B)$ and $\Lambda(B)$ are two nontrivial matrix polynomials such that $|\Xi(B)|$ is a non-zero constant, then $\Phi(B)\mathbf{z}_t = \Theta(B)\mathbf{a}_t$ is equivalent to $\Lambda(B)\mathbf{z}_t = \Xi^{-1}(B)\Theta(B)\mathbf{a}_t$, which is also of finite order. A similar condition applies to $\Theta(B)$.

Existence of exchangeable vector ARMA models leads to alternative specifications of the structure of some associated SCMs. Consider again the simple exchangeable bivariate AR(1) and MA(1) model corresponding to (14.66). For the scalar component $z_{1t} = \boldsymbol{\nu}_0'\mathbf{z}_t$ where $\boldsymbol{\nu}_0 = (1, 0)'$, the structure will be SCM(1, 0) with $\mathbf{g}_1 = (0, -2)'$ for the bivariate AR(1) representation, and it becomes SCM(0, 1) with $\mathbf{h}_1 = (0, 2)'$ for the bivariate MA(1) model. In this case, we shall say that $\boldsymbol{\nu}_0'\mathbf{z}_t$ follows an SCM of exchangeable orders (1,0) and (0,1). As a further illustration, consider the two exchangeable models (14.50) and (14.59). For the second scalar component y_{2t}, the structure is SCM(1,1) for both model (14.50) and model (14.59) but the coefficients are different.

Since exchangeable models have the same probability distribution, they give the same covariance structure and provide the same inference. For this reason, we shall not discuss the conditions needed for unique model representation. Since exchangeable models are synonymous with exchangeable SCMs, we focus instead on methods that can point out the existence of exchangeable SCMs.

Redundant parameters

Redundant parameters are one of the most troublesome features in fitting vector ARMA models. One type of redundant (or unidentifiable) parameters may be eliminated by a careful study of the SCM structure. Consider, for example, the transformed ARMA(2,1) model (14.65). Let $\mathbf{y}_t = (y_{1t}, \ldots, y_{4t})'$, $\mathbf{b}_t = (b_{1t}, \ldots, b_{4t})'$, $\theta_{ij}^{(\ell)}$ and $\phi_{ij}^{(\ell)}$ be the (i, j)th elements of Θ_ℓ and Φ_ℓ, respectively. For the component y_{4t}, we have that

$$y_{4t} = \sum_{j=1}^{4} \phi_{4j}^{(1)} y_{j(t-1)} + \sum_{i=1}^{4} \phi_{4i}^{(2)} y_{i(t-2)} + b_{4t} - \sum_{j=1}^{4} \theta_{4j}^{(1)} b_{j(t-1)}. \tag{14.67}$$

Now, since y_{1t} is white noise, $y_{1(t-1)} = b_{1(t-1)}$. It is clear that in (14.67) $\theta_{41}^{(1)} - \phi_{41}^{(1)}$ is a constant, and we may set either $\theta_{41}^{(1)}$ or $\phi_{41}^{(1)}$ to zero. Next, from the structure of y_{3t} we have

$$y_{3(t-1)} - b_{3(t-1)} = \sum_{i=1}^{4} \phi_{3i}^{(1)} y_{i(t-2)}.$$

Since $y_{i(t-2)}$, for $i = 1, \ldots, 4$, are already in (14.67), it is clear that either $y_{3(t-1)}$ or $b_{3(t-1)}$ can be eliminated and we may set $\theta_{43}^{(1)}$ to zero.

In general, the structure of SCMs for a vector ARMA model makes it particularly convenient to spot redundant parameters. More specifically, consider a transformed vector ARMA(p,q) model for $\mathbf{y}_t$ such that y_{1t} has an SCM(p_1,q_1) structure and y_{2t} has an SCM(p_2,q_2) structure where $p_2 > p_1$ and $q_2 > q_1$. In this case, we can write the SCMs for y_{1t} and y_{2t} as

$$y_{it} - \left\{\phi_i^{(1)} B + \cdots + \phi_i^{(p_i)} B^{p_i}\right\}\mathbf{y}_t = b_{it} - \left\{\theta_i^{(1)} B + \cdots + \theta_i^{(q_i)} B^{q_i}\right\}\mathbf{b}_t \qquad (14.68)$$

where $i = 1, 2$ and $\phi_i^{(\ell)}$ and $\theta_i^{(\ell)}$ are the ith rows of the matrices Φ_ℓ and Θ_ℓ, respectively. Now for $i = 2$ we see from (14.68) that y_{2t} is related to $y_{1(t-1)}, \ldots, y_{1(t-p_2)}$ and $b_{1(t-1)}, \ldots, b_{1(t-q_2)}$ via

$$\left(\phi_{21}^{(1)} B + \cdots + \phi_{21}^{(p_2)} B^{p_2}\right) y_{1t} - \left(\theta_{21}^{(1)} B + \cdots + \theta_{21}^{(q_2)} B^{q_2}\right) b_{1t}. \qquad (14.69)$$

Since

$$B^S(y_{1t} - b_{1t}) = \left\{\phi_1^{(1)} B + \cdots + \phi_1^{(p_1)} B^{p_1}\right\}\mathbf{y}_{t-s} - \left\{\theta_1^{(1)} B + \cdots + \theta_1^{(q_1)} B^{q_1}\right\}\mathbf{b}_{t-s}, \qquad (14.70)$$

it is clear that, if all the $\mathbf{y}$'s and the $\mathbf{b}$'s on the RHS of (14.70) are in the component model for y_{2t}, then either the coefficient of $y_{1(t-s)}$ or that of $b_{1(t-s)}$ is redundant given that the other is in the model. Therefore, if $p_2 > p_1$ and $q_2 > q_1$, then for each pair of parameters $(\phi_{21}^{(s)}, \theta_{21}^{(s)})$ in (14.69) $s = 1, \ldots, \min(p_2 - p_1, q_2 - q_1)$, only one parameter is needed. For convenience, we refer to this rule of spotting redundant parameters as *the rule of elimination*.

In general, by considering a vector ARMA model in its SCM form, and applying the rule of elimination in a pairwise fashion, all such redundant AR or MA parameters can be eliminated.

14.7.3. Model specification via canonical correlation analysis

Tiao and Tsay use canonical correlation analysis to search for SCMs of minimal orders at the model specification stage. The basic idea of the procedure is as follows.

First, for $m \geq 0$ let

$$\mathbf{z}_{m,t} = (\mathbf{z}'_t, \ldots, \mathbf{z}'_{t-m})' \tag{14.71}$$

From (14.63), a SCM of order (p_1,q_1) implies that there exists a linear combination of $\mathbf{z}_{p_1,t}$ which is independent of $\mathbf{a}_{t-j-1}$, and hence of $\mathbf{z}_{t-j-1}$, for $j \geq q_1$. Thus, corresponding to a SCM(p_1, q_1), there will be a zero canonical correlation between the two processes $\mathbf{z}_{p_1,t}$ and $\mathbf{z}_{h,t-j-1}$ for $j \geq q_1$ and $h \geq 0$. In general, for $m = 0, 1, \ldots; j = 0, 1, \ldots;$ and fixed h, the procedure searches for k linearly independent SCMs of minimal orders by examining the number of zero canonical correlations (and corresponding eigenvectors) between $\mathbf{z}_{m,t}$ and $\mathbf{z}_{h,t-j-1}$. The discriminating criteria involve the choice of lags of sample autocovariance matrices and associated χ^2 test statistics. For further details, the reader is referred to the Tiao and Tsay (1989) paper.

14.7.4. An illustrative example

To illustrate, consider again the 5-series U.S. hog data discussed earlier in Section 14.6. Following the SCM search procedure, an overall ARMA(2,1) model for the transformed process with five SCMs of orders (0,0), (0,1), (1,0), (1,1) and (2,0) is tentatively specified:

$$(\mathbf{I} - \Phi_1 B - \Phi_2 B^2)\mathbf{y}_t = \mathbf{c} + (\mathbf{I} - \Theta_1 B)\mathbf{b}_t \tag{14.72}$$

where the detailed structure of Φ_1, Φ_2, and Θ_1 after applying the rule of elimination is

$$\Phi_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times \end{bmatrix},$$

$$\Theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are 29 parameters in the coefficient matrices. In contrast, an unconstrained 5D ARMA(2,1) model would have 75 parameters in these matrices. This illustrates the usefulness of structural specification in multiple time series analysis. In fact, after further simplification in the estimation stage of the transformed model, there are only 16 nonzero coefficients in the final model. For further details; see the 1989 paper.

The five transformed series are shown in Figure 1.13 (in Chapter 1). The first two series are quite stable, while the other three show certain nonstationary behavior. In particular, the third series capture much of the growth of the original data. These features are broadly similar to the first two and the last transformed series in Figure 14.9 using the canonical analysis method of Box and Tiao (1977). Note that the last transformed component in Figure 1.13 seems to capture the periodic behavior of the observed processes. This feature was discussed in Quenouille (1957) but seems less apparent in Figure 14.9.

14.7.5. Some further remarks

In the structural analysis discussed in this and the preceding sections, the method of canonical correlation analysis has been employed. This method is commonly used in statistics for extracting information in a multivariate problem (Hotelling 1936, Anderson 1984). Its potential for time series analysis has long been recognized. See, for instance, Robinson (1973), Akaike (1976), Brillinger (1975, 1981), Cooper and Wood (1982), Jewell and Bloomfield (1983), Jewell et al. (1983), Tsay and Tiao (1985), Velu et al. (1986) and Peña and Box (1987).

Finally, it is worth noting that apart from the two methods presented in this chapter, the principal component analysis is also of value in structural simplification. In particular, it should be applied to the sample covariance of the original series to eliminate redundant series that are exact linear combinations of the other components. It should also be applied to the sample covariance matrics of the residuals after vector autoregressive fitting to uncover exact lagged linear relationship among the series. Apart from finding exact linear relations, principal component analysis can also help achieve structural simplification in multiple time series, as illustrated in Tiao et al. (1993).

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EXAMPLE XVIII: A Generated GARCH(1,1) Series

We here consider a series of 1000 observations generated from the following model

$$Z_t = 5 + .8Z_{t-1} + a_t$$
$$\sigma_t^2 = .5 + .2a_{t-1}^2 + .7\sigma_{t-1}^2$$

***** SCA Output from running ex18.tsk *****

```
call allmacro.file is 'ex18.mac'.
--
input z.file'\b325\garch527.dat'
Z       , A 1000  BY   1 VARIABLE, IS STORED IN THE WORKSPACE
--
iden z.maxl 12

NAME OF THE SERIES . . . . . . . . . .             Z
TIME PERIOD ANALYZED . . . . . . . . .  1  TO  1000
MEAN OF THE (DIFFERENCED) SERIES . . .      25.1033
STANDARD DEVIATION OF THE SERIES . . .       3.7640
T-VALUE OF MEAN (AGAINST ZERO) . . . .     210.9005

AUTOCORRELATIONS

  1- 12      .79  .61  .47  .34  .22  .13  .09  .06  .03  .03  .00 -.01
  ST.E.      .03  .05  .05  .06  .06  .06  .06  .06  .06  .06  .06  .06
   Q         618  995 1219 1338 1385 1402 1410 1413 1415 1415 1415 1415

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .79                         + IX+XXXXXXXXXXXXXXXXXX
  2    .61                         + IX+XXXXXXXXXXXXX
  3    .47                        +  IXX+XXXXXXXXX
  4    .34                        +  IXX+XXXXXX
  5    .22                        +  IXX+XX
  6    .13                        +  IXXX
  7    .09                        +  IXX+
  8    .06                        +  IX +
  9    .03                        +  IX +
 10    .03                        +  IX +
 11    .00                        +  I  +
 12   -.01                        +  I  +

PARTIAL AUTOCORRELATIONS

  1- 12      .79 -.01 -.01 -.05 -.09  .02  .04  .00 -.00  .01 -.06  .01
  ST.E.      .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03
```

```
          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .79                         + IX+XXXXXXXXXXXXXXXXXX
  2   -.01                         + I +
  3   -.01                         + I +
  4   -.05                         +XI +
  5   -.09                         XXI +
  6    .02                         + I +
  7    .04                         + IX+
  8    .00                         + I +
  9    .00                         + I +
 10    .01                         + I +
 11   -.06                         +XI +
 12    .01                         + I +
--
```

tsm mz.model (1)z=c+noise

--

estim mz.hold resi(r)

```
SUMMARY FOR UNIVARIATE TIME SERIES MODEL --    MZ
-----------------------------------------------------------------------
VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
          VARIABLE   OR CENTERED

   Z       RANDOM     ORIGINAL     NONE
-----------------------------------------------------------------------
PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
   LABEL       NAME    DENOM.                TRAINT               ERROR  VALUE

  1    C               CNST      1      0     NONE     5.3878     .4965  10.85
  2            Z        AR       1      1     NONE      .7852     .0196  40.15

TOTAL SUM OF SQUARES . . . . . . . . .   .141679E+05
TOTAL NUMBER OF OBSERVATIONS . . . .            1000
RESIDUAL SUM OF SQUARES. . . . . . .   .541537E+04
R-SQUARE . . . . . . . . . . . . . .            .617
EFFECTIVE NUMBER OF OBSERVATIONS . .             999
RESIDUAL VARIANCE ESTIMATE . . . . .   .542079E+01
RESIDUAL STANDARD ERROR. . . . . . .   .232826E+01

acf r.maxl 12

NAME OF THE SERIES . . . . . . . . . .               R
TIME PERIOD ANALYZED . . . . . . . . .  2  TO  1000
MEAN OF THE (DIFFERENCED) SERIES . . .         .0000
STANDARD DEVIATION OF THE SERIES . . .        2.3283
T-VALUE OF MEAN (AGAINST ZERO) . . . .         .0000

AUTOCORRELATIONS

  1- 12      .01  .01  .03  .04 -.07 -.06 -.01 -.01 -.03  .04 -.02 -.01
  ST.E.      .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03
   Q          .1   .1  1.2  3.0  7.8 12.0 12.2 12.3 13.2 14.6 15.1 15.2
```

```
          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .01                         + I +
  2    .01                         + I +
  3    .03                         + IX+
  4    .04                         + IX+
  5   -.07                         XXI +
  6   -.06                         XXI +
  7   -.01                         + I +
  8   -.01                         + I +
  9   -.03                         +XI +
 10    .04                         + IX+
 11   -.02                         +XI +
 12   -.01                         + I +
--
```
desc z

```
VARIABLE    NAME    IS         Z
NUMBER OF OBSERVATIONS      1000
NUMBER OF MISSING VALUES       0

                   STATISTIC      STD. ERROR     STATISTIC/S.E.

MEAN                25.1033           .1191        210.7950
VARIANCE            14.1821
STD DEVIATION        3.7659
C.  V.                .1500
SKEWNESS             -.1006           .0773
KURTOSIS              .9773           .1545

                   QUARTILE

MINIMUM              9.5381
1ST QUARTILE        22.7292
MEDIAN              25.1783
3RD QUARTILE        27.3967
MAXIMUM             39.4183

                      RANGE

MAX - MIN           29.8802
Q3  - Q1             4.6675
```

desc r

```
VARIABLE    NAME    IS         R
NUMBER OF OBSERVATIONS      1000
NUMBER OF MISSING VALUES       1

                   STATISTIC      STD. ERROR     STATISTIC/S.E.

MEAN                  .0000           .0737           .0000
VARIANCE             5.4262
STD DEVIATION        2.3294
C.  V.       -34880740.0000
SKEWNESS              .1689           .0774
```

```
KURTOSIS                .5210           .1546

                     QUARTILE

MINIMUM               -9.1164
1ST QUARTILE          -1.5224
MEDIAN                 -.0150
3RD QUARTILE           1.4079
MAXIMUM                9.7298

                        RANGE

MAX - MIN             18.8462
Q3  - Q1               2.9303
```

r2=r*r
--
iden r2.maxl 12

```
NAME OF THE SERIES . . . . . . . . . .             R2
TIME PERIOD ANALYZED . . . . . . . . .  2  TO  1000
MEAN OF THE (DIFFERENCED) SERIES . . .       5.4208
STANDARD DEVIATION OF THE SERIES . . .       8.6189
T-VALUE OF MEAN (AGAINST ZERO) . . . .      19.8789

AUTOCORRELATIONS

  1- 12     .27  .22  .19  .15  .19  .09  .09  .05  .06  .12  .02  .02
  ST.E.     .03  .03  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04
   Q       74.9  124  162  185  223  231  239  242  245  260  260  261

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .27                         + IX+XXXXX
  2    .22                         + IX+XXXX
  3    .19                         + IX+XXX
  4    .15                         + IX+XX
  5    .19                         + IX+XXX
  6    .09                         + IXX
  7    .09                         + IXX
  8    .05                         + IX+
  9    .06                         + IXX
 10    .12                         + IX+X
 11    .02                         + IX+
 12    .02                         + I +

PARTIAL AUTOCORRELATIONS

  1- 12     .27  .16  .11  .06  .11 -.02  .01 -.02  .01  .08 -.04 -.03
  ST.E.     .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .27                         + IX+XXXXX
  2    .16                         + IX+XX
```

```
  3    .11                        + IX+X
  4    .06                        + IX+
  5    .11                        + IX+X
  6   -.02                        +XI +
  7    .01                        + I +
  8   -.02                        + I +
  9    .01                        + I +
 10    .08                        + IXX
 11   -.04                        +XI +
 12   -.03                        +XI +
--
```

eacf r2

```
NAME OF THE SERIES . . . . . . . . . .            R2
TIME PERIOD ANALYZED . . . . . . . . .  2  TO  1000
MEAN OF THE (DIFFERENCED) SERIES . . .       5.4208
STANDARD DEVIATION OF THE SERIES . . .       8.6189
T-VALUE OF MEAN (AGAINST ZERO) . . . .      19.8789

THE EXTENDED ACF TABLE

(Q-->)   0    1    2    3    4    5    6    7    8    9   10   11   12
-----------------------------------------------------------------------
(P= 0)  .27  .22  .19  .15  .19  .09  .09  .05  .06  .12  .02  .02  .02
(P= 1) -.44 -.01  .02 -.05  .12 -.07  .03 -.03 -.01  .12 -.05 -.01  .02
(P= 2) -.47 -.41 -.00 -.02  .10 -.01 -.04 -.01 -.01  .11 -.06 -.02  .00
(P= 3) -.41 -.13  .09 -.01  .10 -.02 -.00 -.03 -.01  .11  .05 -.02  .00
(P= 4) -.41 -.35 -.19 -.19  .09 -.02  .00 -.04 -.02  .11  .02 -.03 -.02
(P= 5)  .18 -.02  .13 -.07 -.30  .00  .01 -.02  .02  .04  .04  .02  .03
(P= 6)  .27  .02  .09 -.02 -.30 -.01  .00 -.00  .02  .01  .04 -.03  .02

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)  0  1  2  3  4  5  6  7  8  9 10 11 12
----------------------------------------------
(P= 0)  X  X  X  X  X  X  X  O  O  X  O  O  O
(P= 1)  X  O  O  O  X  O  O  O  O  X  O  O  O
(P= 2)  X  X  O  O  X  O  O  O  O  X  O  O  O
(P= 3)  X  X  X  O  X  O  O  O  O  X  O  O  O
(P= 4)  X  X  X  X  X  O  O  O  O  X  O  O  O
(P= 5)  X  O  X  X  X  O  O  O  O  O  O  O  O
(P= 6)  X  O  X  O  X  O  O  O  O  O  O  O  O
--
```

tsm mr2.model (1)r2=c+(1)noise

--

estim mr2.method exact.hold resi(rr)

```
SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   MR2
-----------------------------------------------------------------------
VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
          VARIABLE   OR CENTERED

   R2      RANDOM     ORIGINAL     NONE
-----------------------------------------------------------------------
 PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
   LABEL       NAME    DENOM.                 TRAINT              ERROR  VALUE
```

```
  1    C               CNST      1      0     NONE      .8635      .2371   3.64
  2             R2      MA       1      1     NONE      .6455      .0567  11.39
  3             R2      AR       1      1     NONE      .8416      .0403  20.86

TOTAL SUM OF SQUARES . . . . . . . .    .742117E+05
TOTAL NUMBER OF OBSERVATIONS . . . .            999
RESIDUAL SUM OF SQUARES. . . . . . .    .654924E+05
R-SQUARE . . . . . . . . . . . . . .           .117
EFFECTIVE NUMBER OF OBSERVATIONS . .            998
RESIDUAL VARIANCE ESTIMATE . . . . .    .656236E+02
RESIDUAL STANDARD ERROR. . . . . . .    .810084E+01
```

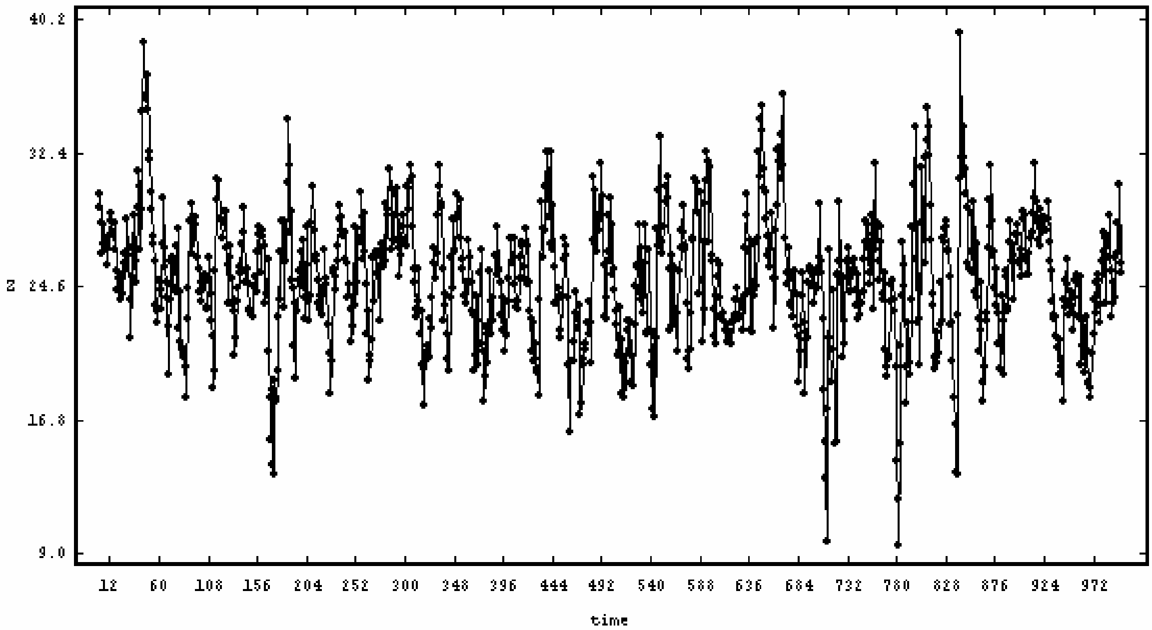
acf rr.maxl 12

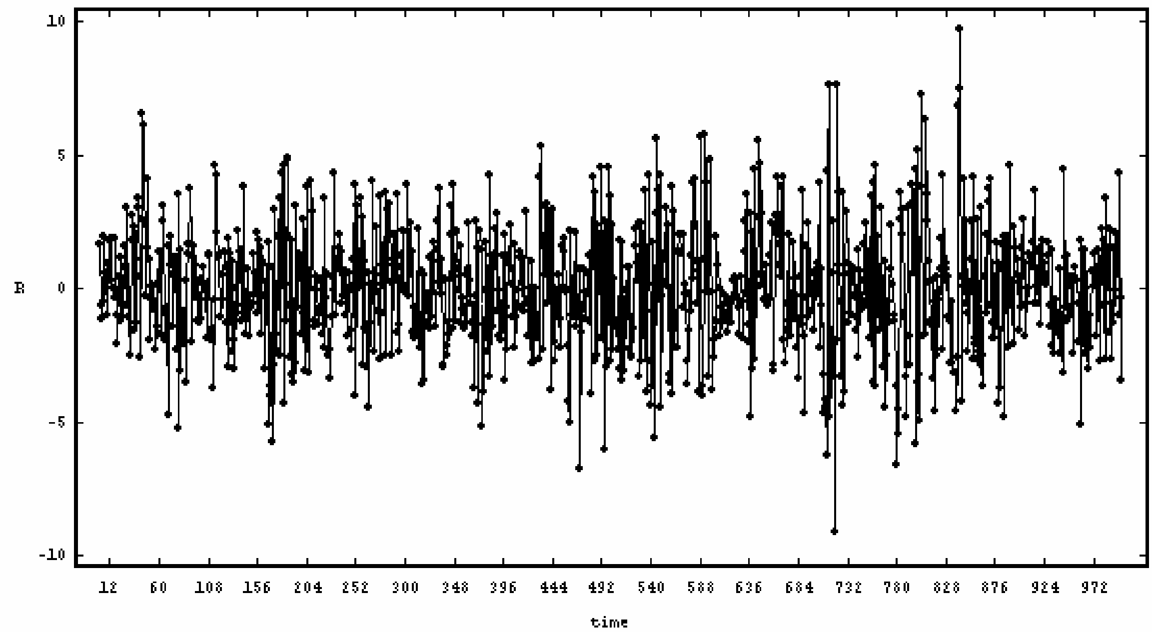
```
NAME OF THE SERIES . . . . . . . . . .           RR
TIME PERIOD ANALYZED . . . . . . . . .  2  TO   999
MEAN OF THE (DIFFERENCED) SERIES . . .       -.0022
STANDARD DEVIATION OF THE SERIES . . .       8.1005
T-VALUE OF MEAN (AGAINST ZERO) . . . .       -.0085

AUTOCORRELATIONS

  1- 12     .00 -.01  .00 -.02  .08 -.03 -.01 -.03 -.01  .09 -.03 -.02
  ST.E.     .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03  .03
   Q         .0   .1   .1   .4  6.3  7.4  7.5  8.7  8.7 16.8 17.7 18.1

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .00                         + I +
  2   -.01                         + I +
  3    .00                         + I +
  4   -.02                         + I +
  5    .08                         + IXX
  6   -.03                         +XI +
  7   -.01                         + I +
  8   -.03                         +XI +
  9   -.01                         + I +
 10    .09                         + IXX
 11   -.03                         +XI +
 12   -.02                         +XI +
```

We give below a comparison of the estimated coefficients:

Parameter	SCA	S-PLUS
c	5.390	5.680
ϕ	.785	.773
α_0	.864	.784
α_1	.196	.244
β_1	.646	.621

The results are fairly close.


40.2
32.4
Z 24.6
16.8
9.0
12 60 108 156 204 252 300 348 396 444 492 540 588 636 684 732 780 828 876 924 972
time


10
5
R 0
-5
-10
12 60 108 156 204 252 300 348 396 444 492 540 588 636 684 732 780 828 876 924 972
time

ex18-7

EXAMPLE: GARCH Model for the SP500 Stock Index

This example consists of 792 monthly returns of of value-weighted S&P 500 stocks from 1926 to 1991.

***** SCA Output from running ex19.tsk *****

```
call allmacro.file is 'ex19.mac'.
--
input z.file'\b325\sp500.dat'
Z       , A  792  BY   1 VARIABLE, IS STORED IN THE WORKSPACE

eacf z

NAME OF THE SERIES . . . . . . . . . .            Z
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   792
MEAN OF THE (DIFFERENCED) SERIES . . .        .0061
STANDARD DEVIATION OF THE SERIES . . .        .0584
T-VALUE OF MEAN (AGAINST ZERO) . . . .       2.9592

THE EXTENDED ACF TABLE

(Q-->)   0    1    2    3    4    5    6    7    8    9   10   11   12
------------------------------------------------------------------------
(P= 0)  .09 -.03 -.13  .02  .07 -.04  .00  .05  .09  .00 -.02 -.00 -.04
(P= 1)  .34 -.02 -.12  .04  .06 -.04  .00  .01  .09  .01 -.02 -.00 -.01
(P= 2) -.26  .48 -.06 -.03  .03  .02  .03 -.00  .08  .02 -.04  .00 -.01
(P= 3)  .31  .42 -.29 -.03  .05 -.01  .03  .04  .08  .00 -.03 -.01 -.01
(P= 4) -.47  .38  .10  .20  .02  .01  .08  .03  .09 -.03  .01 -.02  .01
(P= 5)  .50  .28 -.31 -.03  .14 -.02  .08 -.02  .05  .00 -.03  .00  .02
(P= 6)  .29  .27 -.12 -.06  .21  .34  .02 -.00  .05  .00 -.02  .00  .01

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)  0  1  2  3  4  5  6  7  8  9 10 11 12
-----------------------------------------------
(P= 0)  X  O  X  O  X  O  O  O  X  O  O  O  O
(P= 1)  X  O  X  O  O  O  O  O  X  O  O  O  O
(P= 2)  X  X  O  O  O  O  O  O  O  O  O  O  O
(P= 3)  X  X  X  O  O  O  O  O  O  O  O  O  O
(P= 4)  X  X  O  X  O  O  O  O  O  O  O  O  O
(P= 5)  X  X  X  O  X  O  O  O  O  O  O  O  O
(P= 6)  X  X  X  O  X  X  O  O  O  O  O  O  O

iden z.maxl 12

NAME OF THE SERIES . . . . . . . . . .            Z
TIME PERIOD ANALYZED . . . . . . . . .  1  TO   792
MEAN OF THE (DIFFERENCED) SERIES . . .        .0061
STANDARD DEVIATION OF THE SERIES . . .        .0584
T-VALUE OF MEAN (AGAINST ZERO) . . . .       2.9592
```

```
AUTOCORRELATIONS

  1- 12      .09 -.03 -.13  .02  .07 -.04  .00  .05  .09  .00 -.02 -.00
  ST.E.      .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04
   Q         6.4  7.0 19.8 20.2 24.4 26.0 26.0 27.8 33.9 33.9 34.4 34.4

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .09                         + IXX
  2   -.03                         +XI +
  3   -.13                        X+XI +
  4    .02                         + IX+
  5    .07                         + IXX
  6   -.04                         +XI +
  7    .00                         + I +
  8    .05                         + IX+
  9    .09                         + IXX
 10    .00                         + I +
 11   -.02                         +XI +
 12    .00                         + I +

PARTIAL AUTOCORRELATIONS

  1- 12      .09 -.04 -.12  .04  .06 -.07  .02  .06  .06 -.01  .00  .01
  ST.E.      .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04  .04

          -1.0  -.8  -.6  -.4  -.2   .0   .2   .4   .6   .8  1.0
            +----+----+----+----+----+----+----+----+----+----+
                                     I
  1    .09                         + IXX
  2   -.04                         +XI +
  3   -.12                        X+XI +
  4    .04                         + IX+
  5    .06                         + IXX
  6   -.07                         XXI +
  7    .02                         + IX+
  8    .06                         + IXX
  9    .06                         + IX+
 10   -.01                         + I +
 11    .00                         + I +
 12    .01                         + I +
```

tsm mz1.model (1,2,3)z=c+noise
--
estim mz1.hold resi(r1)

```
SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   MZ1
-----------------------------------------------------------------------
VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
          VARIABLE   OR CENTERED

   Z       RANDOM     ORIGINAL     NONE
-----------------------------------------------------------------------
```

```
 PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-      VALUE       STD      T
   LABEL       NAME    DENOM.                TRAINT                 ERROR  VALUE

  1    C               CNST      1      0     NONE       .0066      .0021   3.17
  2             Z       AR       1      1     NONE       .0882      .0354   2.49
  3             Z       AR       1      2     NONE      -.0233      .0355   -.66
  4             Z       AR       1      3     NONE      -.1230      .0354  -3.48

TOTAL SUM OF SQUARES . . . . . . . .   .270319E+01
TOTAL NUMBER OF OBSERVATIONS . . . .           792
RESIDUAL SUM OF SQUARES. . . . . . .   .263104E+01
R-SQUARE . . . . . . . . . . . . . .          .023
EFFECTIVE NUMBER OF OBSERVATIONS . .           789
RESIDUAL VARIANCE ESTIMATE . . . . .   .333466E-02
RESIDUAL STANDARD ERROR. . . . . . .   .577465E-01
```

eacf r1

```
NAME OF THE SERIES . . . . . . . . . .            R1
TIME PERIOD ANALYZED . . . . . . . . .  4  TO   792
MEAN OF THE (DIFFERENCED) SERIES . . .        .0000
STANDARD DEVIATION OF THE SERIES . . .        .0577
T-VALUE OF MEAN (AGAINST ZERO) . . . .        .0000

THE EXTENDED ACF TABLE

(Q-->)   0    1    2    3    4    5    6    7    8    9   10   11   12
-----------------------------------------------------------------------
(P= 0)  .01  .01 -.01  .04  .08 -.06  .01  .05  .08 -.01 -.03  .01 -.04
(P= 1) -.44  .01  .00  .01  .07 -.05  .01  .01  .09 -.02 -.03 -.01 -.02
(P= 2)  .50 -.19 -.00  .01  .07  .02  .03 -.01  .07  .03 -.05  .01 -.02
(P= 3)  .25 -.08 -.13  .00  .06 -.02  .04  .05  .07  .02 -.03 -.02 -.01
(P= 4) -.39  .23 -.19  .08  .03 -.04  .04  .00  .09 -.03  .00 -.02  .00
(P= 5)  .48  .23 -.22 -.39 -.33 -.02  .04 -.00  .06 -.02 -.01 -.00  .02
(P= 6)  .11  .24 -.10 -.42 -.27 -.19  .01  .03  .07 -.01 -.02 -.01  .01

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)  0  1  2  3  4  5  6  7  8  9 10 11 12
---------------------------------------------
(P= 0)  O  O  O  O  X  O  O  O  X  O  O  O  O
(P= 1)  X  O  O  O  O  O  O  O  X  O  O  O  O
(P= 2)  X  X  O  O  O  O  O  O  O  O  O  O  O
(P= 3)  X  X  X  O  O  O  O  O  O  O  O  O  O
(P= 4)  X  X  X  O  O  O  O  O  O  O  O  O  O
(P= 5)  X  X  X  X  X  O  O  O  O  O  O  O  O
(P= 6)  X  X  X  X  X  X  O  O  O  O  O  O  O
--
```

r2=r1*r1

--

eacf r2

```
NAME OF THE SERIES . . . . . . . . . .            R2
TIME PERIOD ANALYZED . . . . . . . . .  4  TO   792
MEAN OF THE (DIFFERENCED) SERIES . . .        .0033
STANDARD DEVIATION OF THE SERIES . . .        .0104
T-VALUE OF MEAN (AGAINST ZERO) . . . .       9.0382
```

```
THE EXTENDED ACF TABLE

(Q-->)   0    1    2    3    4    5    6    7    8    9   10   11   12
-----------------------------------------------------------------------
(P= 0)  .21  .22  .21  .14  .08  .13  .17  .30  .34  .18  .25  .16  .13
(P= 1) -.50  .01  .05  .03 -.05  .01 -.02  .06  .17 -.12  .13 -.04  .01
(P= 2) -.49  .12  .03  .02 -.07 -.03  .00  .02  .17 -.07  .10  .02 -.01
(P= 3) -.34  .22 -.11  .01 -.06 -.03  .00  .01  .21  .08  .10  .01 -.04
(P= 4)  .26 -.12 -.21 -.07 -.04  .02 -.03  .02  .20 -.12  .15 -.03 -.08
(P= 5)  .22 -.19 -.08 -.20 -.27 -.01 -.01  .01  .22  .08  .17 -.07 -.08
(P= 6) -.42 -.06  .08 -.06 -.22  .09 -.02  .00  .20 -.12  .14  .02 -.07

SIMPLIFIED EXTENDED ACF TABLE (5% LEVEL)

(Q-->)  0  1  2  3  4  5  6  7  8  9 10 11 12
---------------------------------------------
(P= 0)  X  X  X  X  O  X  X  X  X  X  X  X  X
(P= 1)  X  O  O  O  O  O  O  O  X  X  X  O  O
(P= 2)  X  X  O  O  O  O  O  O  X  O  X  O  O
(P= 3)  X  X  X  O  O  O  O  O  X  O  X  O  O
(P= 4)  X  X  X  O  O  O  O  O  X  X  X  O  O
(P= 5)  X  X  X  X  X  O  O  O  X  O  X  O  O
(P= 6)  X  O  X  O  X  O  O  O  X  X  X  O  O
```

tsm mr2.model (1)r2=c+(1)noise

--

estim mr2.method exact.hold resi(rr2)

```
SUMMARY FOR UNIVARIATE TIME SERIES MODEL --   MR2
----------------------------------------------------------------------
VARIABLE   TYPE OF    ORIGINAL     DIFFERENCING
          VARIABLE   OR CENTERED

   R2      RANDOM     ORIGINAL     NONE
----------------------------------------------------------------------
 PARAMETER   VARIABLE  NUM./  FACTOR  ORDER   CONS-     VALUE      STD     T
   LABEL       NAME    DENOM.                 TRAINT              ERROR  VALUE

  1    C               CNST      1      0     NONE      .0001 .6819E-04   1.77
  2             R2      MA       1      1     NONE      .8634     .0284  30.38
  3             R2      AR       1      1     NONE      .9648     .0151  63.94


TOTAL SUM OF SQUARES . . . . . . . .   .847416E-01
TOTAL NUMBER OF OBSERVATIONS . . . .           789
RESIDUAL SUM OF SQUARES. . . . . . .   .736130E-01
R-SQUARE . . . . . . . . . . . . . .          .130
EFFECTIVE NUMBER OF OBSERVATIONS . .           788
RESIDUAL VARIANCE ESTIMATE . . . . .   .934175E-04
RESIDUAL STANDARD ERROR. . . . . . .   .966527E-02
```

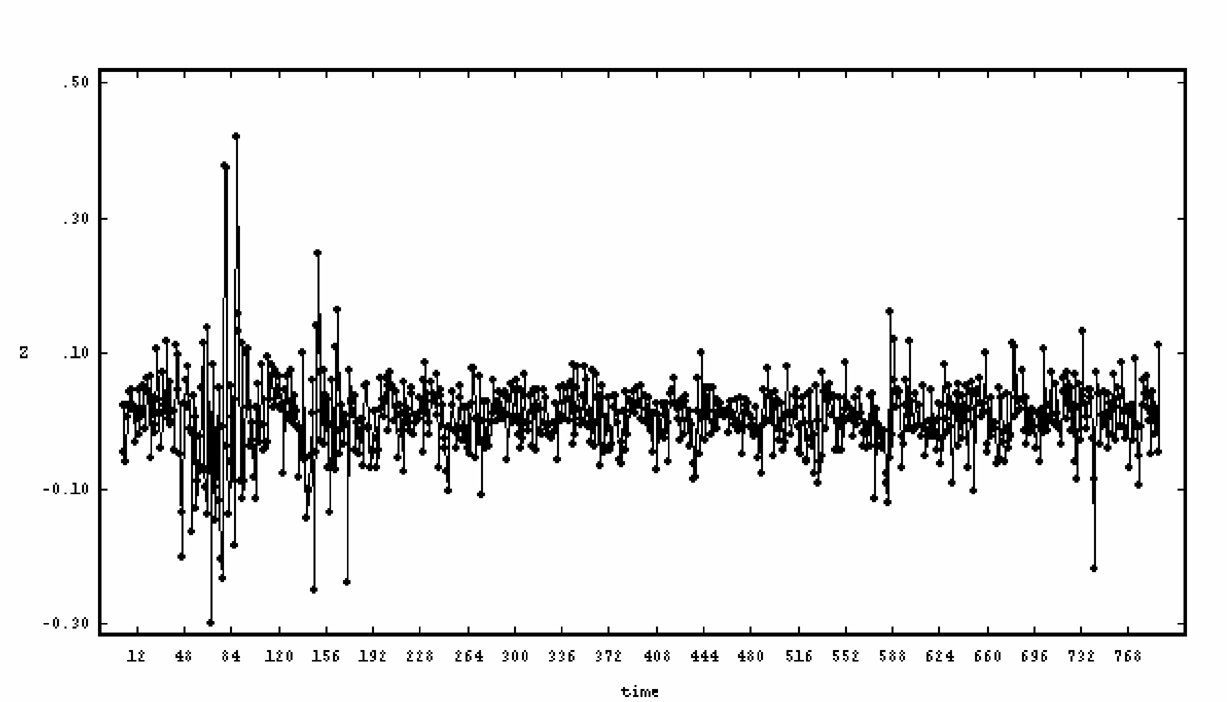
desc z

```
 VARIABLE    NAME    IS         Z
 NUMBER OF OBSERVATIONS       792
 NUMBER OF MISSING VALUES       0

                    STATISTIC      STD. ERROR     STATISTIC/S.E.

 MEAN                   .0061           .0021          2.9573
 VARIANCE               .0034
 STD DEVIATION          .0585
 C.  V.                9.5162
 SKEWNESS               .4106           .0869
 KURTOSIS              9.2692           .1735

                    QUARTILE

 MINIMUM               -.2994
 1ST QUARTILE          -.0208
 MEDIAN                 .0087
 3RD QUARTILE           .0371
 MAXIMUM                .4222

                        RANGE

 MAX - MIN              .7216
 Q3  - Q1               .0579
```

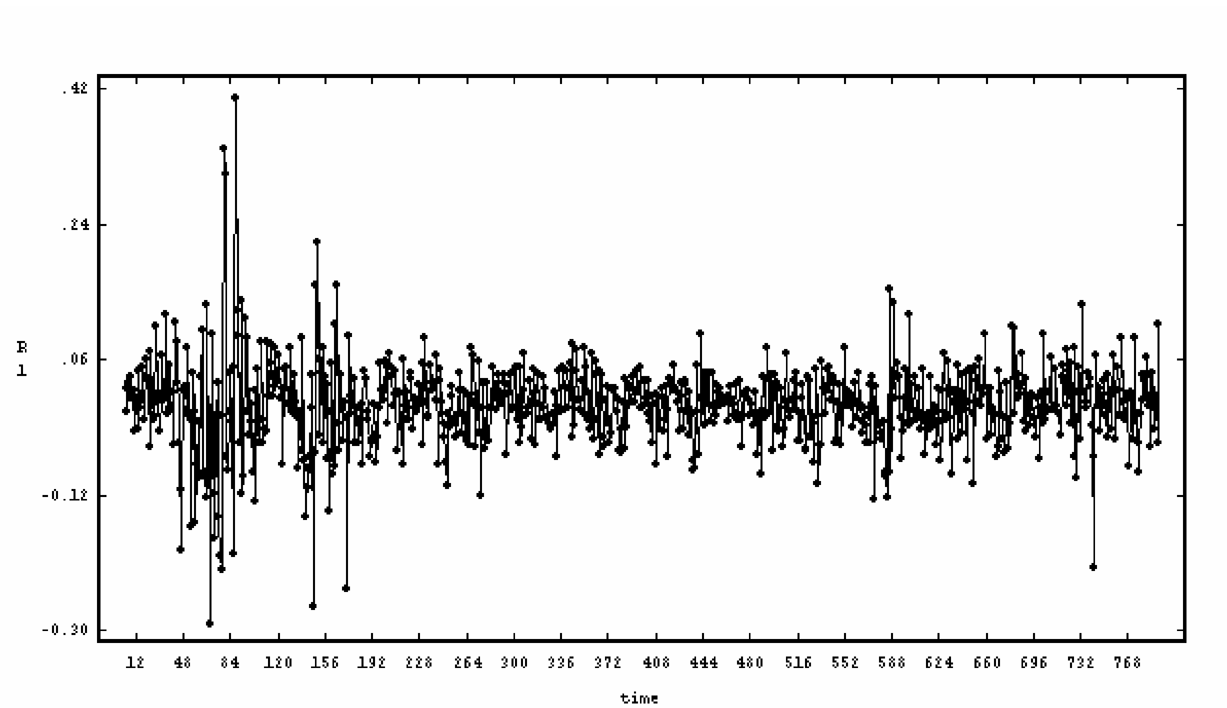
desc r1

```
 VARIABLE    NAME    IS        R1
 NUMBER OF OBSERVATIONS       792
 NUMBER OF MISSING VALUES       3

STATISTIC      STD. ERROR     STATISTIC/S.E.

 MEAN                   .0000           .0021           .0000
 VARIANCE               .0033
 STD DEVIATION          .0578
 C.  V.          3813220.0000
 SKEWNESS               .2411           .0870
 KURTOSIS              7.6317           .1739

                    QUARTILE

 MINIMUM               -.2915
 1ST QUARTILE          -.0264
 MEDIAN                 .0007
 3RD QUARTILE           .0307
 MAXIMUM                .4092

                        RANGE

 MAX - MIN              .7007
 Q3  - Q1               .0572
```

graph z.type tsplot
--



graph r1.type tsplot



We give below a comparison between the estimates obtained from the S-Plus GARCH software and the above results.

Parameter	SCA	S-PLUS
c	.0066	.0078
ϕ_1	.0882	.0327
ϕ_2	-.0233	-.0288
ϕ_3	-.1230	-.0084
α_0	.0001	.0001
α_1	.1014	.1213
α_2	.8634	.8523

Again, the results are close.

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Kurtosis of GARCH and stochastic volatility models with non-normal innovations

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Abstract

Both volatility clustering and conditional non-normality can induce the leptokurtosis typically observed in financial data. In this paper, the exact representation of kurtosis is derived for both GARCH and stochastic volatility models when innovations may be conditionally non-normal. We find that, for both models, the volatility clustering and non-normality contribute interactively and symmetrically to the overall kurtosis of the series.
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Keywords: GARCH; Stochastic volatility; Kurtosis; Mixture normal distribution

1. Introduction

Many financial series, such as returns on stocks and foreign exchange rates, exhibit leptokurtosis and time-varying volatility. These two features have been the subject of extensive studies ever since Mandelbrot (1963) and Fama (1965) first reported them.

The autoregressive conditional heteroscedastic (ARCH) model, Engle (1982), and its generalization, the GARCH model, Bollerslev (1986), provide a convenient framework to study time-varying volatility in financial markets. In practice, a common assumption in applying GARCH models to financial data is that the return series is conditionally normally distributed. We shall refer to this as the normal GARCH model.

It is well known that the normal GARCH model is consistent with volatility clustering patterns typically exhibited in financial and economic time series. However, the

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kurtosis implied by the normal GARCH model tends to be far less than the sample kurtosis observed for most financial return series. For example, Bollerslev (1987) finds evidence of conditional leptokurtosis in monthly S&P 500 Composite Index returns and advocates use of the t-distribution, and Hong (1988) rejects the normality assumption for daily NYSE stock returns. Thus, the normal GARCH model is inconsistent with the large leptokurtosis typically observed in asset returns. Stochastic volatility models have also been advocated for modeling volatility clustering, see Jacquier et al. (1994). Like the normal GARCH model, the conditional normal stochastic volatility model also generates leptokurtosis but often not sufficiently large to explain the sample kurtosis.

This paper derives expressions for the kurtosis of GARCH and stochastic volatility models in the presence of conditionally non-normal leptokurtic innovations. Previously, He and Terasvirta (1999b) examined the forth moment structure of the GARCH(1, 1) model with conditionally non-normal innovations and He and Terasvirta (1999a) extend their results to the GARCH(p, q) model. In both of these papers, the kurtosis is expressed as a function of the underlying model parameters. We take a somewhat different approach in this paper. By working with the well-known ARMA representations of the squares of the error term we are able to extend their results to a broader class of models that include stochastic volatility models. Second, working with the ARMA representation provides an interesting perspective on the interaction between the volatility clustering and the non-normality in the innovation. In particular, we decompose the overall kurtosis into the kurtosis induced by volatility clustering and the kurtosis induced by conditionally non-normal innovations. We find the interesting result that both of these kurtosis enter symmetrically and interactively to determine the overall kurtosis of the process. Additionally, working with the ARMA representations greatly simplifies the derivations.

This paper is organized as follows. We present the results for GARCH model in Sections 2 and 3 with illustrative examples. In Section 4, we study the kurtosis of stochastic volatility models. Section 5 concludes.

2. The kurtosis of GARCH models with non-normal innovations

Consider a GARCH(p, q) model for the time series ε_t,

$$\varepsilon_t = \sqrt{h_t} z_t,$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \tag{1}$$

where h_t is the conditional variance of ε_t given $\mathcal{V}_t = \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$, and z_t are iid with mean 0 and variance 1. The parameters $(\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)$ are restricted such that $h_t > 0$ for all t. We shall assume that the fourth moment of z_t exists. This section derives the kurtosis of ε_t when it exists. In the following, we call the excess kurtosis of z_t the *iid kurtosis* and denote it by $\gamma_2^{(z)}$, and call the excess kurtosis of ε_t the *overall kurtosis* and denote it by $\gamma_2^{(\varepsilon)}$ if it exists.

Rewriting the equation for h_t in (1) with $u_t = \varepsilon_t^2 - h_t$, we have the following well-known ARMA(r,q) representation for ε_t^2:

$$\phi(B)\varepsilon_t^2 = \omega + \beta(B)u_t, \tag{2}$$

where B is the backward shift operator, $\phi(B) = 1 - \sum_{i=1}^{r} \phi_i B^i$, $\phi_i = (\alpha_i + \beta_i)$, $r = \max(p,q)$, and $\beta(B) = 1 - \sum_{i=1}^{q} \beta_i B^i$. It is understood that $\alpha_i = 0$ for $i > p$ and $\beta_i = 0$ for $i > q$. We shall make the following two further assumptions:

(A.1) all the zeroes of the polynomial $\phi(B)$ are lying outside of the unit circle,
(A.2) $0 < (\gamma_2^{(z)} + 2)a_1 < 1$, with $a_1 = \sum_{i=1}^{\infty} \psi_i^2$,

where the $\psi_i's$ are obtained from the relation $\psi(B)\phi(B) = \beta(B)$ with $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$. These two assumptions ensure that the $u_t's$ are uncorrelated with zero mean and finite variance and that the ε_t^2 process is weakly stationary. In this case, the autocorrelation function of ε_t^2 will be exactly the same as that for a stationary ARMA(r,q) model.

If z_t follows a normal distribution, $\gamma_2^{(z)} = 0$ and the process defined by Eq. (1) is the normal GARCH(p,q) process. The excess kurtosis of this normal GARCH process is called the *GARCH kurtosis*, and is denoted by $\gamma_2^{(g)}$ when it exists.

To calculate the GARCH kurtosis $\gamma_2^{(g)}$ and to understand the relationship between the overall kurtosis $\gamma_2^{(\varepsilon)}$ and the volatility clustering and conditional non-normality, we have the following theorem:

Theorem 2.1. *If ε_t follows the GARCH(p,q) process specified by* (1) *and satisfies assumptions* (A.1) *and* (A.2), *then we have*:
(a) $\gamma_2^{(g)} = 6a_1/(1 - 2a_1)$,
(b) $\gamma_2^{(\varepsilon)} = (\gamma_2^{(g)} + \gamma_2^{(z)} + (5/6)\gamma_2^{(g)}\gamma_2^{(z)})\lfloor 1 - (1/6)\gamma_2^{(g)}\gamma_2^{(z)} \rfloor$.

Proof. Assuming $E(h_t)$ and $E(h_t^2)$ exist, the GARCH model in Eq. (1) implies,

$$E(\varepsilon_t^2) = E(h_t), \tag{3}$$

$$E(\varepsilon_t^4) = E(h_t^2 z_t^4) = (\gamma_2^{(z)} + 3)E(h_t^2), \tag{4}$$

$$\operatorname{var}(\varepsilon_t^2) = E(\varepsilon_t^4) - [E(\varepsilon_t^2)]^2 = (\gamma_2^{(z)} + 3)E(h_t^2) - [E(h_t)]^2 \tag{5}$$

and from the definition of u_t,

$$E(u_t) = 0, \tag{6}$$

$$\operatorname{cov}(u_t, u_{t-j}) = E[(\varepsilon_t^2 - h_t)(\varepsilon_{t-j}^2 - h_{t-j})] = 0, \ j > 0, \tag{7}$$

$$\operatorname{var}(u_t) = \operatorname{var}(\varepsilon_t^2 - h_t) = E(\varepsilon_t^4 - h_t^2) = (\gamma_2^{(z)} + 2)E(h_t^2). \tag{8}$$

From the ARMA representation (2), we have, under assumption (A.1),

$$E(\varepsilon_t^2) = E(h_t) = \omega/(1 - \phi_1 - \cdots - \phi_p) \tag{9}$$

and, again assuming that $E(h_t^2)$ exists

$$\operatorname{var}(\varepsilon_t^2) = \operatorname{var}(u_t)(1+a_1) = (\gamma_2^{(z)}+2)E(h_t^2)(1+a_1). \tag{10}$$

Equating the extreme right-hand side of (5) to that of (10) for the two expressions of $\operatorname{var}(\varepsilon_t^2)$, and making use of the expression for $E(h_t)$ in (9), we obtain, under assumption (A.2), the following result for $E(h_t^2)$:

$$E(h_t^2) = \frac{[\omega/(1-\phi_1-\cdots-\phi_p)]^2}{1-(\gamma_2^{(z)}+2)a_1}.$$

Now the overall kurtosis of ε_t is

$$\gamma_2^{(\varepsilon)} = \frac{E(\varepsilon_t^4)}{[E(\varepsilon_t^2)]^2} - 3 = \frac{(\gamma_2^{(z)}+3)E(h_t^2)}{[E(h_t)]^2} - 3$$

$$= \frac{6a_1 + \gamma_2^{(z)}(1+3a_1)}{1-(\gamma_2^{(z)}+2)a_1}. \tag{11}$$

Part (a) of the theorem follows from Eq. (11) by letting $\gamma_2^{(z)} = 0$. Now from part (a) we can express a_1 in terms of $\gamma_2^{(g)}$ as

$$a_1 = \frac{\gamma_2^{(g)}}{2(3+\gamma_2^{(g)})}. \tag{12}$$

Substituting the right-hand side of (12) for a_1 in (11), we obtain part (b) of the theorem.

3. Discussion and examples

We now discuss some interesting issues stemmed from Theorem 2.1. In the theorem, part (a) relates the normal GARCH kurtosis $\gamma_2^{(g)}$ to the GARCH parameters $\alpha's$ and $\beta's$ in (1) characterizing the volatility clustering. Specifically, it confirms the well-known fact that volatility clustering introduces leptokurtosis. For the normal GARCH(1,1) model, the expression reduces to the result in Bollerslev (1986), i.e.,

$$\gamma_2^{(g)} = \frac{6\alpha^2}{1-\phi^2-2\alpha^2}, \tag{13}$$

where $\phi^2 + 2\alpha^2 < 1$, and $\phi = \alpha + \beta$. On the other hand, from (1) and (2), and under conditions (A.1) and (A.2), the autocorrelation function of ε_t^2 for a GARCH(1,1) model is

$$\operatorname{corr}(\varepsilon_t^2, \varepsilon_{t-\ell}^2) = \rho_1 \phi^{\ell-1}, \tag{14}$$

where ρ_1 is the lag-one autocorrelation ($\ell = 1$) and is related to ϕ and α as

$$\rho_1 = \alpha \frac{1-\phi^2+\phi\alpha}{1-\phi^2+\alpha^2} = \alpha\left(1+\alpha\frac{\phi-\alpha}{1-\phi^2+\alpha^2}\right) \tag{15}$$

Thus, ρ_1 determines the lag-one value and ϕ the rate of decay of the autocorrelation function. For financial returns data, the sample autocorrelations are generally positive

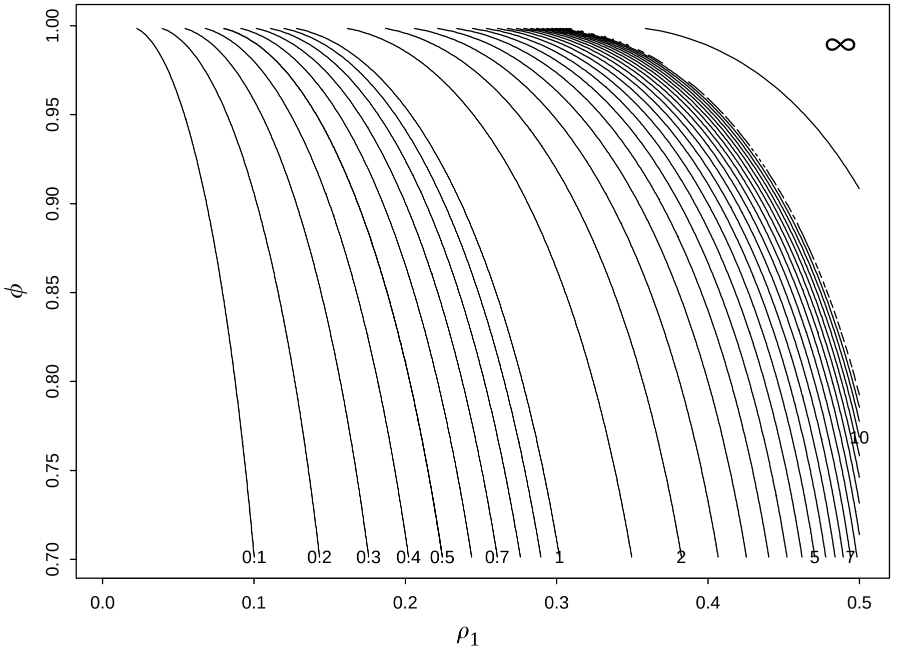

Fig. 1. Contour plot of normal GARCH kurtosis $\gamma_2^{(g)}$.

and persistent, and $\hat{\rho}_1$ is far below 1 with values near 0.2 common. This phenomenon corresponds to the case ρ_1 is about 0.2 and ϕ is large, often close to but less than 1. Using Eq. (15) we can express $\gamma_2^{(g)}$ in Eq. (13) in terms of ρ_1 and ϕ. Fig. 1 shows the contours of $\gamma_2^{(g)}$ in the region $0.00 < \rho_1 < 0.50$ and $0.70 < \phi < 1.00$. Consider the region of $0.10 < \rho_1 < 0.25$ and $0.85 < \phi < 0.99$, in which most financial returns locate. In this region, the implied GARCH kurtosis $\gamma_2^{(g)}$ values between 0.15 and 0.85, which are substantially below the sample excess kurtosis commonly found in the returns data. (See the empirical examples shown below, for instance.) It is well known that under fairly general conditions the parameters of the volatility dynamics can be consistently estimated even when the wrong distribution has been assumed for z_t, see Newey and Steigerwald (1997). Thus, MLE estimates of the normal GARCH(1, 1) model frequently imply autocorrelations of ε_t^2 that match closely the sample autocorrelation function, but is not able to match the large leptokurtosis typically found in the data. This led Bollerslev (1987) to suggest using the t-distribution to match the excessive sample kurtosis, and Engle and Gonzalez-Rivera (1991) proposing a semi-parametric model with non-parametric density for iid innovations z_t. It is simply not possible for the normal GARCH(1, 1) model to match both volatility dynamics and kurtosis. A similar plot and conclusion are given in Terasvirta (1996).

Now, part (b) of Theorem 2.1 shows that the overall kurtosis $\gamma_2^{(\varepsilon)}$ can be expressed as a function of the GARCH kurtosis $\gamma_2^{(g)}$ induced by time-varying volatility, and the iid

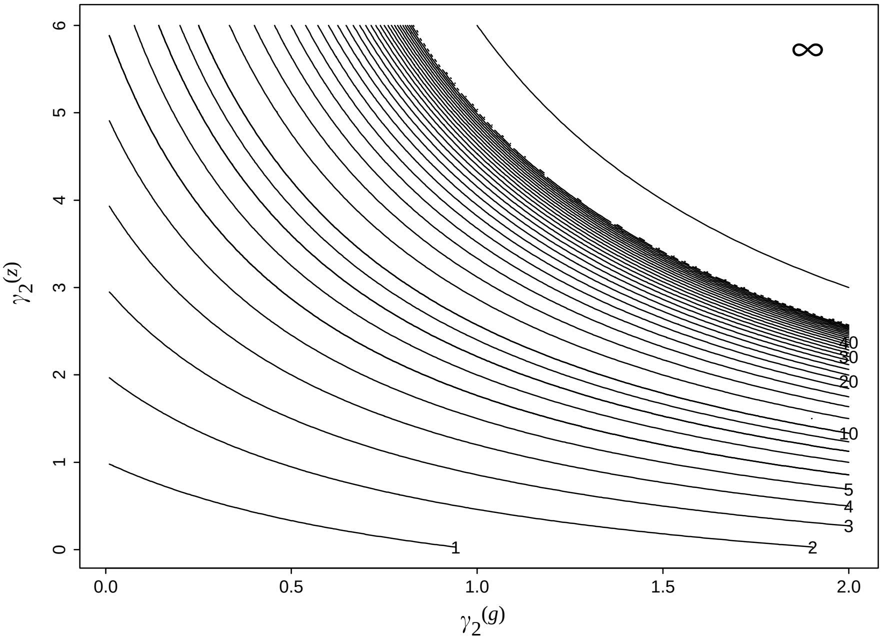

Fig. 2. Contour plot of the overall kurtosis $\gamma_2^{(\varepsilon)}$.

kurtosis $\gamma_2^{(z)}$ of z_t. Specifically, part (b) suggests that $\gamma_2^{(g)}$ and $\gamma_2^{(z)}$ contribute interactively and symmetrically to the increase of the overall kurtosis. To further understand the nonlinear effect, we consider the partial derivative of $\gamma_2^{(\varepsilon)}$ with respect to $\gamma_2^{(g)}$:

$$\frac{\partial \gamma_2^{(\varepsilon)}}{\partial \gamma_2^{(g)}} = \frac{1 + (5/6)\gamma_2^{(z)} + (1/6)[\gamma_2^{(z)}]^2}{[1 - (1/6)\gamma_2^{(g)}\gamma_2^{(z)}]^2}. \tag{16}$$

We first notice from the positivity of the partial derivative $\partial \gamma_2^{(\varepsilon)}/\partial \gamma_2^{(g)}$ that the overall kurtosis $\gamma_2^{(\varepsilon)}$ is increasing with $\gamma_2^{(g)}$. Furthermore, we see that the partial derivative increases with $\gamma_2^{(g)}$ through the denominator but is greatly exacerbated by $\gamma_2^{(z)}$. Similar conclusions can be drawn by considering the partial derivative of $\gamma_2^{(\varepsilon)}$ with respect to $\gamma_2^{(z)}$.

Fig. 2 shows the contours of $\gamma_2^{(\varepsilon)}$ as a function of $\gamma_2^{(g)}$ and $\gamma_2^{(z)}$. From the plot, we see that for each fixed value of $\gamma_2^{(g)}$, leptokurtic departure from normality $\gamma_2^{(z)}$ contributes greatly to the non-linear increase in the overall kurtosis. Typical parameter estimates will produce kurtosis in the upper left corner of the plot. This will be the case for the data we examine below.

To conclude this section, we consider two examples that illustrate the effects of conditional leptokurtosis and volatility clustering on the overall kurtosis. The first series we consider is the IBM daily returns (IBM_t) from January 2, 1990 to December 29,

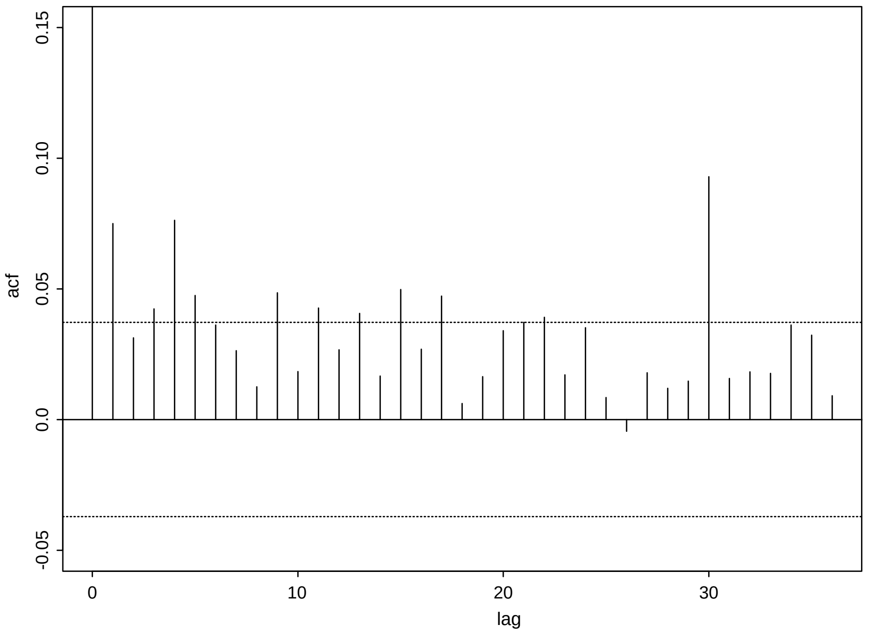

Fig. 3. Plot of ACF function for series IBM_t^2.

2000.[1] This series shows little dynamic dependence in the level, but strong volatility clustering is indicated by the sample autocorrelation function of IBM_t^2 in Fig. 3. The sample excess kurtosis is 6.5306. The second series consists of half-hour foreign exchange rate returns of the Dollar/Deutschemark (DD_t) for the 1 year period January to December, 1996.[2] For simplicity, we concentrate on the volatility clustering and leptokurtosis of DD_t. Here we use the adjusted series $\widetilde{DD}_t$ after accounting for both the deterministic intra-day volatility pattern and an MA(1) structure as in Bai et al. (2001). The sample excess kurtosis for the adjusted return series is 8.1551.

We model these two series first using a normal GARCH(1, 1) model

$$\varepsilon_t = \sqrt{h_t} z_t, \; z_t \sim \text{iid N}(0, 1),$$
$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}, \tag{17}$$

where ε_t can be taken as either the IBM daily returns IBM_t or the adjusted $\widetilde{DD}_t$ series. The maximum likelihood estimation results are given in Table 1.

The values in the parentheses show the estimated standard errors. Diagnostic tests (not provided here for brevity) show no sign of correlation in either the fitted innovation

[1] The IBM daily returns are obtained from CRSP. For estimation convenience, the series are standardized to have unit sample variance.

[2] This data set is obtained from Olsen and Associates. This series shows the usual characteristics of high frequency data, i.e., deterministic intra-day volatility patter and MA(1) structure in the conditional mean. For a detailed description of the data and consequent analysis, see Bai et al. (2001).

Table 1
MLE estimates for normal GARCH(1, 1) model

Series	ω	α	β
IBM	0.0081	0.0350	0.9577
	(0.0016)	(0.0032)	(0.0040)
$\widetilde{\mathrm{DD}}$	0.1655	0.1472	0.6864
	(0.0100)	(0.0066)	(0.0141)

$\hat{z}_t$ and its squares $\hat{z}_t^2$. That is, the volatility clustering for both return series can be captured through the normal GARCH(1, 1) model. However, applying part (a) of Theorem 2.1 to the estimates given in Table 1, we obtain the following implied normal GARCH kurtosis:

$$\gamma_2^{(g)}(\mathrm{IBM}) = 0.6076, \gamma_2^{(g)}(\widetilde{\mathrm{DD}}) = 0.4966,$$

which are much smaller than the sample excess kurtosis of 6.5306 and 8.1551, respectively. It is clear that the normal GARCH(1,1) model is inconsistent with the sample excess kurtosis. The first-order autocorrelation of the squared series implied by the parameter estimates obtained from (15) are 0.1093 and 0.1927 for the IBM and $\widetilde{\mathrm{DD}}$ series, respectively. These values of ρ_1 are inconsistent with the larger sample excess kurtosis as indicated in Fig. 1. To better explain the leptokurtosis observed in the series, we allow for conditional non-normality. Instead of using normal innovations in (17), we assume that z_t are iid and follow a mixture of two normal distributions as in Bai et al. (2001):

$$z_t \sim \text{mixture normal } (\lambda, \eta) = \begin{cases} \mathrm{N}(0, \sigma^2) & \text{with probability } (1-\eta), \\ \mathrm{N}(0, \lambda\sigma^2) & \text{with probability } \eta, \end{cases} \tag{18}$$

where $\sigma^2 = (1 - \eta + \lambda\eta)^{-1}$ so that $\mathrm{var}(z_t) = 1$. For the mixture normal (λ, η) model, the iid kurtosis is

$$\gamma_2^{(z)} = \frac{3\eta(1-\eta)(\lambda-1)^2}{(1-\eta+\lambda\eta)^2}. \tag{19}$$

The resulting GARCH(1, 1) model fits the returns data very well in both cases. Table 2 gives the estimation results.

Applying Eq. (19) and part (b) of Theorem 2.1 to the above estimates, we obtain the estimates of the components and overall kurtosis for the two return series presented in Table 3.

For both series, we notice that the non-normality kurtosis is much larger than the kurtosis induced by volatility clustering and that the overall kurtosis is about twice the sum of the component kurtosis. Thus, once we allow for conditional non-normality, the implied overall kurtosis for IBM daily returns and $\widetilde{\mathrm{DD}}$ exchange rate returns are close to the sample excess kurtosis, indicating that a mixture of normal GARCH(1, 1) model can explain the overall leptokurtosis in addition to volatility clustering.

Table 2
MLE estimates for mixture normal GARCH(1, 1) model

Series	ω	α	β	η	λ
IBM	0.0068	0.0267	0.9656	0.0777	7.2077
	(0.0023)	(0.0047)	(0.0060)	(0.0159)	(0.7621)
$\widetilde{\text{DD}}$	0.1328	0.1296	0.7372	0.1163	6.9950
	(0.0157)	(0.0125)	(0.0230)	(0.0103)	(0.3159)

Table 3
Estimates of kurtosis components for mixture normal GARCH model

Series	Volatility clustering $\gamma_2^{(g)}$	Non-normality $\gamma_2^{(z)}$	Overall $\gamma_2^{(\varepsilon)}$	Sample excess kurtosis
IBM	0.3074	3.7703	6.2510	6.53
$\widetilde{\text{DD}}$	0.4686	3.8469	8.3163	8.16

Table 4
Estimates of kurtosis components for t-distribution GARCH model

Series	Volatility clustering $\gamma_2^{(g)}$	Non-normality $\gamma_2^{(z)}$	Overall $\gamma_2^{(\varepsilon)}$	Sample excess kurtosis
IBM	0.3333	6.5540	13.6937	6.53
$\widetilde{\text{DD}}$	0.5403	16.2657	∞	8.16

We remark here that, for these two data sets, if instead of the mixture normal (λ, η) model we use the t-distribution for the innovation z_t, then the corresponding maximum likelihood estimates for the volatility clustering kurtosis $\gamma_2^{(g)}$ would be fairly close to, but parameter estimates for the degrees of freedom associated with the t-distribution imply that the non-normality kurtosis $\gamma_2^{(z)}$ would far exceed the previous estimates, as shown in Table 4. The above results show that, for both examples, the implied overall kurtosis $\gamma_2^{(\varepsilon)}$are far larger than the sample excess kurtosis, implying that the normal mixture model gives a better fit to the data.

4. The Kurtosis of stochastic volatility models

An alternative model consistent with volatility clustering and excess kurtosis is the stochastic volatility model. The application of this type of models to financial data has accelerated due to their pervasive use in theoretical finance and recent advances in estimation techniques, see for example, Kim et al. (1998). In this section, we extend the results in Section 2 on the GARCH(p, q) model to the following stochastic volatility

(SV) model for the observable time series ε_t,

$$\varepsilon_t = \sqrt{h_t} z_t,$$

$$h_t = f(\{u_s\colon s = t, t-1, \ldots\}), \tag{20}$$

where $(z_t, u_t)'$ is a bivariate unobservable time series, $z_t's$ are iid with mean 0 and unit variance, and $u_t's$ are iid and independent of the $z_t's$. Similar to the GARCH process, h_t can be regarded as the variance of ε_t conditional on the information set $\{u_t, u_{t-1}, u_{t-2}, \ldots\}$. We shall assume that the fourth moment of z_t exists. In this section, we derive the kurtosis of ε_t when it exists. Following the convention in Section 2, $\gamma_2^{(z)}$ denotes the iid kurtosis of z_t, and $\gamma_2^{(\varepsilon)}$ the overall kurtosis of ε_t. If z_t follows a normal distribution, the process defined by Eq. (20) is called a conditional normal SV process, or simply a normal SV process. In the following, we call the excess kurtosis for the normal SV process the SV kurtosis and denote it by $\gamma_2^{(\mathrm{sv})}$ when it exists. The following theorem relates the overall kurtosis $\gamma_2^{(\varepsilon)}$ of the process to the iid kurtosis $\gamma_2^{(z)}$ and SV kurtosis $\gamma_2^{(\mathrm{sv})}$.

Theorem 4.1. *If ε_t follows the process defined by Eq.* (20), *then the overall kurtosis $\gamma_2^{(\varepsilon)}$ exists when both the iid kurtosis $\gamma_2^{(z)}$ and SV kurtosis $\gamma_2^{(\mathrm{sv})}$ exist. Furthermore, we have*

$$\gamma_2^{(\varepsilon)} = \gamma_2^{(z)} + \gamma_2^{(\mathrm{sv})} + \tfrac{1}{3}\gamma_2^{(z)}\gamma_2^{(\mathrm{sv})}.$$

Proof. The SV model in Eq. (20) implies that $E(\varepsilon_t) = 0$, and under the assumption on the indepence between $\{z_t\}$ and $\{u_t\}$,

$$E(\varepsilon_t^2) = E(h_t), \tag{21}$$

$$E(\varepsilon_t^4) = (\gamma_2^{(z)} + 3)E(h_t^2). \tag{22}$$

Thus,

$$\gamma_2^{(\varepsilon)} = (\gamma_2^{(z)} + 3)\frac{E(\varepsilon_t^4)}{[E(\varepsilon_t^2)]^2} - 3 = (\gamma_2^{(z)} + 3)\frac{E(h_t^2)}{[E(h_t)]^2} - 3. \tag{23}$$

Letting $\gamma_2^{(z)} = 0$, we find that

$$\gamma_2^{(\mathrm{sv})} = 3\frac{E(h_t^2)}{[E(h_t)]^2} - 3, \tag{24}$$

$$\frac{E(h_t^2)}{[E(h_t)]^2} = \frac{1}{3}\gamma_2^{(\mathrm{sv})} + 1. \tag{25}$$

Substituting $E(h_t^2)/[E(h_t)]^2$ by $(\frac{1}{3}\gamma_2^{(\mathrm{sv})} + 1)$ in Eq. (23), we obtain the theorem.

As part (b) of Theorem 2.1 for the GARCH process, Theorem 4.1 relates the overall kurtosis $\gamma_2^{(\varepsilon)}$ of an SV process to the volatility clustering property through the normal SV kurtosis $\gamma_2^{(\mathrm{sv})}$, and to the conditional non-normality property through the iid kurtosis $\gamma_2^{(z)}$. Interestingly, although SV and GARCH processes have different data generating mechanisms, their overall kurtosis are determined symmetrically and interactively by two component kurtosis one of which represents the conditional non-normality property and the other the volatility clustering property. It is also interesting to note that, for both models, the overall kurtosis is determined solely by the two component kurtosis. Similar to the GARCH case, Theorem 4.1 says that $\gamma_2^{(\mathrm{sv})}$ and $\gamma_2^{(z)}$ have a non-linear combined effect on the overall kurtosis $\gamma_2^{(\varepsilon)}$. However, unlike the previous case, when one of the $\gamma_2^{(\mathrm{sv})}$ and $\gamma_2^{(z)}$ is fixed, the overall kurtosis $\gamma_2^{(\varepsilon)}$ is a linear function of the other.

Finally, notice that Theorem 4.1 only requires that the associated SV kurtosis $\gamma_2^{(\mathrm{sv})}$ exists, and does not depend on specific volatility generating process $f(\{u_s\colon s=t,t-1,\cdots\})$. In particular, it readily applies to the exponential SV model where $\log(h_t)$ follows an ARMA model. In Jacquier et al. (1994), they have considered the model

$$\begin{aligned}\varepsilon_t &= \sqrt{h_t} z_t,\\ \log(h_t) &= \alpha + \delta \log(h_{t-1}) + u_t,\end{aligned} \tag{26}$$

where the $z_t's$ are iid $N(0,1)$ and the $u_t's$ are iid $N(0,\sigma_u^2)$, and have shown that the SV kurtosis is $\gamma_2^{(\mathrm{sv})} = 3(e^{\sigma_u^2/(1-\delta^2)} - 1)$. Extending their result, a closed-form solution for the overall kurtosis $\gamma_2^{(\varepsilon)}$ when $\log(h_t)$ follows a normal ARMA(p,q) model and z_t has finite fourth moment in closed form is readily obtained from Theorem 4.1.

5. Conclusion

This paper examines the kurtosis of time-series models which exhibits both volatility clustering and conditional non-normality. For the GARCH model and the stochastic volatility model, we find that the overall kurtosis of the process is a non-linear symmetric function of the kurtosis due to volatility clustering and that due to conditional non-normality. In other words, volatility clustering and conditional non-normality contribute symmetrically and non-linearly to the overall kurtosis. For the classical normal GARCH model, we find that although the typical parameters can fit the volatility clustering pattern very well, the implied kurtosis is far less than the sample kurtosis observed in financial data. By allowing for non-normality in the innovations, a non-normal GARCH model not only can fit the dynamic relation well but also can give kurtosis estimate much closer to the sample kurtosis. Through the two examples in Section 3, we have demonstrated that a conditional non-normal GARCH model can capture both the volatility clustering and the exceedingly large kurtosis typically observed in the financial series.

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Some Reflections on Decomposition and Seasonal Adjustment of Economic Time Series

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Useful References

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Issues Involved with the Seasonal Adjustment of Time Series

A. Maravall (in Handbook of Applied Economics, 1993) TRAMO/SEATS

Unobserved Components in Economic Time Series

Findley, Monsell, Bell, Otto, Chen (JBES, 1998)

New Capabilities and Methods of the X-12-ARIMA Seasonal Adjustment program

Outline of topics:

- Additive decomposition

  Deterministic components—Regression
  Random components
  Symmetric moving filters

- A model for X-11
- Model-based decomposition—pros and cons
- Canonical decomposition
- X-12-ARIMA vs. TRAMO/SEATS
- Forecasting components
- Variances of errors of component forcasts
- Unifying X-12-ARIMA and model-based decomposition

Seasonal Adjustment

$\{Z_t\}$ observable time series

(for to-day, t = month)

Additive decomposition

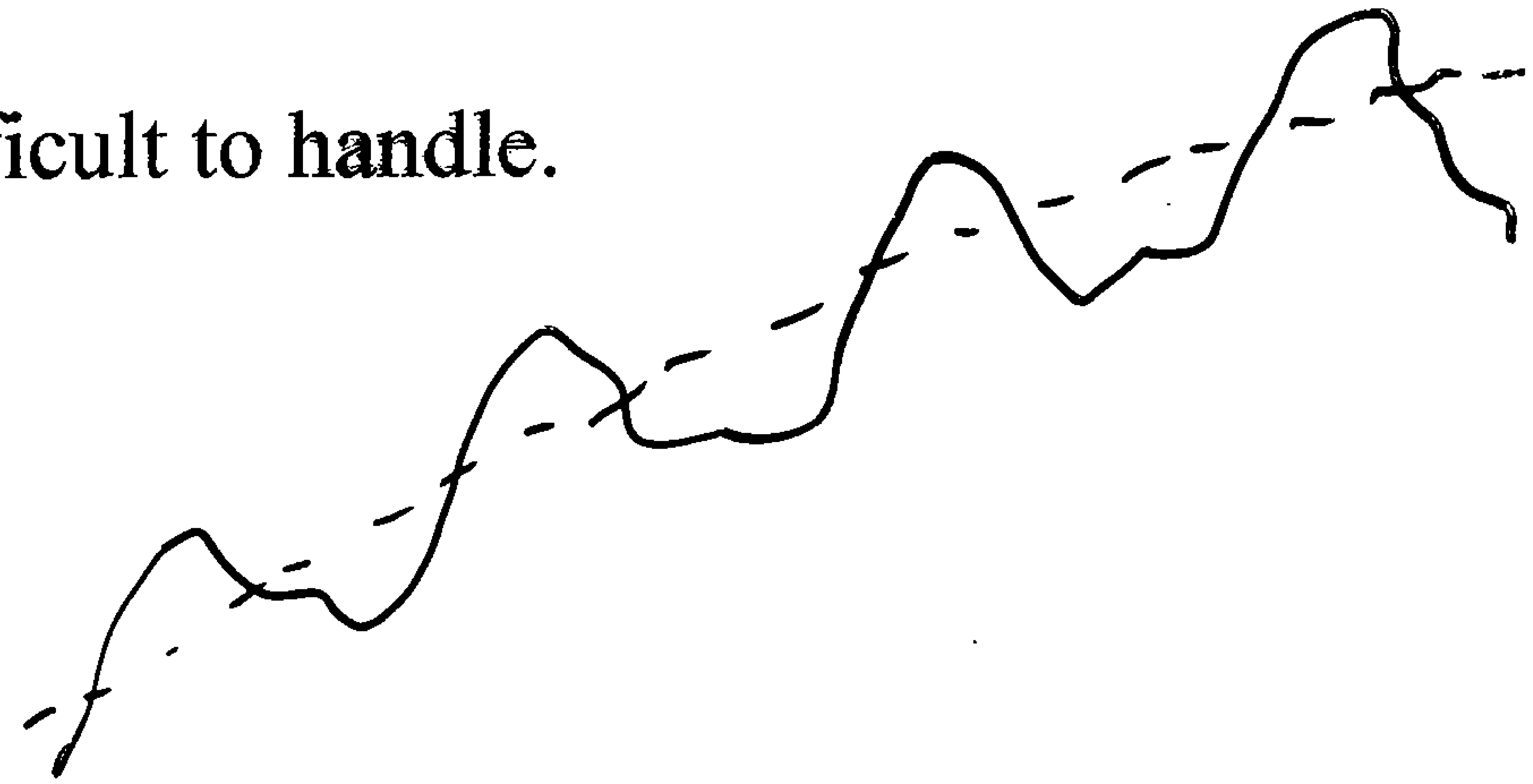
$$Z_t = D_t + R_t$$

D_t: deterministic components including
trading day, day of week, holiday, interventions, missing observations, outliers, level shifts and other special effects

→ R_t: random components including trend, seasonality, cycle and irregular

The deterministic components are typically taken care of via linear and nonlinear regression.

The random components are, however, more difficult to handle.



$$Z_t = D_t + R_t$$

$$R_t = T_t + S_t + N_t$$

T_t: *stochastic trend*

S_t: *stochastic seasonal*

N_t: *random noise*

(unobserved component models, state space models, structural models)

The question is: how to estimate the seasonal component S_t and subtract it from the data Z_t to get the 'seasonally adjusted' series.

U.S. Bureau of the Census has a Computer program, called X-11, to do the seasonal adjustment

$$Z_t = T_t + N_t$$

Signal noise

Symmetric filter

$$M: \quad \delta_0 + \delta_1(B+F) + \delta_2(B^2+F^2) + \cdots$$

where B is the back-shift operator and $F=B^{-1}$ the forward-shift operator.

$$MZ_t = \delta_0 Z_t + \delta_1(Z_{t-1} + Z_{t+1}) + \delta_2(Z_{t-2} + Z_{t+2}) + \cdots$$

In X-11, specific filters are used (with options) to estimate S_t and T_t:

$$\hat{S}_t = M^{(s)} Z_t \quad \textit{and} \quad \hat{T}_t = M^{(T)} Z_t$$

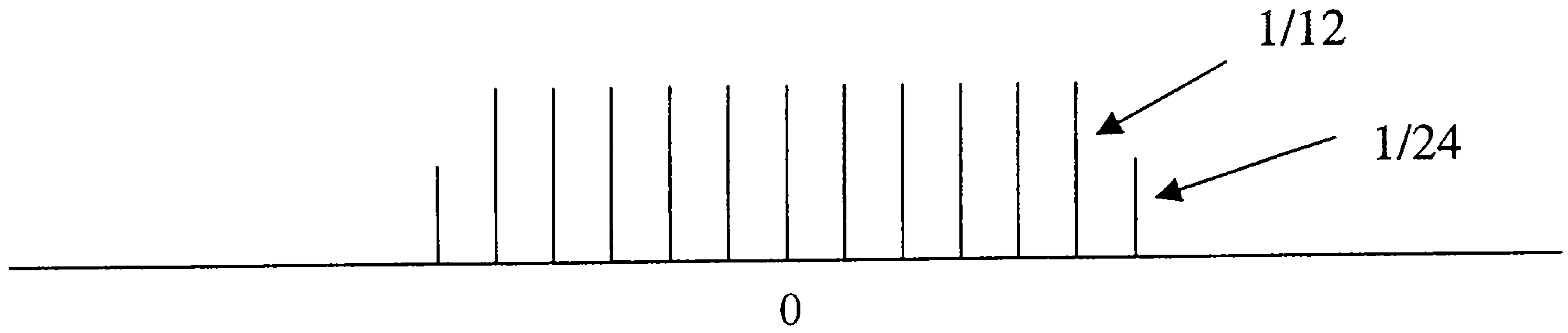
empirically developed

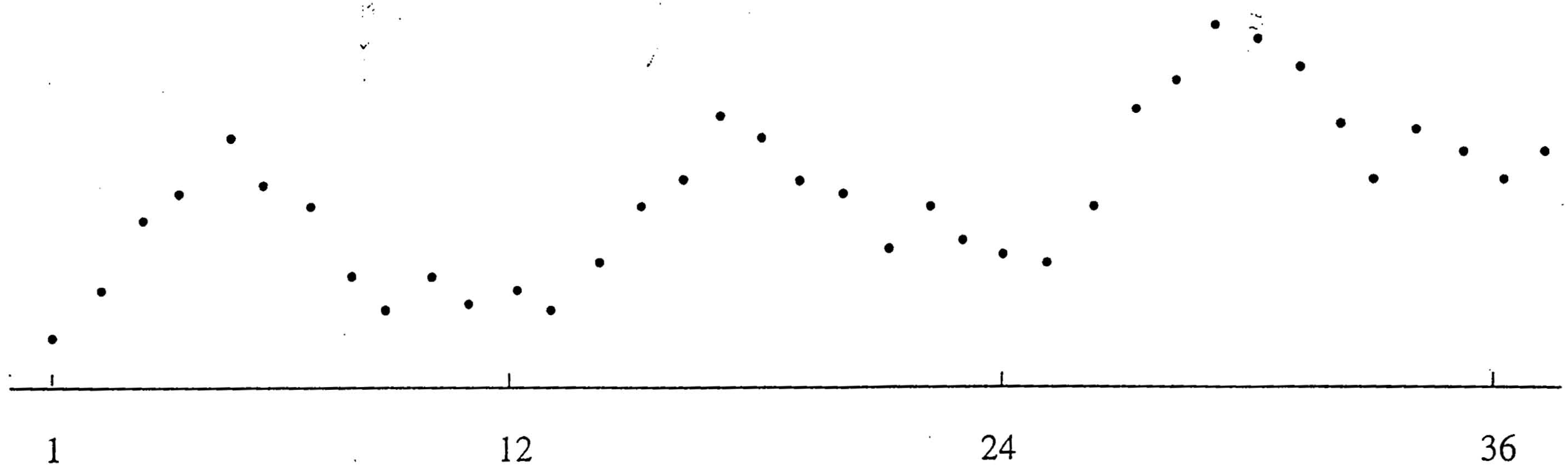
with considerable successes.

Z_t

Census Filters

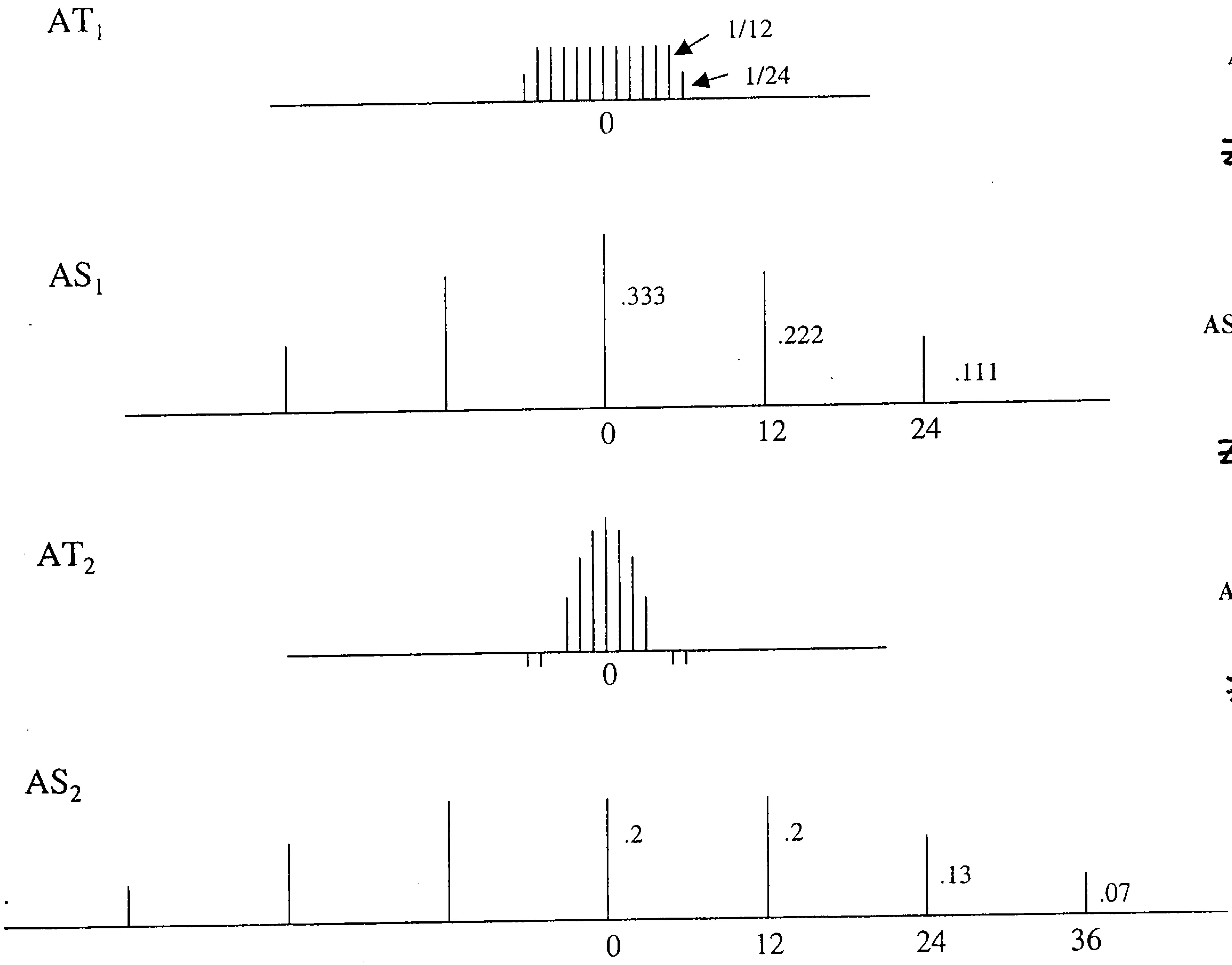
$AT_1 Z_t$ = preliminary trend





1
12
24
36

Filters



Census Operators

$AT_1 Z_t = \tilde{T}_t$ prelim. trend

$Z_t - \tilde{T}_t$ prelim. detrended series

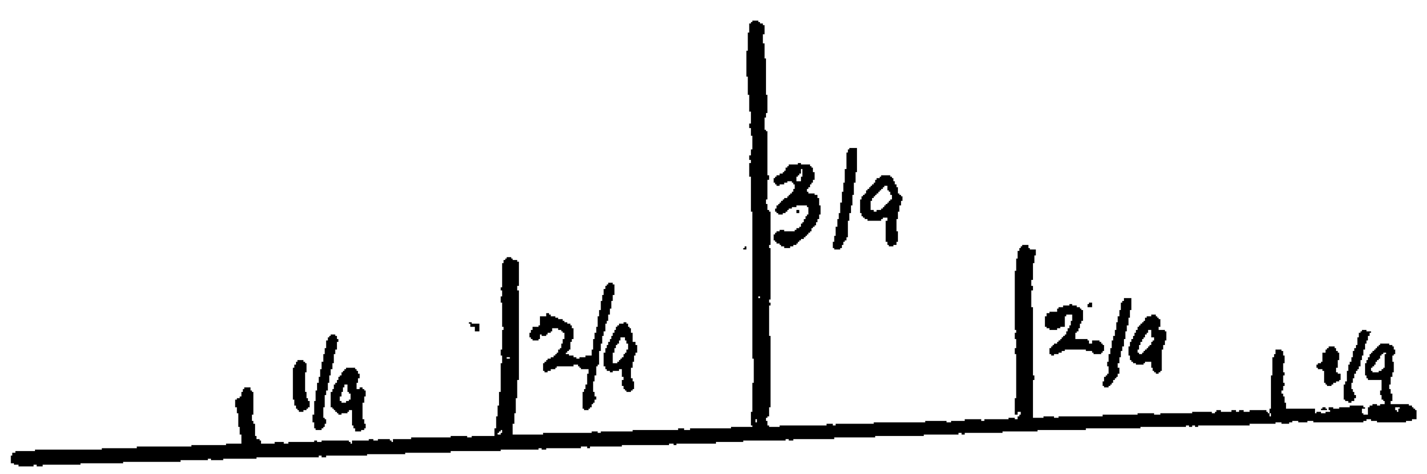
$AS_1(Z_t - \tilde{T}_t) = \tilde{S}_t$ prelim. seasonal factor

$Z_t - \tilde{S}_t$ prelim. deseasonalized series

$AT_2(Z_t - \tilde{S}_t) = \hat{T}_t$ trend factor

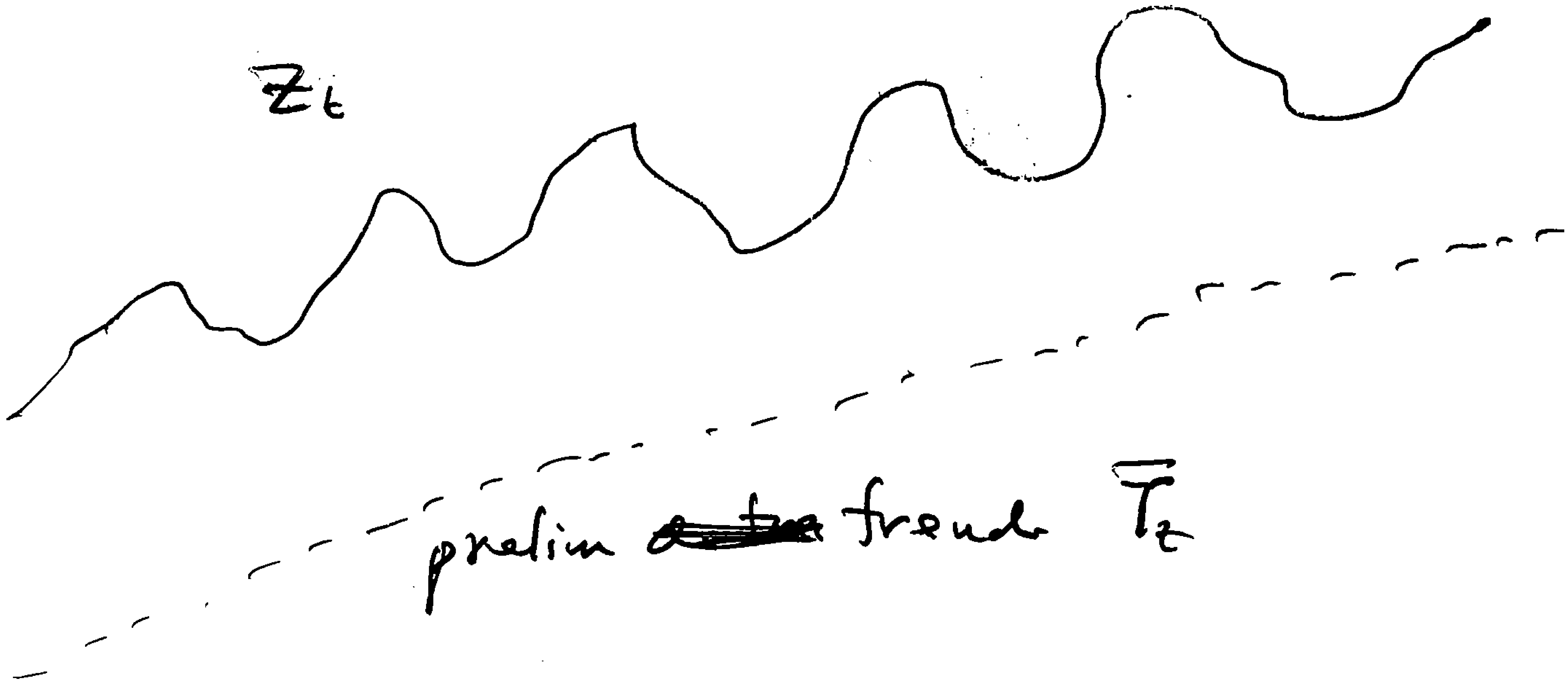
$Z_t - \hat{T}_t$ detrended series

$AS_2(Z_t - \hat{T}_t) = \hat{S}_t$ seasonal factor



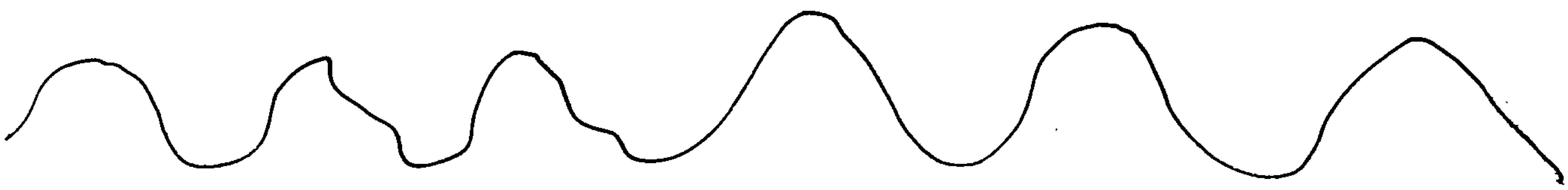
X

Z_t

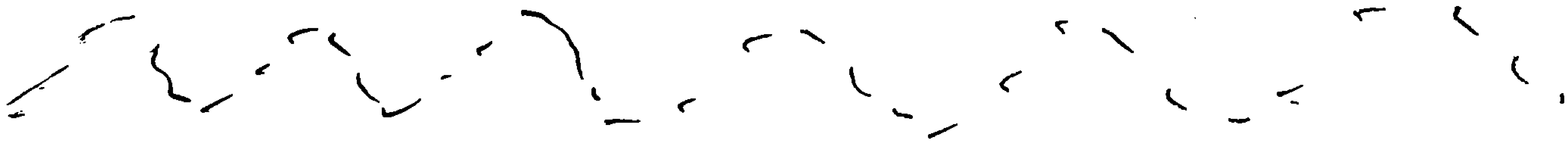


prelim ~~...~~ trend $\bar{T}_t$

$Z_t - \bar{T}_t$

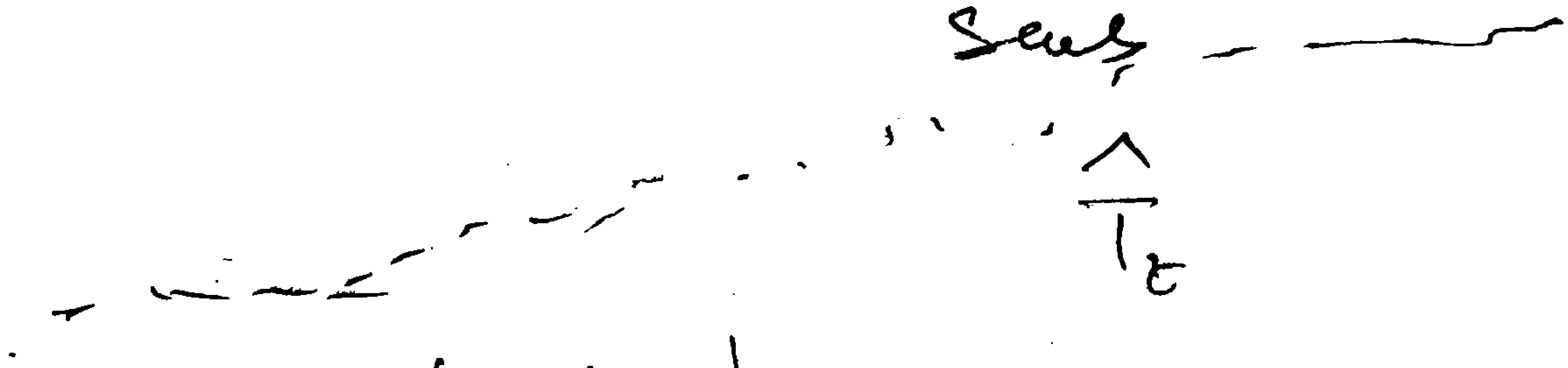


preliminary detrended series.



preliminary seasonal $\tilde{S}_t$

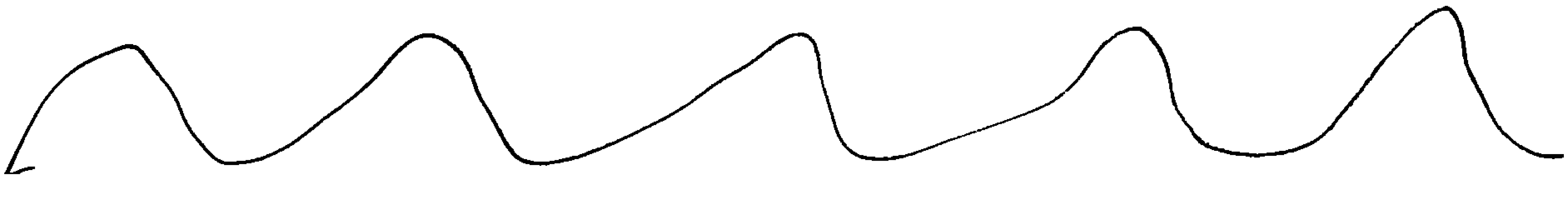
$Z_t - \tilde{S}_t$ prelim. deseasonalized series



$\hat{T}_t$

estimate trend

$Z_t - \hat{T}_t$ detrended series



$$Z_t = T_t + S_t + N_t$$ (back to p. 5 bottom)

A model for X-11

In joint work with Cleveland (1976 JASA), we postulated a seasonal ARIMA model for S_t , a regular ARIMA model for T_t and white noise for N_t:

$$(1-B)^2 T_t = (1-\eta_1 B - \eta_2 B^2) b_t$$
$$(1-B^{12}) S_t = (1-\delta_1 B^{12} - \delta_2 B^{24}) c_t$$
$$N_t = \xi_t$$

(b_t, c_t, ξ_t) independent Gaussian noise with variances $(\sigma_b^2, \sigma_c^2, \sigma_\xi^2)$

$B \equiv L$

Backshift or lag operator

and found that, for specific choice of the parameters,

$$(\eta_1 = -.49,\ \eta_2 = .49);\quad (\delta_1 = -.64,\ \delta_2 = -.83)$$
$$\sigma_c^2 / \sigma_b^2 = 1.3,\quad \sigma_\xi^2 / \sigma_b^2 = 14.4$$

go to Notes 189-192 in unobserved [illegible] (component models

the conditional expectations

$$\tilde{T}_t = E(T_t | Z) \cong \hat{T}_t \quad and \quad \tilde{S}_t = E(S_t | Z) \cong \hat{S}_t$$

Thus, we have an approximate unobserved component model for the X-11 filters

From this unobserved component model, the overall (or marginal) model for Z_t is
(Reduced form)

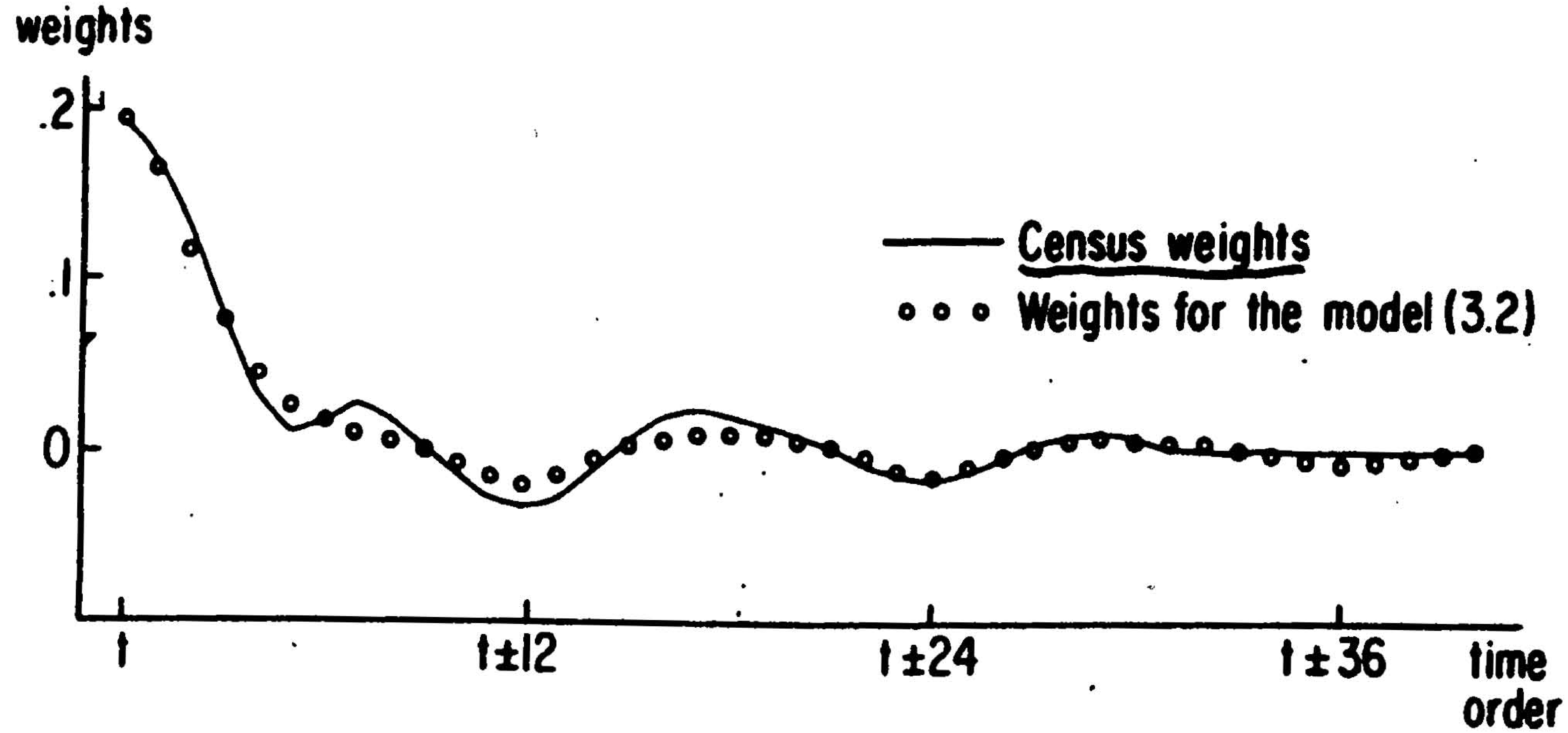
$$
\begin{aligned}
(1-B)(1-B^{12})Z_t = (1 &- .34B + .14B^2 + .14B^3 + .14B^4 + .14B^5 \\
&+ .13B^6 + .13B^7 + .12B^8 + .11B^9 + .09B^{10} \\
&+ .08B^{11} - .42B^{12} + .23B^{13} + .04B^{24} - .02B^{25})a_t
\end{aligned}
$$

which is 'close' to the model for the famous 'airline data' of Box and Jenkins

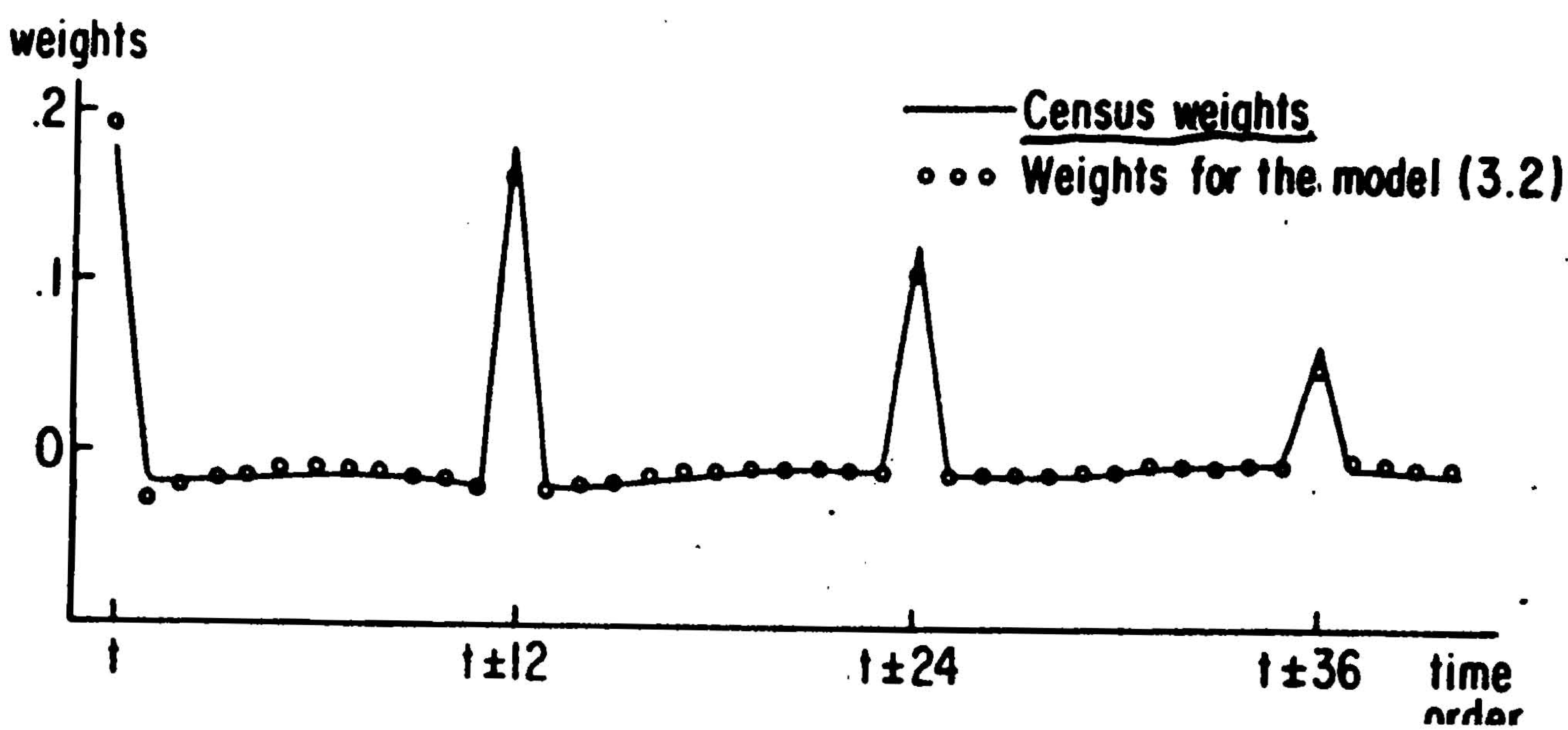
$$(1-B)(1-B^{12})Z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

○○○ From JASA

A. Weight Functions for the Trend Component



B. Weight Functions for the Seasonal Component



7-a

S. Hillmer (late 70's) $Z_t = T_t + S_t + N_t$

This got us into the model-based approach for decomposition:

Based on the overall model built for the data Z_t, back out models for (T_t, S_t, N_t) and estimate the unobserved components accordingly

Some advantages of the model-based approach:

- Make assumptions explicit
- Can make proper inferences about the components
- At least partially verifiable

On the other hand, like many unobserved linear component models in the literature, we run into the problem of

identifiability

Reduced form

i.e. more than one choice of component models leading to the same overall model

272

Simple Example

$$Z_t = T_t + N_t$$

$$(1-B)T_t = b_t \qquad b_t \text{ iid } N(0, \sigma_b^2)$$

$$N_t \quad \text{iid } N(0, \sigma_N^2)$$

$$\Rightarrow \quad (1-B)Z_t = b_t + (1-B)N_t$$

$$= a_t - \theta a_{t-1}$$

but if

$$(1-B)T_t = (1-\eta B)b_t$$

Then

$$(1-B)Z_t = (1-\eta B)b_t + (1-B)N_t$$

$$= a_t - \theta a_{t-1}$$

$\therefore$ given (θ, σ_a^2) infinite choices of $(\eta, \sigma_b^2, \sigma_N^2)$

8x

Canonical decomposition (joint work with Hillmer, *Biometrika* 1978)

$$Z_t = T_t + N_t$$

where N_t is Gaussian white noise independent of T_t. Consider the spectrum

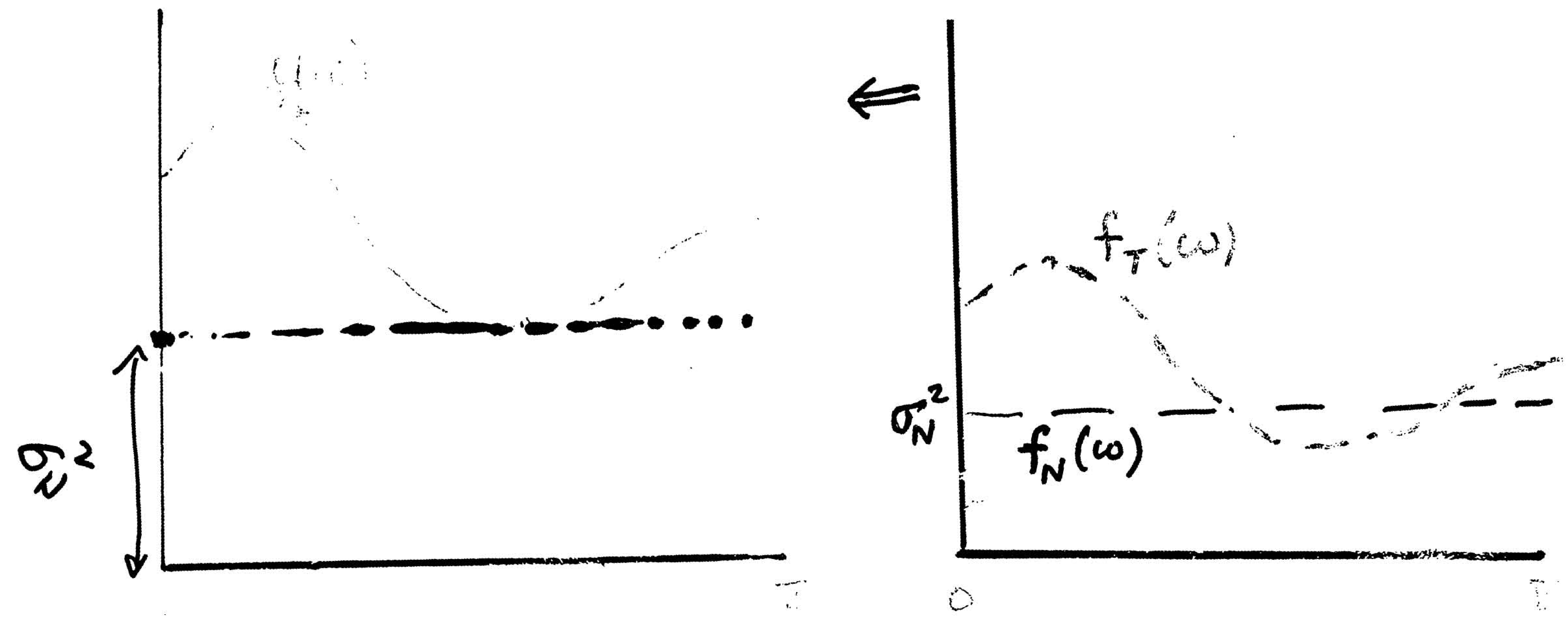
$$f_Z(\omega) = f_T(\omega) + \sigma_N^2$$

If the models for T_t and N_t are known, we can derive the corresponding overall model for Z_t. On the other hand, if only the overall model for Z_t is known, any choice of σ_N^2 in the range

$$0 \le \sigma_N^2 \le \overline{\sigma}_N^2$$

where $\overline{\sigma}_N^2 = \min_\omega f_Z(\omega)$ gives an acceptable decomposition, since $f_T(\omega) = f_Z(\omega) - \sigma_N^2 \ge 0$

$$Z_t = T_t + N_t$$



Overall

Components

The choice

$$\bar{\sigma}_N^2 = \min_\omega f_Z(\omega), \quad \bar{f}_T(\omega) = f_Z(\omega) - \bar{\sigma}_N^2$$

and this resulting $Z_t = \bar{T}_t + \bar{N}_t$ decomposition is called the canonical decomposition.

U. S. Inflation Rate of CPI 1/64 to 12/72 (Box, Hillmer & Tiao, 1978):

$$(1-B)Z_t = (1-\theta B)a_t$$
$$\theta = .84; \quad \sigma_a^2 = .0019$$

$\boxed{\text{If } Z_t = T_t + N_t}$

This implies that the model for T_t must be of the form

$$(1-B)T_t = (1-\eta B)b_t \qquad -1 < \eta < .84$$

and $$(1-\theta B)(1-\theta F)\sigma_a^2 = (1-\eta B)(1-\eta F)\sigma_b^2 + (1-B)(1-F)\sigma_N^2$$

Any choice of $(\eta, \sigma_b^2, \sigma_N^2)$ satisfying the above is an acceptable decomposition.

The canonical decomposition corresponds to

$$\bar{\sigma}_N^2 = \frac{(1+\theta)^2}{4}\sigma_a^2, \quad \bar{\eta} = -1, \quad \bar{\sigma}_b^2 = \frac{(1-\theta)^2}{4}\sigma_a^2$$

The estimate of T_t is

$$\tilde{T}_t = \left(1 - \alpha\frac{(1-B)(1-F)}{(1-\theta B)(1-\theta F)}\right)Z_t, \quad \alpha = \frac{\sigma_N^2}{\sigma_a^2}$$

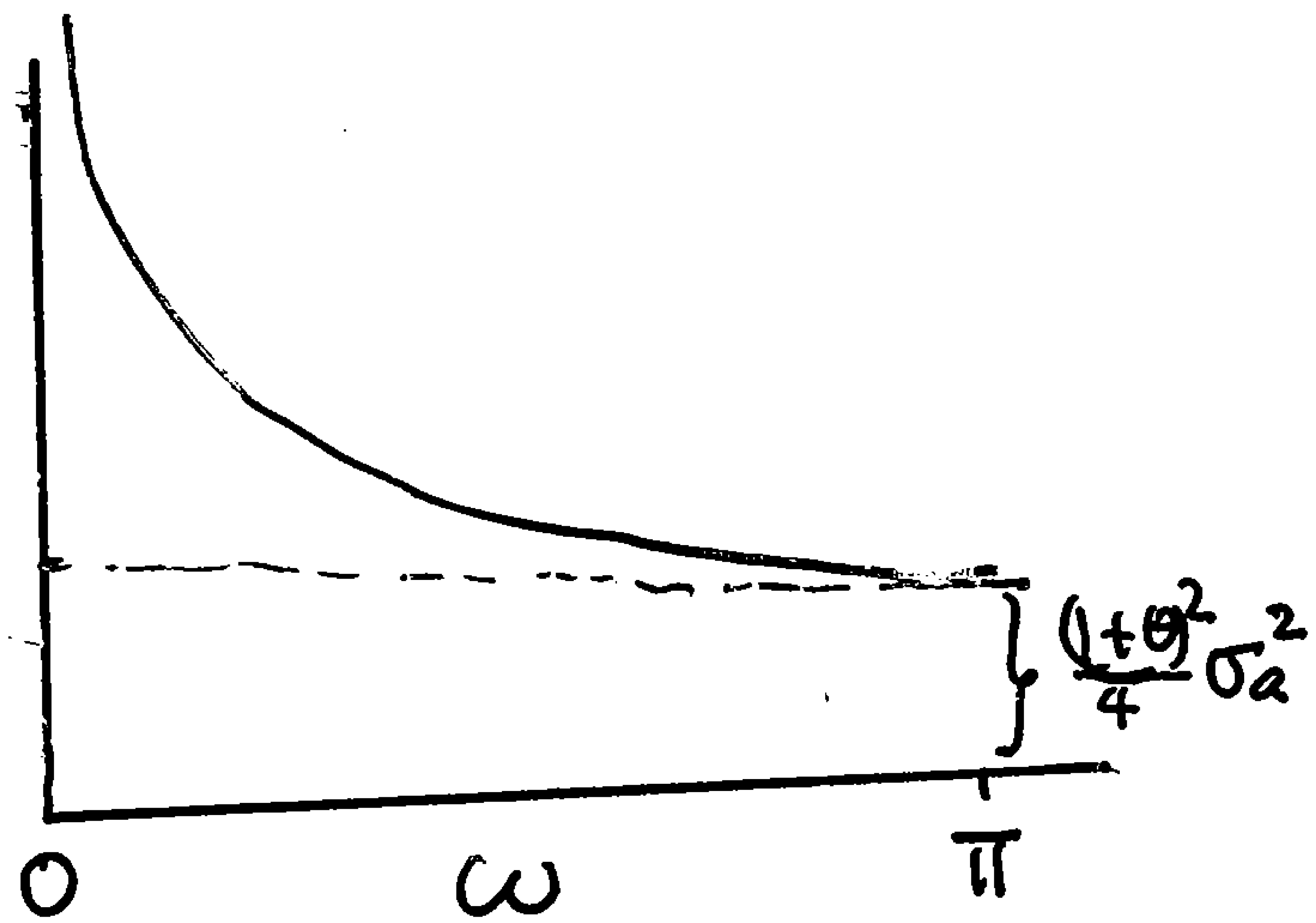
Symmetric Filter.

The canonical decomposition corresponds to $\alpha = \dfrac{(1+\theta)^2}{4} = .846$

e.g. $$(1-B)z_t = (1-\theta B)a_t$$

$$f_z(\omega) = \frac{(1-\theta B)(1-\theta F)}{(1-B)(1-F)}\sigma_a^2$$

$$B = e^{-i\omega}$$



$$\bar{\sigma}_N^2 = \frac{(1+\theta)^2}{4}\sigma_a^2$$

Canonical decomposition

$$\Rightarrow \quad \bar{f}_T(\omega) = \frac{(1+B)(1+F)}{(1-B)(1-F)}\sigma_b^2 \qquad \sigma_b^2 = \frac{1}{4}(1-\theta)^2\sigma_a^2$$

(11a)

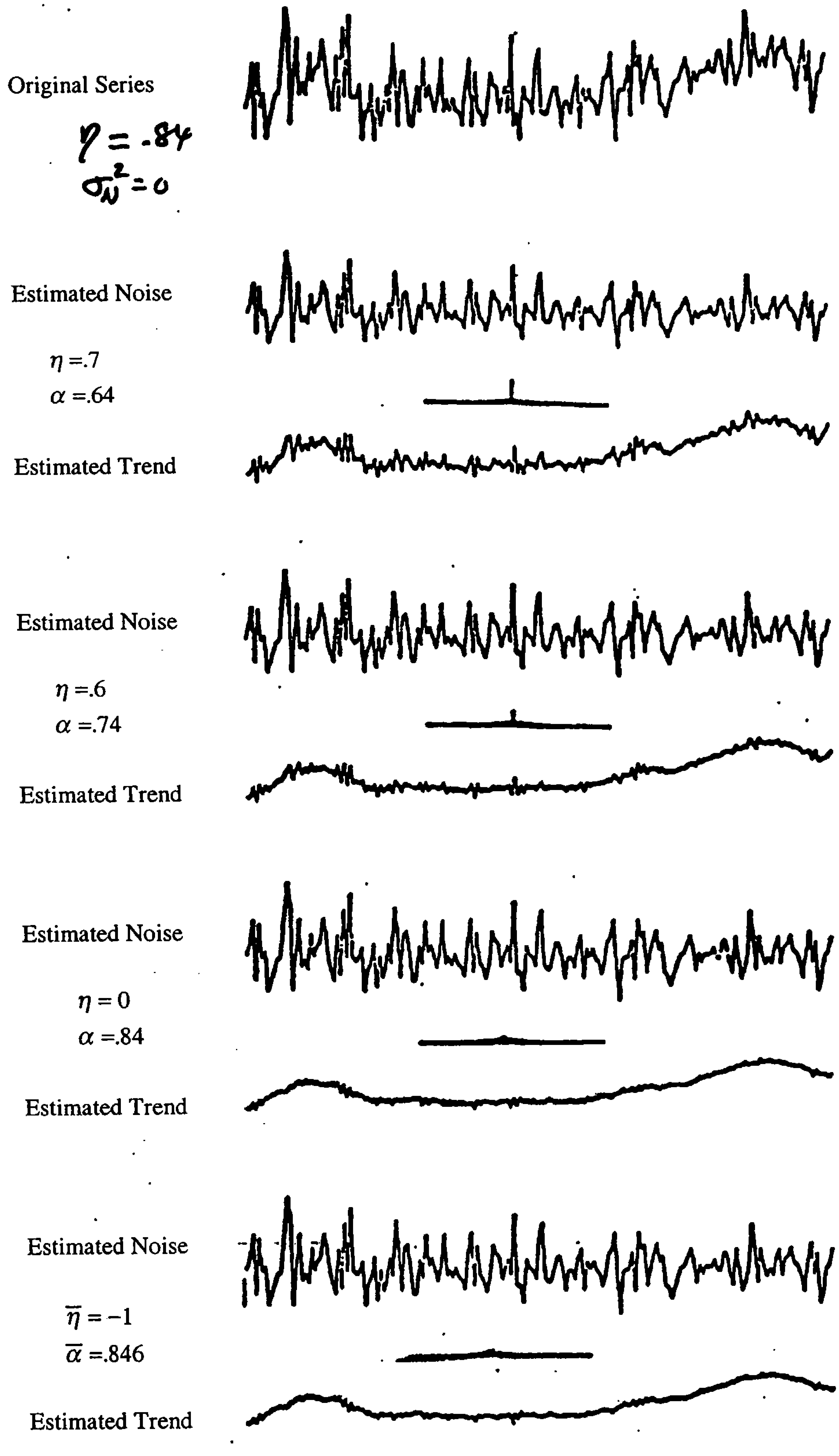
$\theta = .84$

Figure 1. SOME POSSIBLE DECOMPOSITIONS INTO TRENDS AND NOISE

CONSUMER PRICE INDEX EXAMPLE

$(1-B)Z_t = (1-\theta B)a_t$

$Z_t = T_t + N_t$



Canonical noise

Canonical Trend

$= \bar{\sigma}_N^2$

11b

(Reduced form)

From this example, given the overall model

- Canonical decomposition gives the most deterministic trend component
- It attributes the largest possible variance to the noise component
- The popular choice in econometrics literature, $\eta = 0$, gives a estimated trend component very close to that from the canonical decomposition
- But all other acceptable decompositions are of equal claim
- The trend component is clearly not 'robust' over the entire range of acceptable decomposition,

but robust over $-1 \leq \eta \leq 0$.

This formulation was generalized in joint work with Hillmer (1982 JASA) to

$$Z_t = S_t + T_t + N_t$$

Trend model $(1-B)^d T_t = \eta(B) b_t$, $U(B) S_t = \delta(B) c_t$ ← Seasonal component model

$$U(B) = 1 + B + \cdots + B^{s-1}$$

consistent with an overall model for Z_t of the form

$$(1-B)^d U(B) Z_t = \theta(B) a_t$$

Canonical decomposition gives the most 'deterministic' evolving stochastic seasonal and trend components. See examples in Hillmer, Bell, and Tiao(1983) 2nd Census Conference Vol.

A. Maravall

X-12-ARIMA and TRAMO/SEATS

- Many similarities.
- Each consists of two main parts: a regression-ARIMA modeling program and a decomposition program.
- The regression-ARIMA features are very close. Both employ an iterative outlier and level-shift detection procedure initiated in Ih Chang's thesis(1983) [TRAMO strong in automatic modeling; X-12 strong in diagnostics]
  Chang, Tiao + Chen (1988 Technometrics)
- The two decomposition programs are built on different foundations; X-12 retains the structure of the X-11-ARIMA (Dagum), an empirically developed procedure; SEATS is a model-based procedure closely related to canonical decomposition.
- For many economic time series, the decomposition results from these two programs are again very close. This is not too surprising since the X-11 and the 'airline model' are closely related.

$$Z_t = T_t + N_t$$

Forecasting unobserved trend and seasonal components

For the model

$$(1-B)Z_t = (1-\theta B)a_t$$

$$(\theta = .84; \quad \sigma_a^2 = .0019 \text{ for the CPI data})$$

the estimate of T_t,

$$\tilde{T}_t = \left(1 - \alpha\frac{(1-B)(1-F)}{(1-\theta B)(1-\theta F)}\right)Z_t, \quad \boxed{\alpha = \frac{\sigma_N^2}{\sigma_a^2}}$$

depends on the variance ratio α.

Suppose we have data through t_0, i.e. $(Z_{t_0}, Z_{t_0-1}, \ldots)$

$$\tilde{T}_{t_0} = (1-\alpha)Z_{t_0} + \alpha\hat{Z}_{t_0-1}(1) = Z_{t_0} - \alpha a_{t_0}$$

where $\hat{Z}_t(1)$ is the usual 1-step ahead forecast, independent of α. This is the estimate of the current signal T_{t_0}.

For $\ell > 0$

$$\boxed{\tilde{T}_{t_0+\ell} = \hat{Z}_{t_0}(\ell)}$$

Thus, point forecast of $T_{t_0+\ell}$ is the same as forecast of the future observation $Z_{t_0+\ell}$ irrespective of the value of α. Thus, while the component T_t is not identifiable, the forecast is the same for all α. Also, for this model all forecasts are the same.

The above result, $\tilde{T}_{t_0+\ell} = \hat{Z}_{t_0}(\ell)$, is true for all trend plus noise model

$$Z_t = T_t + N_t$$

Thus, if the main interest centers on the best point estimate of future trend $T_{t_0+\ell}$, then there is ***no need*** for decomposition.

284

For the model

$$(1-B^s)Z_t = (1-\theta B^s)a_t$$

$$Z_t = S_t + T_t + N_t$$

it can be shown that forecasting future trend and seasonal components

$$(1-B)\widetilde{T}_{t_0+\ell} = 0$$

$$U(B)\widetilde{S}_{t_0+\ell} = 0$$

$$\widetilde{T}_{t_0+\ell} + \widetilde{S}_{t_0+\ell} = \hat{Z}_{t_0}(\ell)$$

Thus

$$\widetilde{T}_{t_0+\ell} = s^{-1}[\hat{Z}_{t_0}(1)+\cdots+\hat{Z}_{t_0}(s)]$$
$$\widetilde{S}_{t_0+\ell} = \hat{Z}_{t_0}(\ell) - s^{-1}[\hat{Z}_{t_0}(1)+\cdots+\hat{Z}_{t_0}(s)]$$

which is a natural decomposition of the forecasting function $\hat{Z}_t(\ell)$ for the overall model of Z_t.

For the 'airline model'

$$(1-B)(1-B^s)Z_t = (1-\theta B)(1-\Theta B^s)a_t$$

Box, Pierce & Newbold (1987)
JASA

$$(1-B)^2 \widetilde{T}_{t_0+\ell} = 0$$

$$U(B)\widetilde{S}_{t_0+\ell} = 0$$

$$\widetilde{T}_{t_0+\ell} + \widetilde{S}_{t_0+\ell} = \hat{Z}_{t_0}(\ell)$$

$$(1-B)(1-B^s)\hat{Z}_{t_0}(\ell) = 0$$

The above implies that, if the goal of decomposition is to forecast future values of the unobserved components, we can simply decompose the forecast function for the overall model to get the optimum point forecasts.

The variance depends of course on the choice of decomposition

Variances of errors of component forecasts

For the model

$$(1-B)Z_t = (1-\theta B)a_t$$

$$Z_t = T_t + N_t \qquad \Rightarrow (1-B)T_t = (1-\eta B)b_t$$

$$\rightarrow \mathrm{var}(T_{t_0+\ell} - \widetilde{T}_{t_0+\ell}) = (\sigma_a^2 - \sigma_N^2) + (\ell-1)(1-\eta)^2\sigma_b^2$$

$$\sigma_a^2 = \frac{1}{(1+\theta)^2}[\sigma_b^2(1+\eta)^2 + 4\sigma_N^2]$$

well known in terms of component model parameters ($\eta, \sigma_b^2, \sigma_N^2$)

- The second term represents the linear growth, coming from the model of the trend component
- The first term depend on both the component noise and the trend models

288

$$\mathrm{var}(T_{t_0+\ell} - \tilde{T}_{t_0+\ell}) = (\sigma_a^2 - \sigma_N^2) + \underline{(\ell-1)(1-\eta)^2\sigma_b^2}$$

For forecasting future observations

$$\mathrm{var}(Z_{t_0+\ell} - \hat{Z}_{t_0}(\ell)) = \sigma_a^2 + (\ell-1)(1-\theta)^2\sigma_a^2$$

$$\underline{(\ell-1)(1-\eta)^2\sigma_b^2}$$

Since $\boxed{(1-\theta)^2\sigma_a^2 = (1-\eta)^2\sigma_b^2}$

$$\mathrm{var}(Z_{t_0+\ell} - \hat{Z}_{t_0}(\ell)) = \sigma_a^2 + (\ell-1)(1-\eta)^2\sigma_b^2$$

To measure <u>how fast the linear growth</u> will <u>catch up with σ_N^2</u>

$$(\ell-1)(1-\theta)^2\sigma_a^2 = \sigma_N^2$$

$$\ell-1 = \sigma_N^2 / (1-\theta)^2\sigma_a^2$$

For canonical decomposition

$$\ell - 1 = \frac{1}{4}\left(\frac{1+\theta}{1-\theta}\right)^2$$

a time constant, and we can use

$$\lambda = \left(\frac{1-\theta}{1+\theta}\right)$$

as a useful measure of the 'strength' of the trend signal in the data. (see p. 11)

This approach can be extended to the trend plus seasonal decomposition

signal / noise ratio for seasonal
for trend.

All from overall model.

290

Concluding Remarks:

- Model Based Procedures
  - make assumptions explicit
  - inferential process straightforward
  - lend itself to further improvement
- Canonical decomposition exposes explicitly
  - robustness of seasonal component
  - robustness of trend component
- Convergence of various procedures
  - esp. in regression part
- For forecasting future components **NO NEED** for adjustment
- measures of strength of seasonal signal and trend signal may be useful

X-12 filters (22a)

Beveridge Nelson Decomposition (1981) and Signal extraction

$$(1-B)z_t = (1-\theta B)a_t$$

$$z_t = (1-\theta)(a_{t-1} + a_{t-2} + a_{t-3} + \cdots) + a_t \quad \text{(Muth, 1960)}$$

B-N decomposition

$$z_t = P_t + \epsilon_t$$

$$\begin{cases} P_t = (1-\theta)(a_t + a_{t-1} + a_{t-2} + \cdots) & \text{permanent component} \\ \epsilon_t = z_t - P_t = \theta a_t & \text{transitory component.} \end{cases}$$

Now we can write

$$P_t = z_t - \theta a_t$$

From p.15 $Z_t = T_t + N_t$

$$\tilde{T}_{t_0} = Z_{t_0} - \alpha a_{t_0}$$ estimate of current signal

$$\alpha = \sigma_N^2/\sigma_a^2$$

Here $P_{t_0} = Z_{t_0} - \theta a_{t_0}$

$$\Rightarrow \quad \theta = \frac{\sigma_N^2}{\sigma_a^2}$$

what model for T_t will give rise to this?

$$\Rightarrow \quad (1-B) T_t = b_t$$ random walk!

i.e. an random walk + noise decomposition provided $\theta > 0$

Implications:

- P_t is then the so-called "concurrent" estimate of the random walk signal T_t
- When data beyond t become available, P_t is not the best estimate for T_t
- There are other choices of models for T_t

unifying X-12-ARIMA and model-based procedure

Yea-Jane Chu, Thesis

model based procedure

$$Z_t = S_t + T_t + N_t$$

X-12 various filters (seasonal) options

choose a filter to min average

MSE between a X-12 filter and S_t

over the range of acceptable decompositions

Example to compare X-12-ARIMA + Tramo-Seats

Monthly Logged Airline Passenger Totals Data (1949.Jan – 1960.Dec)

Parameter	SCA	X-12-ARIMA	TRAMO
$\hat{\theta}$	.402(.079)*	.401(.079)	.400(.082)
$\hat{\Theta}$	.560(.071)	.560(.075)	.562(.086)
$\hat{\sigma}_a$	.0368	.0373	.0375

* standard error of estimates   P 323 324

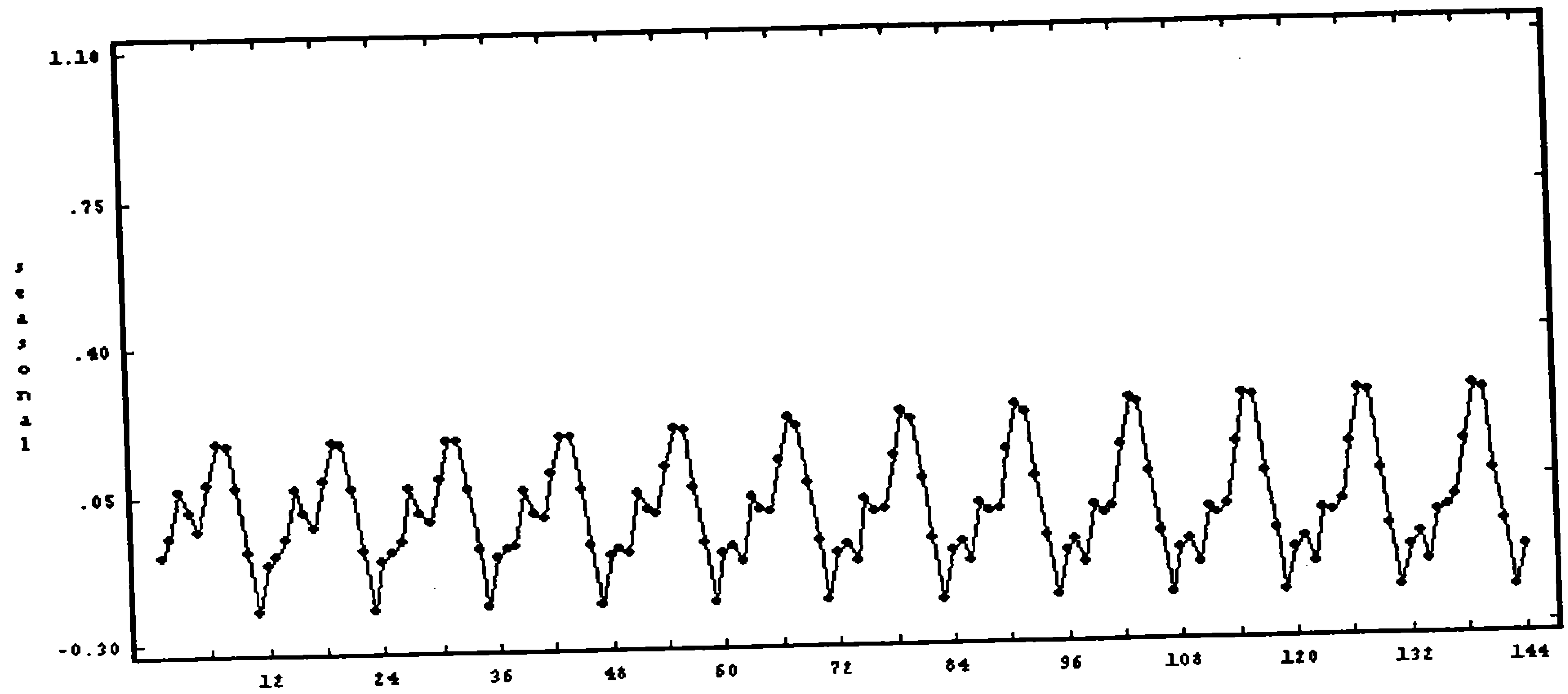
No outlier is detected for all methods.

The "Airline Model"

$$(1-B)(1-B^{12})z_t = (1-\theta B)(1-\Theta B^{12})a_t$$

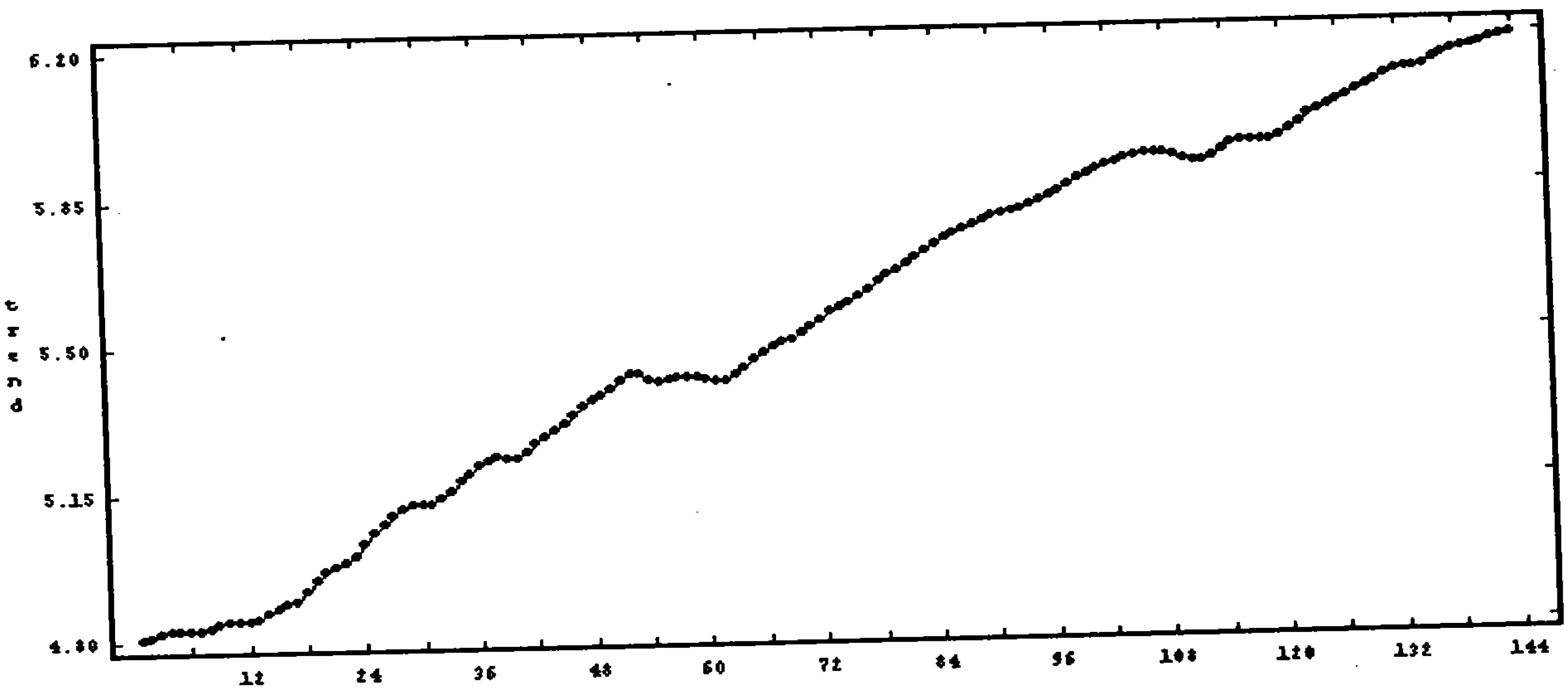
Same scaling

Canonical Seasonal Component (Tramo/Seats)



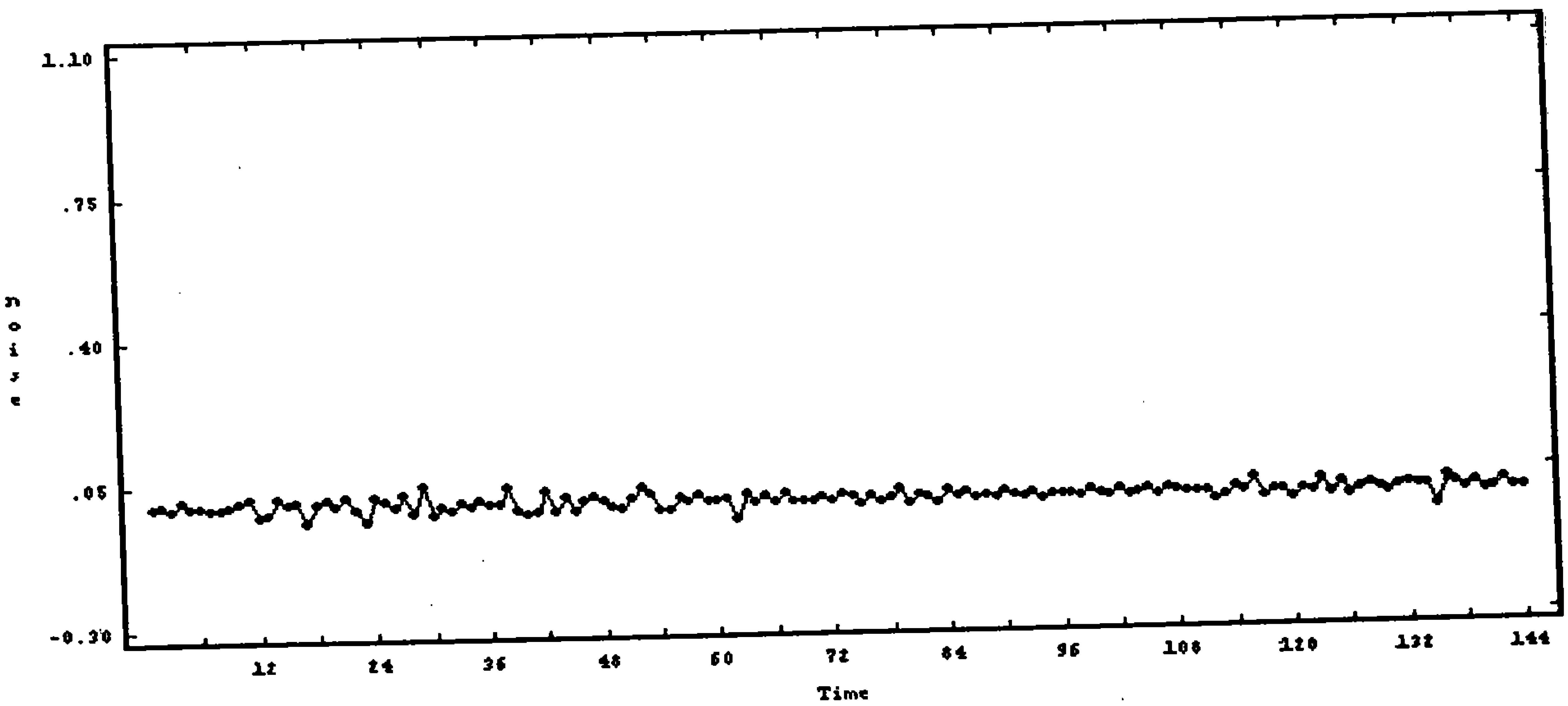
Canonical Trend Component

4



Canonical Noise Component

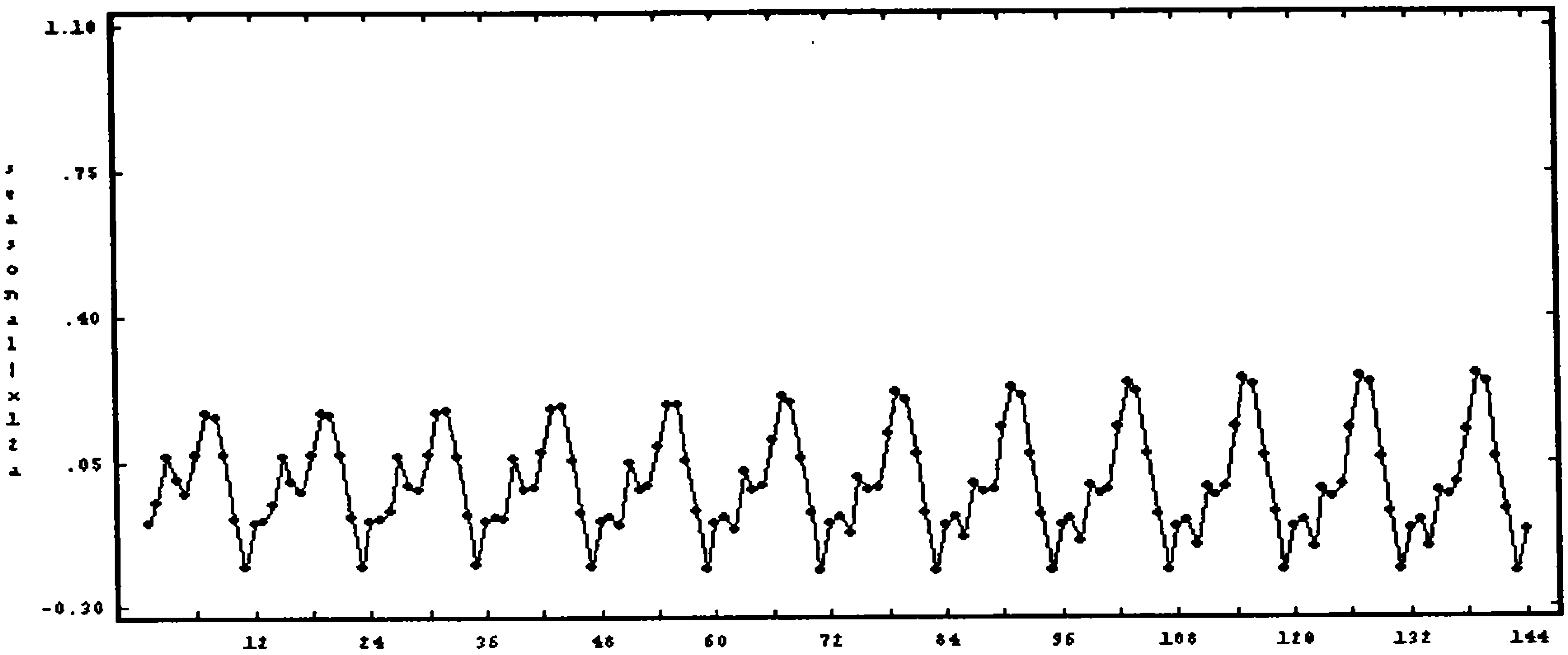
1.4



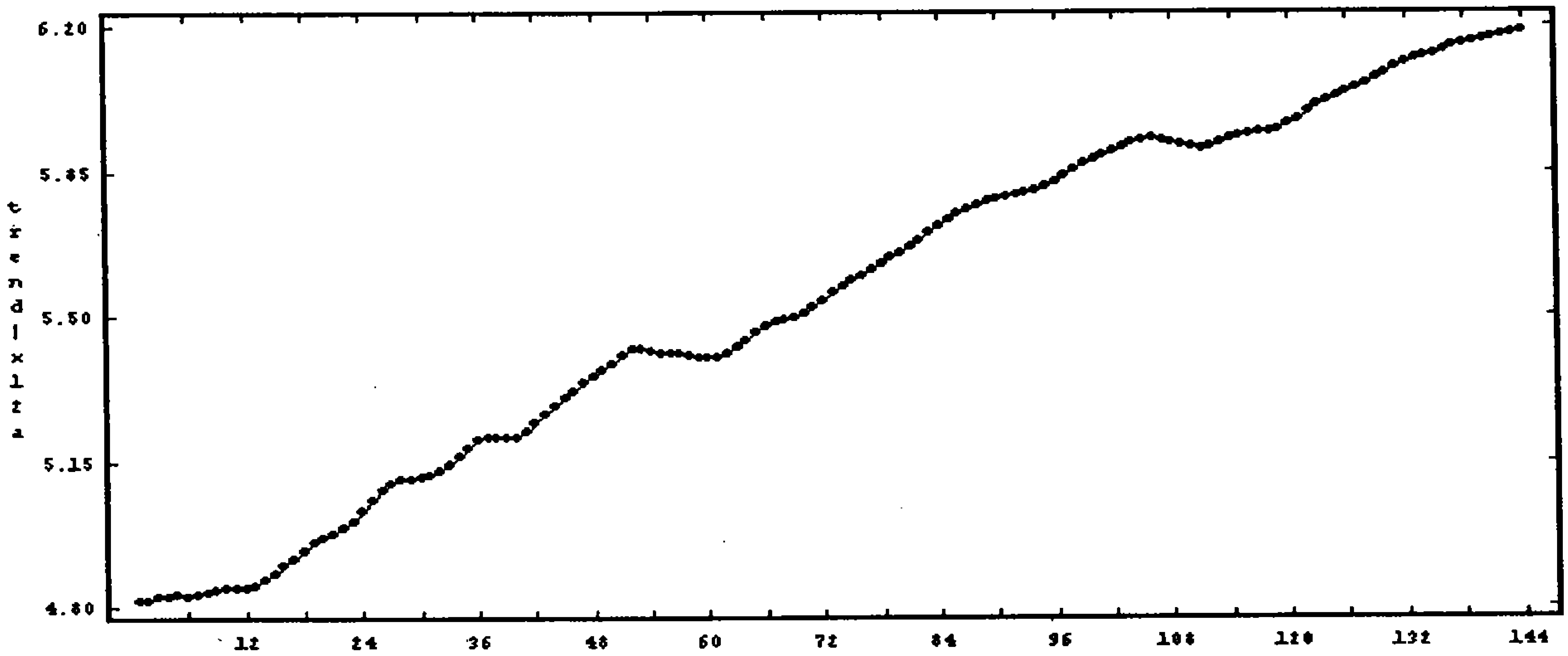
2-4

Same scaling

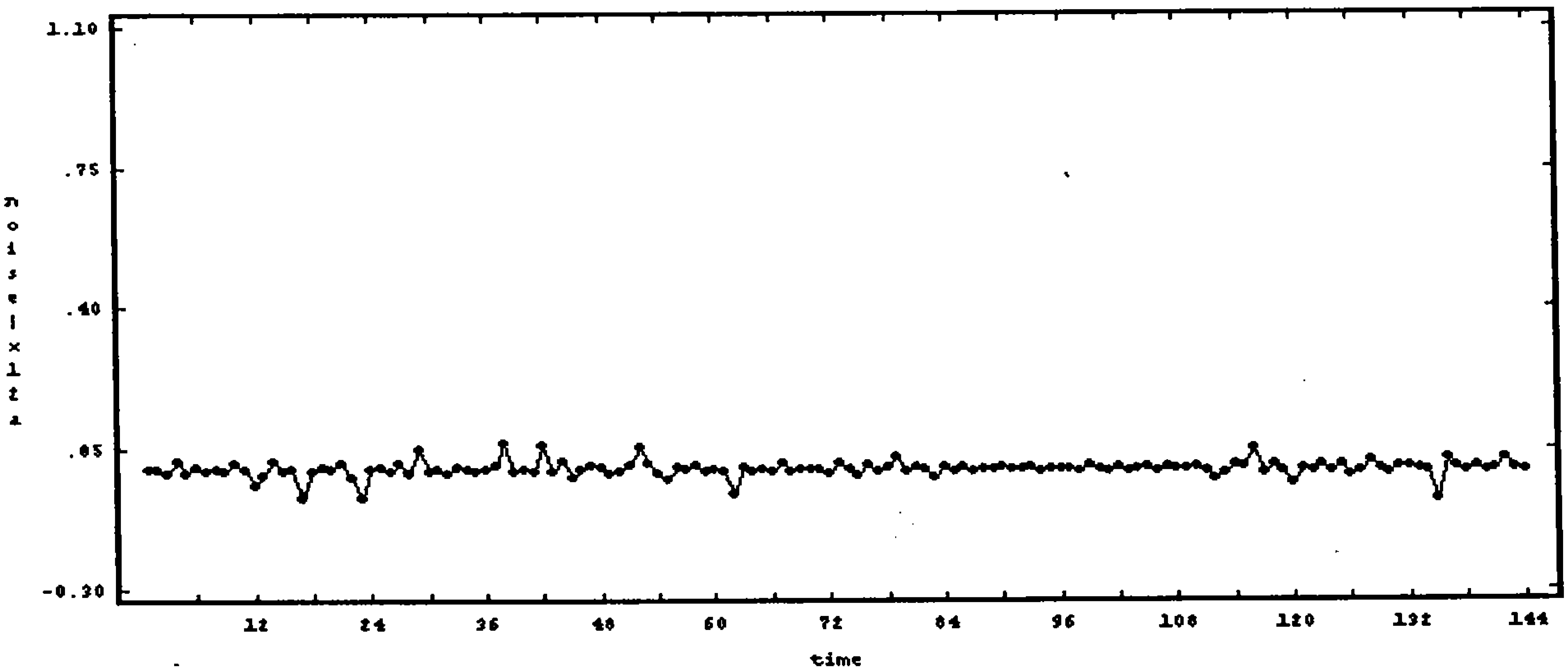
Seasonal Component (X-12-ARIMA)



Trend Component

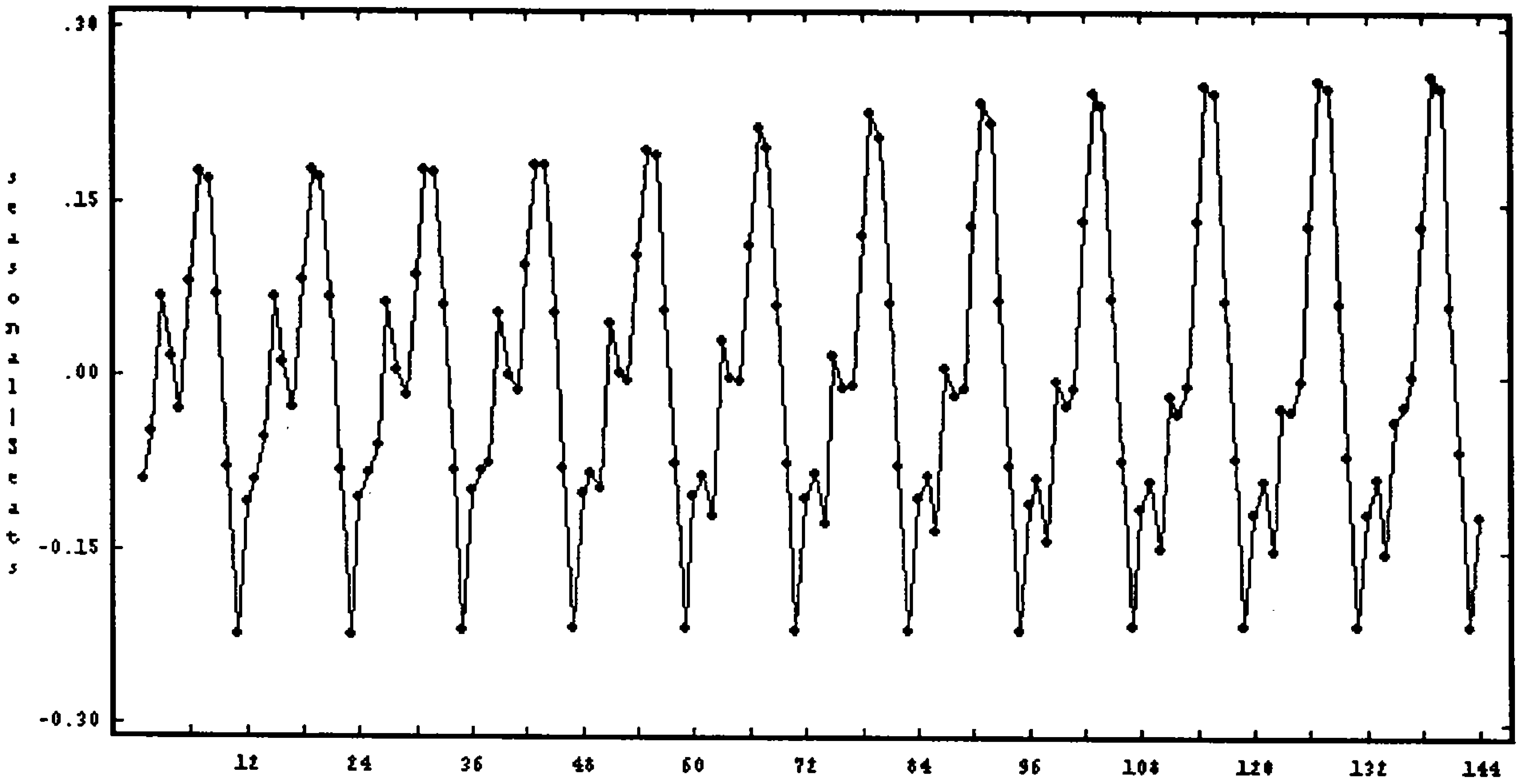


Noise Component

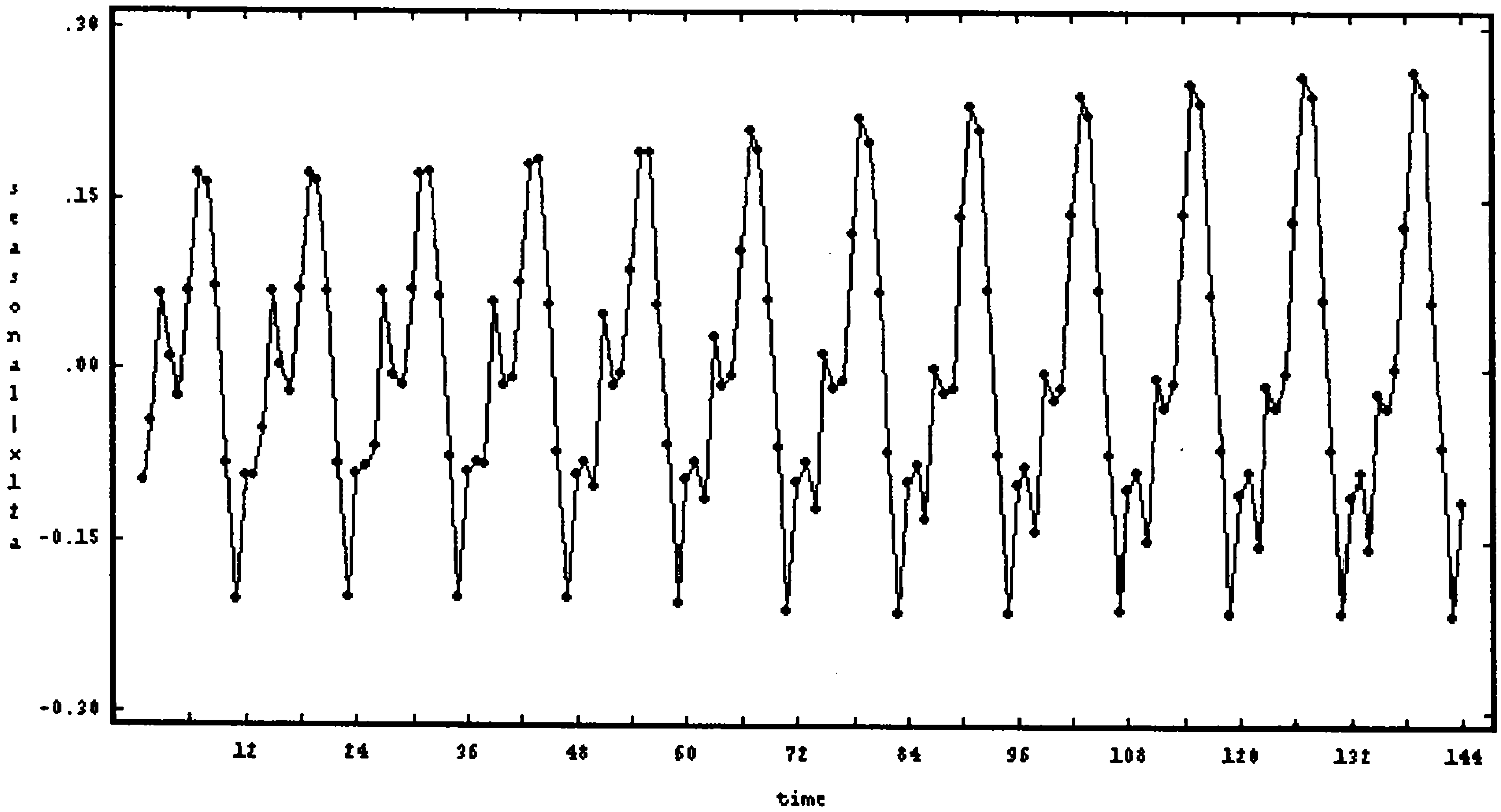


2-8

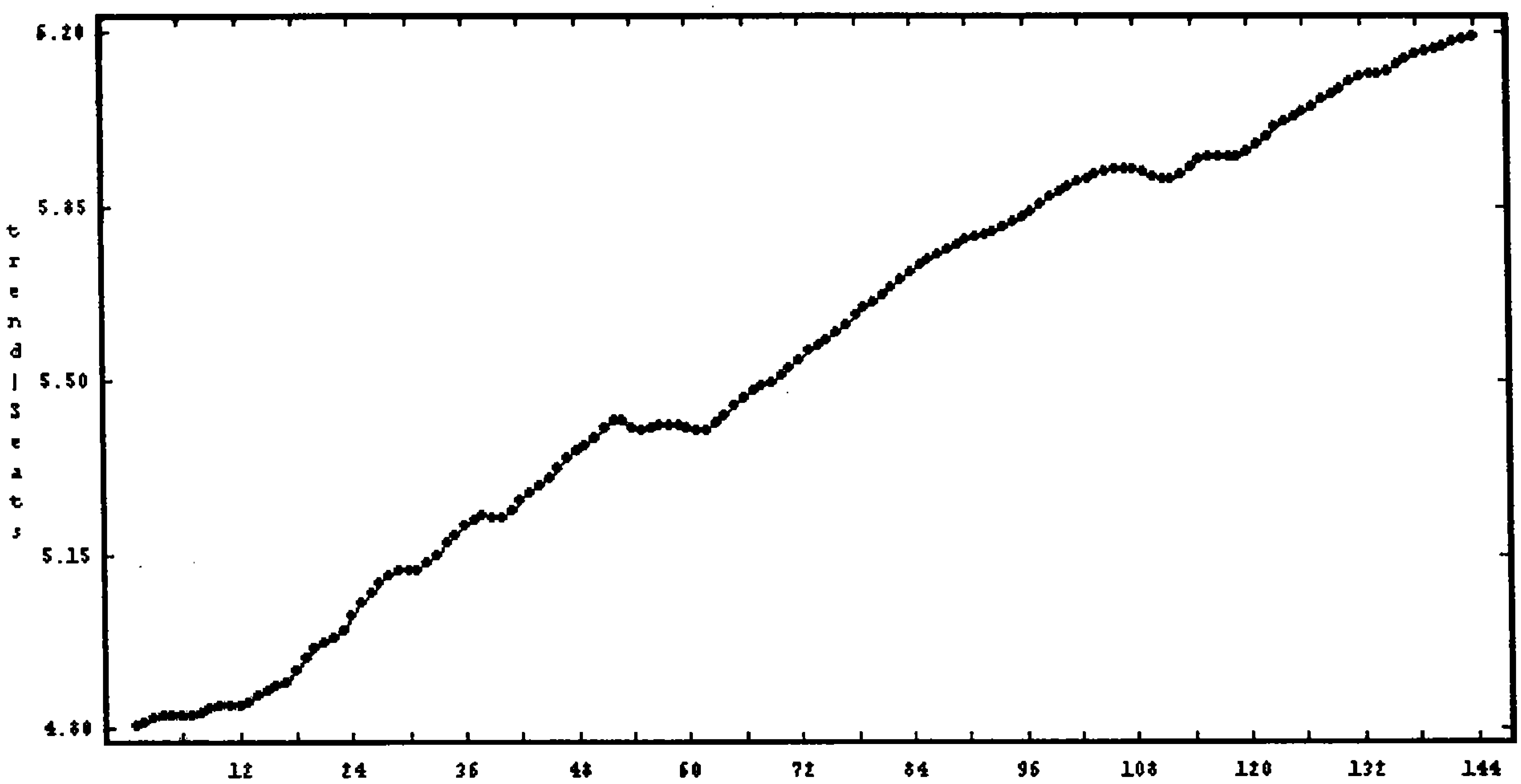
Canonical Seasonal Component (Tramo/Seats)



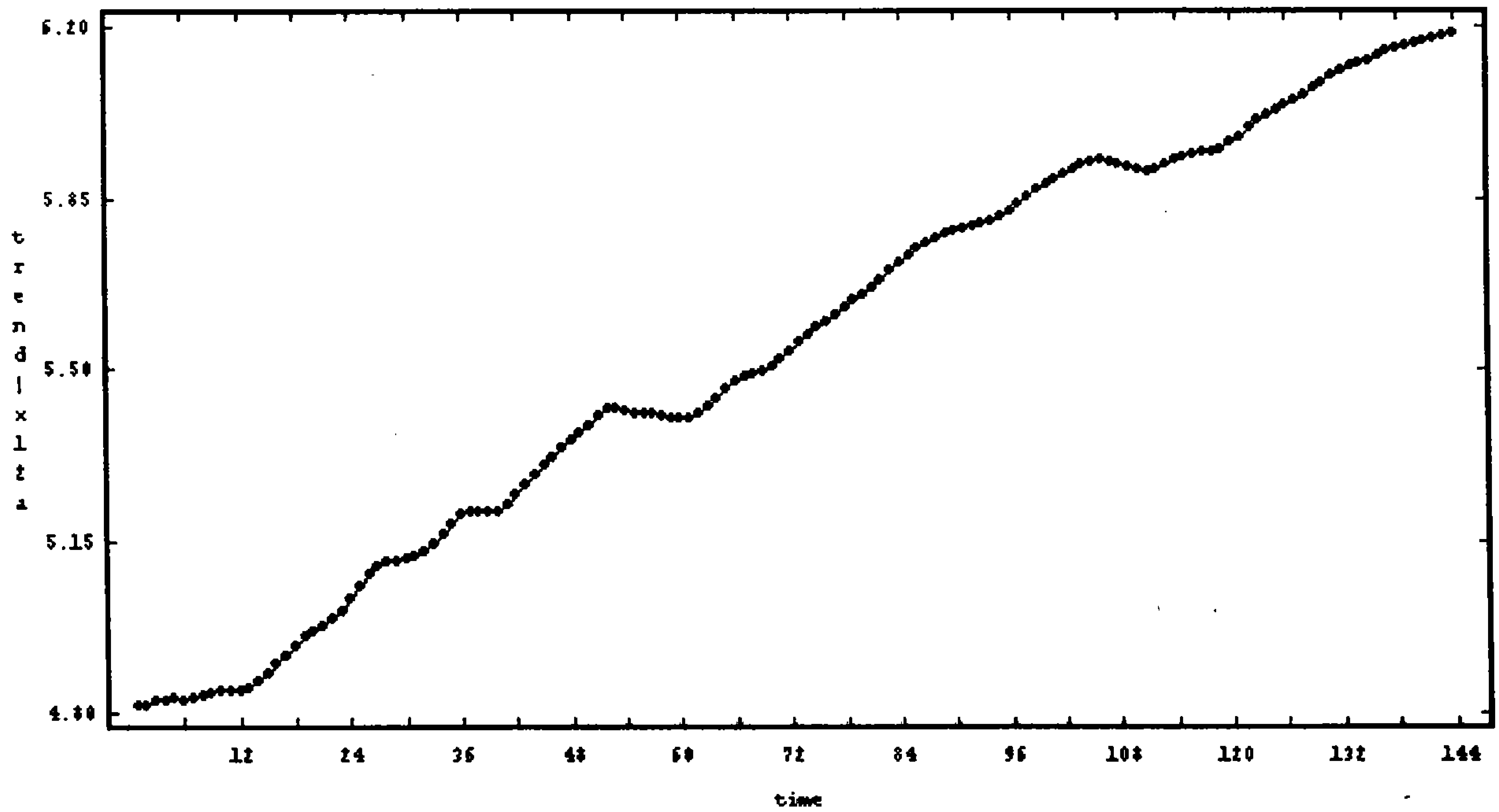
Seasonal Component (X-12-ARIMA)



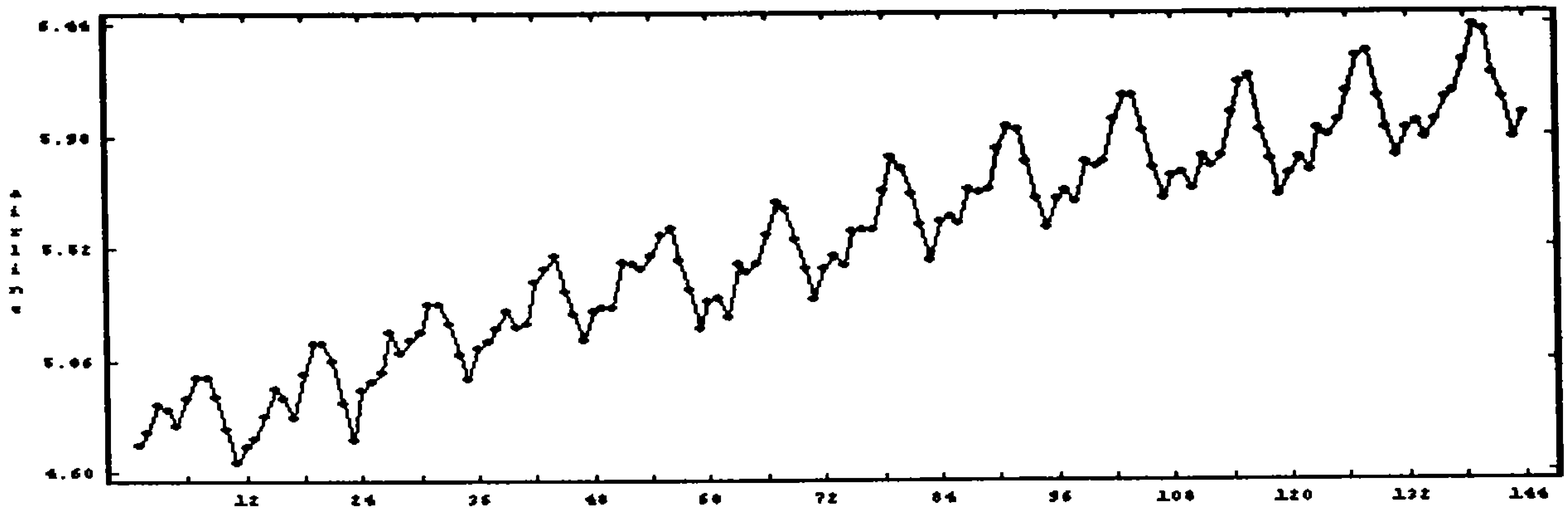
Canonical Trend Component (Tramo/Seats)



Trend Component (X-12-ARIMA)



Logged Monthly Airline Passenger Totals (49:1 - 60:12)



Decomposition of Seasonal Time Series: A Model for the Census X-11 Program

W. P. CLEVELAND and G. C. TIAO*

This paper shows that the linear filter version of the Census X-11 program for time-series decomposition can be approximately justified in terms of an additive model with stochastic trend, seasonal and noise components. Optimal estimates of the trend and seasonal components are obtained from the model and found to be in close agreement with the corresponding estimates for the Census procedure. This approach makes it possible to assess the appropriateness of the Census method. Two examples are given, one showing that the use of the X-11 procedure is largely appropriate and the other much less so.

1. INTRODUCTION

One of the goals people often have in approaching a seasonal time series is removal of the seasonal component. This may be undertaken for several reasons. It leaves a series with a simpler pattern to be studied for its implications. It is particularly helpful in revealing the latest trends, since averaging by eye is not as easy in this nonsymmetric situation. Another common reason for wanting a deseasonalized series is to make comparisons of series with different seasonal patterns.

One way of effecting the decomposition of a series into seasonal and trend components is to fit a model which has deterministic seasonal and nonseasonal parts to the data. Another is to apply moving-average filters to the data to separate out these components. The filters are often designed on the basis of allocating various parts of the power spectrum of the series to the seasonal and trend components.

The Bureau of the Census has developed a computer program for seasonal adjustment which has been widely used in government and industry. The basic feature of the program is that it uses a sequence of moving average filters to decompose a series into a seasonal component, a trend component, and a noise component. The strength and weakness of the program lies in its use of roughly the same filters for most series. Such a decomposition has the superficial advantage of uniform interpretation of the seasonal and trend components of most series. On the other hand, if one believes that observed phenomena are generated according to the physical circumstances of the problem, one could certainly be misled as to their nature by the results of the census program if no checks on its adequacy are made. Such checks would, however, be difficult to make unless one had some ideas as to the kinds of *underlying mechanisms* for which the census program would be appropriate.

This paper proposes a stochastic model for which the census procedure is nearly optimal. Specifically, we suppose that an observed time series y_t consists of three additive random components: a seasonal component s_t, a trend component p_t, and a noise component e_t, such that if particular autoregressive integrated moving average models are given for these components, the optimum estimators for s_t and p_t turn out to be very close to those obtained from the basic set of moving average filters employed in the census program. The results shed considerable light on both the merits and the demerits of the census program. In particular, one can see why the census method is generally satisfactory for a variety of series, why the residuals of the census decomposition will have smaller variance than the residuals of an autoregressive integrated moving average model fit to the same series, and why the census residuals may be correlated.

2. THE BUREAU OF THE CENSUS PROGRAM

The census procedure is summarized in Shiskin and Eisenpress [12] and described fully in Shiskin, Young, and Musgrave [13]. The program assumes the additive decomposition[1]

$$y_t = p_t + s_t + e_t \ , \tag{2.1}$$

where y_t is the observed series, p_t is the trend component, s_t is the seasonal component, and e_t is the noise. For each series the estimates of p_t and s_t, denoted, respectively, by $\hat{p}_t$ and $\hat{s}_t$, may be obtained. The basic device used to estimate the seasonal and trend components is the symmetric moving-average operator (filter) with weights summing to one. General properties of moving average operators are discussed in Kendall and Stuart [9, p. 366] and in Whittaker and Robinson [16, p. 288]. A symmetric moving average operator $S(\delta, k)$ generates a series

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[1] There is also a multiplicative version of the program. By taking logarithms, the multiplicative version is essentially the same as the version discussed.

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Applications Section

x_t from a series y_t according to the relation

$$x_t = S(\delta, k)y_t = \sum_{j=-k}^{k} \delta_j y_{t-j} \tag{2.2}$$

where k is a non-negative integer, and δ_j are the weights (constants) used in averaging y_t such that $\delta_{-j} = \delta_j$.

For a series $y_1, \ldots, y_T$ of length T, x_t cannot be computed according to (2.2) for $t = 1, \ldots, k$ and $t = T - k + 1, \ldots, T$. The operator $S(\delta, k)$ must, therefore, be modified at both ends of the series according to some extrapolation principle, as is done in the census procedure. In the analysis here, we shall be concerned mainly with version (2.2). However, the results developed can be logically extended to cover the end effects of the series.

Wallis [15] describes the sequence of moving average operators used in the census program. Although it includes procedures for adjusting outliers and for making trading day adjustments, the basic linear filter feature of the program is of the form (2.2). Also, the census program has a built-in procedure to select the 9-term, 13-term, or 23-term Henderson trend filters. In what follows we have assumed the 13-term filter because it seems to be the one applicable to the majority of series, and also because it would be rather difficult to model the procedure for selecting this one from the other two alternatives. The model in Section 3 applies to this linear filter version of the program.

3. A MODEL FOR THE BUREAU OF THE CENSUS PROGRAM

Without a specific underlying model the Bureau of the Census program lends itself to all the criticisms of moving-averages in general. If the observed series were a polynomial in time plus a trigonometric series plus noise, one should do a regression. If the preceding with time changing coefficients were the model, one could parameterize the pattern of changes in the coefficients and still do a regression. Models which fit a polynomial over a finite number of points to estimate the center point are not consistent, in the sense that different polynomials represent the same points at different times.

For a discussion of the properties of moving average filters and criticisms of the Census procedure in the context of spectral theory, the reader is referred to [3–8, 10, 11].

The best justification of moving-average filters seems to be in terms of stochastic models. For the additive decomposition in (2.1), we employ the autoregressive integrated moving average (ARIMA) processes [1] for the components. Specifically, we suppose that

$$y_t = p_t + s_t + e_t \tag{3.1}$$

with

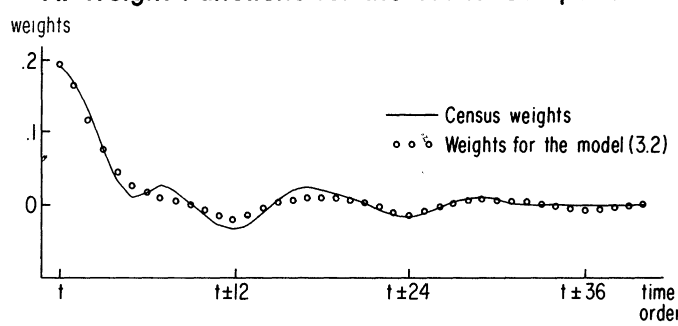
$$\phi_{r_1}(B)(p_t - \mu) = \psi_{q_1}(B)b_{1t}$$
$$\phi_{r_2}(B)s_t = \psi_{q_2}(B)b_{2t} \ ,$$

where μ is a constant, B is the backward shift operator such that $Bx_t = x_{t-1}$, and $\psi_{q_1}(B)$, $\psi_{q_2}(B)$, $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ are real polynomials in B of degrees q_1, q_2, r_1 and r_2, respectively. We shall require that the zeroes of $\psi_{q_1}(B)$ and $\psi_{q_2}(B)$ lie outside the unit circle and those of $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ lie on or outside the unit circle. In (3.1), e_t, b_{1t} and b_{2t} are three independent white noise processes, normally distributed with zero means and variances σ_e^2, $\sigma_{b_1}^2$ and $\sigma_{b_2}^2$, respectively. In this framework, the minimum mean square error estimators of p_t and s_t are, respectively, the conditional expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ where $\mathbf{y} = (y_1, \ldots, y_T)'$ is the vector of observations. It is shown in the appendix that, for t not close to either end of the series, $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ are, to a close approximation, symmetric moving averages of the observations $\mathbf{y}$. Thus, if one can find a model for which the conditional expectations give the same weights as those of particular symmetric moving average filters, it may then be argued that this model represents an underlying stochastic mechanism for those filters.

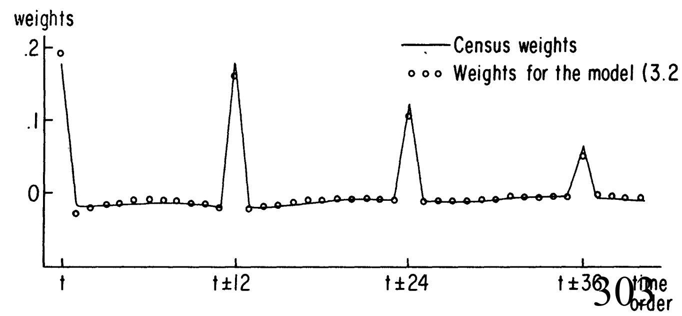
In searching for a model of the form (3.1) which would give weight functions similar to those of the census program, we kept several things in mind. When $\phi_{r_1}(B)$ contains the factor $(1 - B)$ to a degree at least one higher than $\phi_{r_2}(B)$, the weights in $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ sum to one and zero, respectively, as do the Census weights for $\hat{p}_t$ and $\hat{s}_t$. The use of a moving average filter for $\hat{p}_t$ rather than a polynomial suggests that one should allow for changing trend slopes. Similarly, the seasonal estimator should allow for changing amplitude and phase. Otherwise, a trigonometric series could be used.

The model stated in (3.2) is consistent with these principles and gives good correspondence to the census program using a minimal number of parameters. The weight functions for $E(p_t|\mathbf{y})$ and $\hat{p}_t$ are exceedingly close, as are the weight functions for $E(s_t|\mathbf{y})$ and $\hat{s}_t$. These are illustrated in Figures A and B.

A. Weight Functions for the Trend Component



B. Weight Functions for the Seasonal Component



$$y_t = p_t + s_t + e_t \ ,$$
$$(1 - B)^2 p_t = (1 - \psi_{11}B - \psi_{12}B^2)b_{1t} \ ,$$
$$(1 - B^{12})s_t = (1 - \psi_{21}B^{12} - \psi_{22}B^{24})b_{2t} \ , \qquad (3.2)$$
$$\psi_{11} = -.49 \ , \ \psi_{12} = .49 \ , \ \psi_{21} = -.64 \ , \ \psi_{22} = -.83 \ ,$$
$$\sigma_{b_2}^2/\sigma_{b_1}^2 = 1.3 \quad \text{and} \quad \sigma_e^2/\sigma_{b_1}^2 = 14.4 \ .$$

Models differing from (3.2) in the order of differencing and in the numbers of autoregressive and moving average parameters in the two component models were tried. This model was found to be as good as models with more parameters and superior to others, particularly in matching $E(s_t|\mathbf{y})$ to the corresponding census weights.

A general purpose nonlinear least-squares program was used to estimate the parameter values for each model. For a given set of parameter values, the generating functions (A.20) and (A.21) in the appendix were expanded for the first 42 terms on either side of the center. The program used these as predictions of the corresponding weights in the linear filter version of the census program, and it adjusted the parameter values iteratively until the sum of squared deviations over the seasonal and trend weights was minimized.

While we used the sum of squares as one measure in choosing among candidate models, we also visually inspected the nature of the discrepancies in the weight functions, as displayed in Figures A and B, when evaluating the adequacy of a model. Occasionally tails of the census and model weights did not appear to be converging or only the trend weights and not the seasonal were being well approximated.

3.1 The Overall Model for y_t

From (3.2), we may write

$$\begin{aligned}(1 - B)(1 - B^{12})y_t &= ((1 - B^{12})/(1 - B))(1 - \psi_{11}B - \psi_{12}B^2)b_{1t} \\ &+ (1 - B)(1 - \psi_{21}B^{12} - \psi_{22}B^{24})b_{2t} \\ &+ (1 - B)(1 - B^{12})e_t \ , \qquad (3.3)\end{aligned}$$

so that the autocorrelations of $w_t = (1 - B)(1 - B^{12})y_t$ are

k	ρ_k									
1–10	−.25	.13	.12	.11	.09	.08	.07	.05	.04	.03
11–20	.18	−.35	.16	.00	.00	.00	.00	.00	.00	.00
21–25	.00	.00	−.01	.03	−.01					

with $\rho_k = 0$ for $k > 25$. The explicit overall model for y_t is found to be,

$$\begin{aligned}(1 - B)(1 - B^{12})y_t &= (1 - .337B + .144B^2 + .141B^3 + .139B^4 \\ &+ .136B^5 + .131B^6 + .125B^7 + .117B^8 + .106B^9 \\ &+ .093B^{10} + .077B^{11} - .417B^{12} + .232B^{13} \\ &- .001B^{20} - .003B^{21} - .004B^{22} - .006B^{23} \\ &+ .035B^{24} - .021B^{25})c_t \ . \qquad (3.4)\end{aligned}$$

where c_t is a white noise process, normally distributed with zero mean and variance σ_c^2.

It is important not to confuse the basic additive model (3.2) and the overall form (3.4). In particular,

1. the overall form is a logical consequence of the basic model,
2. the conditional expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$ are determined not by (3.4) but by (3.2),
3. different additive models can lead to the same overall model,
4. the overall form is, however, important in that it can be identified from the data and, hence, provides a means to partially assess the appropriateness of the census procedure for a given set of data.

The model in (3.2) and its consequent overall form (3.4) helps explain why the census program fits series as well as it often does. A nonstationary seasonal pattern is presumed, and the model for p_t allows for a trend of changing slope. The autoregressive part of the overall model for y_t is $(1 - B)(1 - B^{12})$. This suggests that a series obeying (3.4) or at least $(1 - B)(1 - B^{12})y_t = \theta(B)c_t$ for some polynomial $\theta(B)$ might be fairly accurately analyzed by the census program. The overall model is broadly similar to the multiplicative model discussed by Box and Jenkins [1, Ch. 9], though not the same.

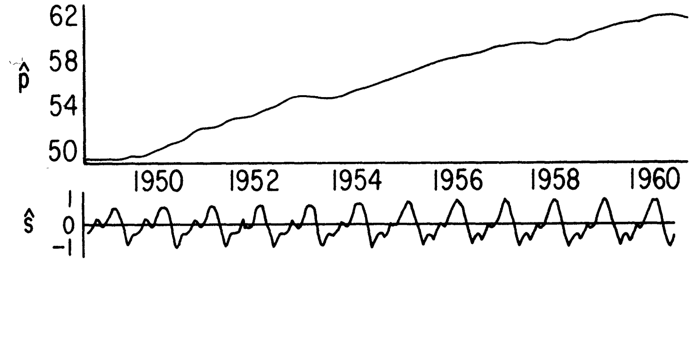
4. A COMPARISON OF THE CENSUS PROGRAM WITH OVERALL ARIMA SEASONAL MODELS FOR TWO SERIES

Two series are presented which have been fit with ARIMA models according to the procedures suggested in Box and Jenkins [1]. In doing so, the models sought were the ones that fit best in the sense of matching the correlation structure of the observed series and having uncorrelated residuals. The goal in each case was a model for forecasting rather than seasonal decomposition. It is, however, of interest to compare the results of this procedure with those of the census program.

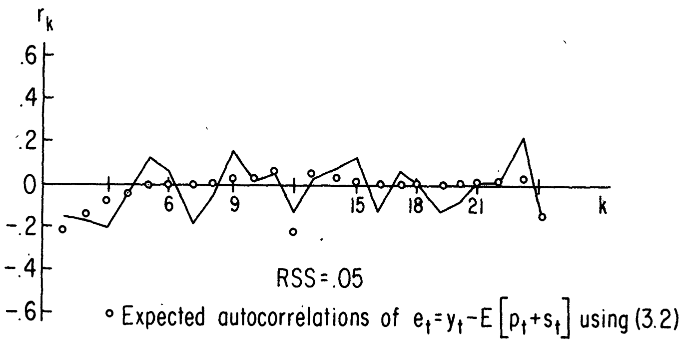
4.1 The Airline Data

These data may be found in [2, p. 429], or in [1, p. 304]. They represent logarithms of monthly passenger totals. Figures C and D show the census estimates of its components, $\hat{\mathbf{p}}$ and $\hat{\mathbf{s}}$, and the autocorrelation pattern of the residuals, $\mathbf{y} - \hat{\mathbf{p}} - \hat{\mathbf{s}}$. Also shown in Figure D are the expected residual autocorrelations, which will be explained later. The immediate impression of the series is that it exhibits a regular seasonal pattern with almost

C. Estimates of the Seasonal and Trend Components of the Airline Data from the Bureau of the Census Program



D. Residual Autocorrelations of the Airline Data from the Bureau of the Census Program



linear slope. The plots of the component estimates bear this out to a certain extent, but some evolution of the seasonal pattern and deviation from linearity are indicated. Some of the sample autocorrelations of the residuals appear to be rather large in magnitude, but the autocorrelation function as a whole exhibits no discernible pattern. The residual sum of squares, RSS, is .05.

The model for the airline data obtained by Box and Jenkins is

$$(1 - B)(1 - B^{12})y_t = (1 - \theta_1 B)(1 - \theta_{12}B^{12})c_t$$
$$\hat{\theta}_1 = .4\ ,\quad \hat{\theta}_{12} = .6\ ,\quad \hat{\sigma}_c^2 = .00134\ ,\quad \mathrm{RSS} = .175\ . \tag{4.1}$$

The autoregressive part of the model in (4.1) is the same as that of (3.4). Also, the estimates of θ_1 and θ_{12} in (4.1) are reasonably close to the coefficients associated with B and B^{12} on the right side of (3.4). One is, however, immediately struck by the fact that the RSS in (4.1), .175, is much larger than the corresponding number, .05, for the census program.

To see why, consider a simpler case where $y_t = z_t + e_t$ and the model for z_t is $(1 - B)z_t = (1 - \psi B)b_t$ where $|\psi| < 1$. This gives

$$(1 - B)y_t = (1 - \psi B)b_t + (1 - B)e_t$$
$$= (1 - \theta B)c_t\ ,\quad \text{where } |\theta| < 1\ . \tag{4.2}$$

Thus,

$$c_t = [(1 - \psi B)/(1 - \theta B)]b_t + [(1 - B)/(1 - \theta B)]e_t\ . \tag{4.3}$$

By expanding the right side of (4.3) we find

$$\sigma_c^2 = \left\{1 + \frac{(\theta - \psi)^2}{1 - \theta^2}\right\}\sigma_b^2 + \frac{2}{1 + \theta}\sigma_e^2 \tag{4.4}$$

so that

$$\sigma_c^2 > (2/(1 + \theta))\sigma_e^2 > \sigma_e^2\ .$$

From (A.16) in the appendix, the variance of the residuals $\hat{e}_t = y_t - E(z_t|\mathbf{y})$ is

$$\mathrm{Var}\,(\hat{e}_t) = \sigma_e^2 \cdot \frac{\sigma_e^2(2/(1 + \theta))}{\sigma_c^2} \cdot \tag{4.5}$$

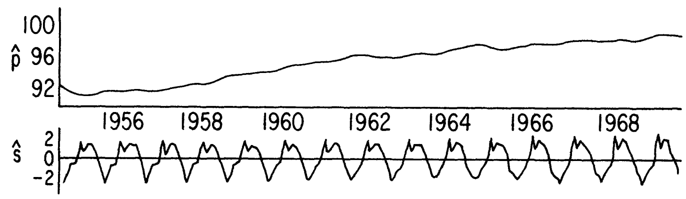
Hence, $\sigma_c^2 > \mathrm{Var}\,(\hat{e}_t)$. Now the residuals from Box and Jenkins's method of model fitting are $\hat{c}_t$ which are different from the residuals $\hat{e}_t$. Thus the difference in RSS is not in this situation evidence of one method being better than the other. Two different kinds of residuals are involved. As we have said, the model for y_t in (4.1) has the same differencing structure as (3.4), indicating that the census program is a nearly appropriate decomposition procedure for the airline series.

Further evidence that the census procedure works reasonably well for the airline data is provided by the following analysis. The theoretical autocorrelations of the residuals $y_t - E(p_t + s_t|\mathbf{y})$ corresponding to the underlying model (3.2) may be computed using the general result (A.16) in the appendix. These correlations are given by the circles in Figure D. The pattern of the expected correlations is fairly close to that computed from the data using the census procedure, and the agreement in sign of the correlations is nearly perfect.

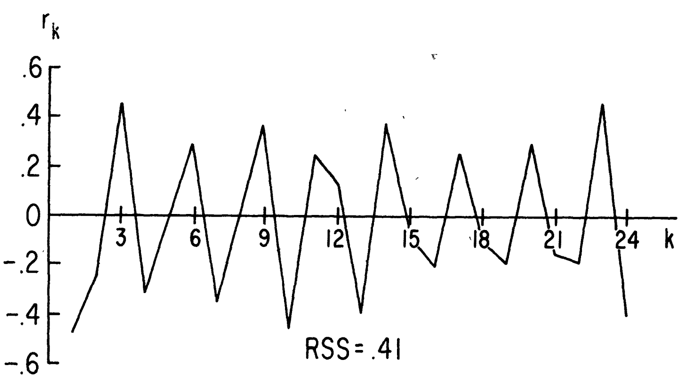
4.2 The Outward Telephone Data

The data in [14] are logarithms of monthly telephone disconnections, which the telephone companies call outward movement. Figures E and F show the census estimates of its components, and the residual autocorrelations. Note that the autocorrelations in this case exhibit a strong cyclical pattern. It seems clear that not all of the seasonality in the data has been accounted for by the program.

E. Estimates of the Seasonal and Trend Components of the Telephone Data from the Bureau of the Census Program



F. Residual Autocorrelations of the Telephone Data from the Bureau of the Census Program



The telephone data were analyzed by Thompson and Tiao and found to obey the model

$$(1 - \phi_3 B^3)(1 - \phi_{12}B^{12})y_t = (1 - \theta_9 B^9 - \theta_{12}B^{12} - \theta_{13}B^{13})c_t$$
$$\hat{\phi}_3 = .49\ ,\quad \hat{\phi}_{12} = 1.005\ ,\quad \hat{\theta}_9 = .23\ , \tag{4.6}$$
$$\hat{\theta}_{12} = .334\ ,\quad \hat{\theta}_{13} = .17\ ,\quad \hat{\sigma}_c^2 = .0035\ ,\quad \mathrm{RSS} = .5$$

3905

The autoregressive part of (4.6) is now quite different from that in the model (3.4) underlying the census program. Again the residual sum of squares, RSS, for this model is larger than that of the census program (.59 to .41). However, the census residuals exhibit a strong quarterly correlation pattern, while the residuals $\hat{c}_t$ do not. This is probably due to the differences between the two models (3.2) and (4.6), suggesting that using the census program on this series is not justified.

5. CONCLUSIONS

In the preceding sections, it was shown that the census procedure can be approximately justified in terms of an additive model which consists of a stochastic trend and seasonal components. The specific models for the components are given in (3.2), and the resulting overall model for the observations in (3.4).

If this model could be *assumed* for a given series, the census program could be used to estimate the unobserved trend and seasonal components during the observational period. In addition, the components could be forecast by calculating the conditional distributions of their future values given the observations. However, this result also raises a number of serious questions. First, should this model or minor deviations from it, corresponding to the use of the alternative 9-term or 23-term trend filters, be assumed for all series as implied by the census program? Second, is the census procedure robust to departures from this model? The answer to the first question is clearly no. Some clues to the answer for the second are provided by the analysis of the airline and telephone data.

The model (4.1) fitted by Box and Jenkins to the airline data has the same autoregressive part as the model (3.4), but the moving average parts are somewhat different. If we take the model fitted by Box and Jenkins to be correct, the appropriateness of the decomposition (3.2) is thrown in doubt. Yet the census program appears to perform quite well on this set of data, suggesting a certain degree of robustness. On the other hand, the census residuals from the telephone data show a strong quarterly correlation pattern. Since both the autoregressive and moving-average parts of (3.4) are quite different from those of the model (4.6) fitted to the telphone data, one suspects that this robustness does not extend to models which differ markedly from (3.4).

It might be argued that the robustness of the census procedure would be enhanced if the use of the alternative 9- or 23-term trend filters were taken into account. While this is undoubtedly true to a certain extent, one ought to remember that these alternatives were designed mainly for somewhat different evolutionary trend patterns. Thus, one would expect that the models corresponding to these alternative filters would be similar to (3.2) but with different values for the parameters, so that the use of these alternative filters on the telephone data would lead to a similar pattern of the residual autocorrelations as that shown in Figure F.

APPENDIX

For the general additive model (3.1), we here derive the conditional expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$, as well as the variance and autocorrelations of the residuals $\hat{e}_t = y_t - E(p_t + s_t|\mathbf{y})$.

To facilitate presentation, we first consider the situation

$$y_t = z_t + e_t \ , \quad t = 1, \ldots, T \ , \tag{A.1}$$

where

$$\phi_r(B)(z_t - \mu) = \psi_q(B)b_t \ ,$$

e_t are i.i.d. $N(0, \sigma_e^2)$, b_t are i.i.d. $N(0, \sigma_b^2)$ and independent of e_t, $\psi_q(B)$ is a polynomial in B of degree q having its zeroes lying outside the unit circle, and $\phi_r(B)$ is a polynomial in B of degree r having its zeroes lying on or outside the unit circle. Now, z_t can be written as a function of r values in the past (see, e.g., [1, p. 115]).

$$z_t = c_t + \sum_{j=m}^{t} b_j\pi_{t-j} \ , \quad m \leq -\max(r, q) + 1 \tag{A.2}$$

where (i) $c_t = \sum_{l=1}^{r} A_l\alpha_l^{t-m} + \mu$, α_l^{-1} are the zeroes of $\phi_r(B)$ assumed distinct, and A_l depend on the starting values of the series at time m; (ii) π_j are obtained by equating coefficients of B^j from the relation

$$\phi_r(B)(1 + \pi_1 B + \pi_2 B^2 + \ldots) = \psi_q(B) \ . \tag{A.2a}$$

Thus the vector $\mathbf{z}$, where $\mathbf{z} = (z_1, \ldots, z_T)'$, is distributed as normal with mean vector $\mathbf{n} = (c_1, \ldots, c_T)'$ and covariance matrix $\sigma_b^2\mathbf{\Sigma}_z$ whose elements can be readily obtained from (A.2).

A.1 The Conditional Distribution of z Given y

Given the observation vector $\mathbf{y} = (y_1, \ldots, y_T)'$, $\mathbf{z}$ is distributed as normal with

$$E(\mathbf{z}|\mathbf{y}) = (\mathbf{I} + v\mathbf{\Sigma}_z^{-1})^{-1}(\mathbf{y} + v\mathbf{\Sigma}_z^{-1}\mathbf{n})$$

and

$$\operatorname{cov}(\mathbf{z}|\mathbf{y}) = \sigma_e^2(\mathbf{I} + v\mathbf{\Sigma}_z^{-1})^{-1} \tag{A.3}$$

where $v = \sigma_e^2\sigma_b^{-2}$ and $\mathbf{I}$ is the identity matrix. Note that in $E(\mathbf{z}|\mathbf{y})$, $(\mathbf{I} + v\mathbf{\Sigma}_z^{-1})^{-1}(\mathbf{h} + v\mathbf{\Sigma}_z^{-1}\mathbf{h}) = \mathbf{h}$, where $\mathbf{h}$ is a vector of ones, so that for each z_t, $E(z_t|\mathbf{y})$ is a weighted average of $\mathbf{y}$ and $\mathbf{n}$ with weights summing to one.

In practice, it is often appropriate to suppose that the series began at some remote past point of time. That is, m is a large negative integer. We now distinguish between two situations.

(I) *All Zeroes of $\phi_r(B)$ Lie Outside the Unit Circle.* In this case, $\mathbf{z}$ can be thought of as a vector of observations from a stationary Gaussian process. That is, for large $-m$, $\mathbf{n}$ approaches $\mathcal{H}\mu$, and $\mathbf{\Sigma}_z$ tends to its stationary value $\dot{\mathbf{\Sigma}}_z$ the (i, j)th element of which is $\sigma_{ij} = \sum_{l=0}^{\infty} \pi_l\pi_{l-(i-j)}$ where $\pi_0 = 1$ and $\pi_l = \pi_{-l}$. Thus, the expressions in (A.3) become

$$E(\mathbf{z}|\mathbf{y}) = (\mathbf{I} + v\dot{\mathbf{\Sigma}}_z^{-1})^{-1}(\mathbf{y} + v\dot{\mathbf{\Sigma}}_z^{-1}\mathbf{h}\mu)$$

and

$$\operatorname{cov}(\mathbf{z}|\mathbf{y}) = (\mathbf{I} + v\dot{\mathbf{\Sigma}}_z^{-1})^{-1}\sigma_e^2 \ . \tag{A.4}$$

(II) *Zeroes of $\phi_r(B)$ Lie on or Outside the Unit Circle.* To see the implications here, consider first the simplest case $\phi_r(B) = \phi_{r^*}(B)(1 - B)$ where $r^* = r - 1$ and the zeroes of $\phi_{r^*}(B)$ lie outside the unit circle. Then c_t in (A.2) is of the form

$$c_t = A_1 + \sum_{l=2}^{r} A_l\alpha_l^{t-m} \ . \tag{A.5}$$

Consider now the transformation

$$\begin{bmatrix} z_1 \\ \mathbf{w} \end{bmatrix} = J\mathbf{z} \ , \quad J = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix} ,$$

i.e., $\mathbf{w} = (w_2, \ldots, w_T)'$ where $w_t = z_t - z_{t-1}$ follow the model

$$\phi_{r^*}(B)w_t = \psi_q(B)b_t \ . \tag{A.6}$$

Making use of the partitioned inverse of a matrix, we obtain

$$\boldsymbol{\Sigma}_z^{-1} = J'QJ \ , \quad Q = \begin{bmatrix} \sigma_{z_1}^{-2} - \sigma_{z_1}^{-4}\boldsymbol{\sigma}_{z_1w}'G\boldsymbol{\sigma}_{z_1w} & -\sigma_{z_1}^{-2}\boldsymbol{\sigma}_{z_1w}'G \\ -\sigma_{z_1}^{-2}G\boldsymbol{\sigma}_{z_1w} & G \end{bmatrix}$$

where $\sigma_{z_1}^{-2} = \sigma_b^2/\mathrm{Var}\,(z_1)$, $\boldsymbol{\sigma}_{z_1w}' = \sigma_b^{-2}\,\mathrm{cov}\,(z_1\mathbf{w}')$,

$$G = [\boldsymbol{\Sigma}_w - \sigma_{z_1}^{-2}\boldsymbol{\sigma}_{z_1w}\boldsymbol{\sigma}_{z_1w}']^{-1}$$

and $\boldsymbol{\Sigma}_w$ is the covariance matrix of w. Letting

$$\phi_{r^*}(B)(1 + \bar{\pi}_1B + \bar{\pi}_2B^2 + \ldots) = \psi_q(B) \ ,$$

we have from (A.2a) that $\pi_j - \pi_{j-1} = \bar{\pi}_j$, $j \geq 1$. It readily follows from (A.2) that

$$\mathrm{Var}\,(z_1) = \sigma_b^2 \sum_{j=0}^{-m} (\sum_{l=0}^{j} \bar{\pi}_l)^2 \ , \quad \bar{\pi}_0 = 1$$

and

$$\mathrm{cov}\,(z_1w_t) = \sigma_b^2 \sum_{j=t-1}^{t-1-m} \bar{\pi}_j\pi_{j-(t-1)} \ .$$

Note that since the zeroes of $\phi_{r^*}(B)$ lie outside the unit circle, $\sum_{l=0}^{\infty} |\bar{\pi}_j| < \infty$. Thus, for large $-m$, $\sigma_{z_1}^{-2}$ and $\sigma_{z_1}^{-2}\,\mathrm{cov}\,(z_1w_t)$ are of order m^{-1}; also, $\mathbf{G} = \dot{\boldsymbol{\Sigma}}_w^{-1} + \mathbf{R}$, where $\mathbf{R}$ is a matrix whose elements are of order m^{-1} and $\dot{\boldsymbol{\Sigma}}_w^{-1}$ is the stationary value of $\boldsymbol{\Sigma}_w$, the (ij)th element of which is $\sum_{l=0}^{\infty} \bar{\pi}_l\bar{\pi}_{l-(i-j)}$ with $\bar{\pi}_{-l} = \bar{\pi}_l$. Hence, to order O(1),

$$\boldsymbol{\Sigma}_z^{-1} = \dot{\boldsymbol{\Sigma}}_z^{-1} = J' \begin{bmatrix} 0 & 0 \\ 0 & \dot{\boldsymbol{\Sigma}}_w^{-1} \end{bmatrix} \mathbf{J} \tag{A.7}$$

To this order of approximation, then, $\dot{\boldsymbol{\Sigma}}_z^{-1}$ is finite, nonnegative definite and its rank is $T - 1$. It follows from (A.5) and (A.7) that, for large $-m$, the expressions in (A.3) becomes

$$E(\mathbf{z}\,|\,\mathbf{y}) = (I + v\dot{\boldsymbol{\Sigma}}_z^{-1})^{-1}y \ ,$$

and

$$\mathrm{cov}\,(\mathbf{z}\,|\,\mathbf{y}) = \sigma_e^2(I + v\dot{\boldsymbol{\Sigma}}_z^{-1})^{-1} \ . \tag{A.8}$$

The preceding result can be readily extended to the situation in which $\phi_r(B)$ in (A.1) takes the form

$$\phi_r(B) = \phi_{r^*}(B)(1 - B)^{d_1}(1 - B^c)^{d_2} \tag{A.9}$$

where d_1, d_2, c are positive integers and $r^* + d_1 + cd_2 = r$. In this case, for large $-m$, the expressions in (A.8) still hold except that the rank of $\dot{\boldsymbol{\Sigma}}_z^{-1}$ will be reduced to $T - (d_1 + cd_2)$.

A.2 The Asymptotic Form of $\dot{\boldsymbol{\Sigma}}_z^{-1}$

When the number of observations T is large, the asymptotic elements of $\dot{\boldsymbol{\Sigma}}_z^{-1}$ can be obtained from the generating function

$$\frac{\phi_r(B)\phi_r(F)}{\psi_q(B)\psi_q(F)} = X_0 + \sum_{k=1}^{\infty} X_k(B^k + F^k) \tag{A.10}$$

where $F = B^{-1}$. Specifically, let σ^{ij} be the (i, j)th element of $\dot{\boldsymbol{\Sigma}}_z^{-1}$. Then, for i not close to 1 or T, $\lim_{T\to\infty} \sigma^{ij} = X_{|i-j|}$. This result was obtained by Wise [18] for when the model of z_t in (A.1) is stationary and invertible. It also holds when $\phi_r(B)$ takes the form (A.9). This can be seen by considering the simplest nonstationary case $\phi_r(B) = \phi_{r^*}(B)(1 - B)$. Since w_t follows the stationary model in (A.6), the generating function of the asymptotic elements of $\dot{\boldsymbol{\Sigma}}_w^{-1}$ can be written

$$\frac{\phi_{r^*}(B)\phi_{r^*}(F)}{\psi_q(B)\psi_q(F)} = \tilde{x}_0 + \sum_{k=1}^{\infty} \tilde{x}_k(B^k + F^k) \ , \quad \text{say} \ . \tag{A.11}$$

From (A.7), the elements σ^{ij} of $\dot{\boldsymbol{\Sigma}}_z^{-1}$ and the elements $\tilde{\sigma}^{ij}$ of $\dot{\boldsymbol{\Sigma}}_w^{-1}$ are related by

$$\sigma^{ij} = \tilde{\sigma}^{ij} - \tilde{\sigma}^{i+1,j} - \tilde{\sigma}^{i,j+1} + \tilde{\sigma}^{i+1,j+1} \ .$$

Whence, from (A.11), for i not close to 1 or T

$$\lim_{T\to\infty} \sigma^{ij} = 2\tilde{x}_{|j-i|} - \tilde{x}_{|j-i-1|} - \tilde{x}_{|j-i+1|} \ .$$

Thus, the generating function of σ^{ij} is

$$2(\tilde{x}_0 - \tilde{x}_1) + \sum_{k=1}^{\infty} (2\tilde{x}_k - \tilde{x}_{k-1} - \tilde{x}_{k+1})(B^k + F^k)$$
$$= \frac{(1 - B)\phi_{r^*}(B)\phi_{r^*}(F)(1 - F)}{\psi_q(B)\psi_q(F)} = \frac{\phi_r(B)\phi_r(F)}{\psi_q(B)\psi_q(F)}$$

which gives (A.10). In a similar way, one can show that the generating function holds for any $\phi_r(B)$ of the form (A.9).

A.3 The Asymptotic Form of $E(z_t|y)$ and the Covariance Generating Function of $\hat{e}_t = y_t - E(z_t|y)$

From (A.1), the overall model of y_t can be written

$$\phi_r(B)(y_t - \mu) = \theta_u(B)c_t \tag{A.12}$$

where the c_t are i.i.d. $N(0, \sigma_c^2)$ and $\theta_u(B)$ is a polynomial in B of degree u, $u \leq \max\,(p, q)$, having its zeroes lying outside the unit circle. The quantities $\theta_u(B)$, σ_c^2, $\psi_q(B)$, $\phi_r(B)$, σ_b^2 and σ_e^2 are related by the covariance generating function of $\phi_r(B)(y_t - \mu)$, namely,

$$\psi_q(B)\psi_q(F)\sigma_b^2 + \phi_r(B)\phi_r(F)\sigma_e^2 = \theta_u(B)\theta_u(F)\sigma_c^2 \ . \tag{A.13}$$

Making use of (A.10), it follows from (A.4) and (A.8) that, for large T the asymptotic form of the conditional expectation of z_t given y is

$$E(z_t|\mathbf{y}) = [\theta_u(B)\theta_u(F)\sigma_c^2]^{-1}[\sigma_b^2\psi_q(B)\psi_q(F)y_t + \sigma_e^2\phi_r(B)\phi_r(F)\mu] \ . \tag{A.14}$$

In particular, if $\phi_r(B)$ contains the factor $(1 - B)$, then $\phi_r(B)\mu = 0$ and

$$E(z_t|\mathbf{y}) = [\theta_u(B)\theta_u(F)\sigma_c^2]^{-1}\sigma_b^2\psi_q(B)\psi_q(F)y_t \ , \tag{A.15}$$

which is a symmetric moving average of y_t. Further, by setting $B = 1$ in (A.13) we see that $[\psi_q(1)]^2\sigma_b^2 = [\theta_u(1)]^2\sigma_c^2$ so that the weights of (A.15) sum to one.

For the autocorrelations of $\hat{e}_t = y_t - E(z_t|y)$ it is readily verified from (A.13) and (A.14) that the autocovariance generating function of $\hat{e}_t$ is

$$\mathrm{C.G.F.}\,(\hat{e}_t) = \sigma_e^2 \frac{\sigma_e^2\phi_r(B)\phi_r(F)}{\sigma_c^2\theta_u(B)\theta_u(F)} \tag{A.16}$$

from which the variance and autocorrelations can be readily calculated.

If z_t is stationary, expressions (A.14) and (A.16) are given in [17, p. 58] for $\mu = 0$. We have shown that these expressions are equally applicable when z_t is nonstationary with $\phi_r(B)$ assuming the form (A.9).

A.4 The Conditional Expectations $E(p_t|\mathbf{y})$ and $E(s_t|\mathbf{y})$

In (3.1), let $z_t = p_t + s_t$ and let $\phi^*(B)$ be the factor common to $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ and write

$$\phi_{r_1}(B) = \phi_{r^*_1}(B)\phi^*(B) \ , \quad \phi_{r_2}(B) = \phi_{r^*_2}(B)\phi^*(B) \tag{A.17}$$

where

$$r_1^* \leq r_1 \quad \text{and} \quad r_2^* \leq r_2 \ .$$

It follows that the model for z_t in (A.1) is related to the models for p_t and s_t by

$$\phi_r(B) = \phi_{r^*_1}(B)\phi_{r^*_2}(B)\phi^*(B) \tag{A.18}$$

and

$$\sigma_b^2\psi_q(B)\psi_q(F) = \sigma_{b_2}^2\phi_{r^*_1}(B)\phi_{r^*_1}(F)\psi_{q_2}(B)\psi_{q_2}(F) + \sigma_{b_1}^2\phi_{r^*_2}(B)\phi_{r^*_2}(F)\psi_{q_1}(B)\psi_{q_1}(F) \ ,$$

where $q \leq \max\,(r_1^* + q_2, r_2^* + q_1)$.

We now obtain the conditional expectation $E(s_t|\mathbf{y})$, by first finding the expectation $E(s_t|\mathbf{z})$. From (3.1) and adopting an approach analogous to the derivation of (A.14), the asymptotic form is

$$E(s_t|z) = [\psi_q(B)\psi_q(F)\sigma_b^2]^{-1}\sigma_{b_2}^2\psi_{q_2}(B)\psi_{q_2}(F)\phi_{r^*_1}(B)\phi_{r^*_1}(F)(z_t - \mu) \tag{A.19}$$

Since p_t, s_t and e_t are assumed independent, we see that

$$E(s_t|\mathbf{y}) = E[E(s_t|\mathbf{z})|\mathbf{y}] \ .$$

On substituting $E(z_t|\mathbf{y})$ of (A.14) for z_t on the right side of (A.19), we obtain

$$E(s_t|\mathbf{y}) = \frac{\sigma_{b_2}^2\psi_{q_2}(B)\psi_{q_2}(F)\phi_{r^*_1}(B)\phi_{r^*_1}(F)}{\sigma_c^2\theta_u(B)\theta_u(F)}(y_t - \mu) \ , \tag{A.20}$$

where $\sigma_c^2\theta_u(B)\theta_u(F)$ can be readily obtained from (A.13) and (A.18).

Finally, the conditional expectation $E(p_t|\mathbf{y})$ is simply the difference,

$$E(p_t|\mathbf{y}) = E(z_t|\mathbf{y}) - E(s_t|\mathbf{y}) \ . \tag{A.21}$$

Note that when $\phi_{r^*_1}(B)$ contains the factor $(1 - B)$, the constant μ in (A.14) and (A.20) vanishes. In this case, (i) $E(s_t|\mathbf{y})$ is a symmetric moving average with weights summing to zero, (ii) $E(z_t|\mathbf{y})$ and therefore $E(p_t|\mathbf{y})$ are symmetric moving averages with weights summing to one. This situation occurs for the model (3.2) because $\phi_{r_1}(B)$ and $\phi_{r_2}(B)$ have a common factor $(1 - B)$ so that $\phi_{r^*_1}(B) = (1 - B)$, and it corresponds precisely to the nature of the moving average operators for $\hat{s}$ and $\hat{p}$ in the census procedure.

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