L13: Time series essentials

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Today we are going to learn...

1. **The lag operators**
2. **The difference operators**
3. **Linear models for stationary time series**
4. **Stationary**
5. **White noise**

The lag operators

- Suppose X_t is the GDP for past ten years $(t = 1, 2, ..., 10)$.
- The X_{t-1} is called the GDP with a lapse of time (i.e. **a lag**).
- In time series analysis, the **lag operator** or **backshift (**L **is the notation) operator** operates on an element of a time series to produce the previous element.
    - Given some time series $X = \{X_1, X_2, \dots\}$
    - then $LX_t = X_{t-1}$ for all $t > 1$
    - or equivalently $X_t = LX_{t+1}$ for all $t \geqslant 1$
    - and this also works $L^{-1}X_t = X_{t+1}$
    - and $L^k X_t = X_{t-k}$.
- How many lags can we have maximumly?

Why lags?

- Psychological reasons
  - Those who become instant millionaires by winning lotteries may not change the lifestyles intermediately.
  - People do not change their consumption habits immediately following a price decrease or an income increase.
- Technological reasons
  - We obtained the data from the stock market maybe always 5 seconds behind real time due to technological reasons.
  - The data obtain from authorities maybe always delayed due to confidential reasons.
- Institutional reasons
  - Employers often give their employees a choice among several health insurance plans, but once a choice is made, an employee may not switch to another plan for at least 1 year.
  - You are only allowed to take the re-exam next year if you fail this time.

The difference operator

- Assume your yearly salaries are X_t, how much do you earn compared to previous year?
- That should be $\Delta_t X_t = X_t - X_{t-1}$ which is called **the first difference** operator in time series analysis.
- It could be written in terms of lag operators $\Delta_t X_t = X_t - X_{t-1} = (1 - L)X_t$
- Similarly, the second difference operator works as follows:

$$\begin{aligned}
\Delta(\Delta X_t) &= \Delta X_t - \Delta X_{t-1} \\
\Delta^2 X_t &= (1 - L)\Delta X_t \\
\Delta^2 X_t &= (1 - L)(1 - L)X_t \\
\Delta^2 X_t &= (1 - L)^2 X_t \ .
\end{aligned}$$

Autocovariance and Autocorrelation Functions

- The covariance between y_t and its value at another time period, say, y_{t+k} is called the **autocovariance** at lag k,

$$\gamma_k = Cov(y_t, y_{t+k}) = E((y_t - \mu)(y_{t+k} - \mu))$$

- The collection of the values of γ_k, $k = 0, 1, 2, ...$ is called the **autocovariance function**.
- The autocovariance at lag $k = 0$ is just the variance of the time series;
- The **autocorrelation coefficient** at lag k is

$$\rho_k = \frac{Cov(y_t, y_{t+k})}{Var(y_t)} = \frac{\gamma_k}{\gamma_0}$$

- Note that by definition $\rho_0 = 1$.
- The collection of the values of ρ_k. $k = 0.1.2....$ is called the **autocorrelation function** (ACF).
- The ACF is independent of the scale of measurement of the time series.
- The autocorrelation function is symmetric around zero $\rho_k = \rho_{-k}$.

Sample autocorrelation function particial autocorrelation I

- It is necessary to estimate the autocovariance and autocorrelation functions from a time series of finite length. The usual estimate of the autocovariance function is

$$c_k = \hat{\gamma}_k = \frac{1}{T} \sum_{t=1}^{T-k} (y_t - \bar{y})(y_{t+k} - \bar{y})$$

- The autocorrelation function is estimated by the **sample autocorrelation function**

$$r_k = \hat{\rho}_k = \frac{c_k}{c_0}$$

Sample autocorrelation function particial autocorrelation II

- **The partial correlation** is the correlation between two variables after being adjusted for a common factor that may be affecting them.
- The partial correlation between X and Y after adjusting for Z is defined as

$$\mathrm{Corr}(X - \hat{X}, Y - \hat{Y})$$

  where $\hat{X} = a_1 + b_1 Z$ and $\hat{Y} = a_2 + b_2 Z$
- The **partial autocorrelation function** between y_t and y_{t-k} is the autocorrelation between y_t and y_{t-k} after adjusting for y_{t-1}, y_{t-2},...,y_{t-k+1} and y_{t-k}.

Linear models for stationary time series

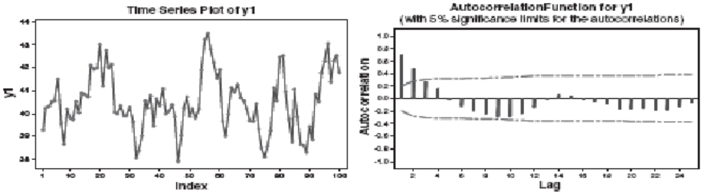
- Consider a linear operation from one time series x_t to another time series y_t

$$y_t = \sum_{i=-\infty}^{\infty} \psi_i x_{t-i}$$

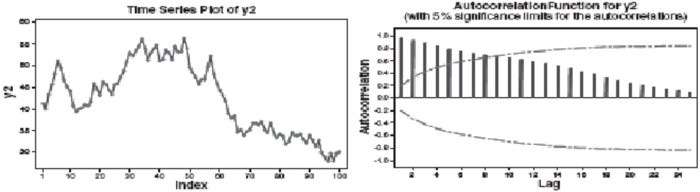
  which is called a **linear filter**.
- The linear filter should have the flowing properties
  - **Time-invariant**: ψ do not depend on time.
  - **Stable** if $\sum_{i=-\infty}^{\infty} |\psi_i| < \infty$

Stationary

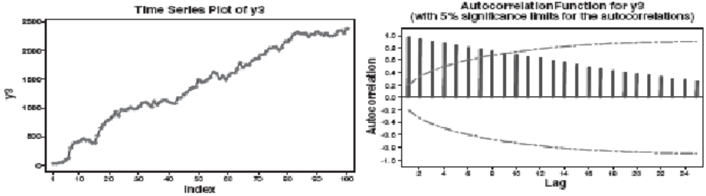
- A **stationary time series** exhibits similar "statistical behavior" in time and this is often characterized as a **constant** probability distribution (in terms of mean, variance, skewness, kurtosis, or even higher moments) in time.
- If we only consider the first two moments of the time series, we are talking about **weak stationarity** which is defined
  - The expected value of the time series does not depend on time.
  - The autocovariance function defined as $Cov(y_t, y_{t-k})$ for any lag k is only a function of k and not time t.
- If the time series is not stationary, it can be examined by observing **autocorrelation function (ACF)** and **particial autocorrelation function (PACF)**.



(a) $y_{1,t} = 10 + 0.75 y_{1,t-1} + \varepsilon_t$



(b) $y_{2,t} = 2 + 0.95 y_{2,t-1} + \varepsilon_t$



(c) $y_{3,t} = 20 + y_{3,t-1} + \varepsilon_t$

Example: Calculating ACF with R.

White noise

- If a time series consists of uncorrelated observations and has constant variance. we say that it is **white noise**.
- If in addition, the observations in this time series are normally distributed, the time series is **Gaussian white noise**.
- If a time series is white noise, the distribution of the sample autocorrelation coefficient at lag k in large samples is approximately normal with mean zero and variance $1/T$.

Stationary time series

- Many time series do not exhibit a stationary behavior.
- The stationarity is in fact a rarity in real life.
- However it provides a foundation to build upon since (as we will see later on) if the time series in not stationary, its first difference $(y_t - y_{t-1})$ will often be stationary.

Stationary time series

- For a time-invariant and stable linear filter and a stationary input time series x_t

$$y_t = \sum_{i=-\infty}^{\infty} \psi_i x_{t-i}$$

  with $\mu_x = E(x_t)$ and $\gamma_x(k) = Cov(x_t, x_{t+k})$.
- The output time series y_t is also a stationary time series where

$$E(y_t) = \mu_y = \sum_{-\infty}^{\infty} \psi_i \mu_x$$

$$\gamma_y(k) = Cov(y_t, y_{t+k}) = \sum_{-\infty}^{\infty} \psi_i \psi_j \gamma_x(i - j + k)$$

Stationary time series

- The following stable linear process with white noise time series, ϵ_t,

$$y_t = \mu + \sum_{-\infty}^{\infty} \psi_i \epsilon_{t-i}$$

is also stationary where ϵ_t has $E(\epsilon_t) = 0$ and

$$\gamma_\epsilon(k) = Cov(\epsilon_t, \epsilon_{t+k}) = \begin{cases} \sigma^2 & k = 0 \\ 0 & otherwise \end{cases}$$

- The autocovariance function of y_t is

$$\gamma_y(k) = Cov(y_t, y_{t+k}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \psi_i \psi_j \gamma_\epsilon(i - j + k)$$

$$= \sigma^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$$

Reading material

Introduction to Time Series Analysis and Forecasting by Montgomery, Jennings and Kulahci (**Chapter 5**)

Available at http://feng.li/files/ec2013fall/ARIMA-Models.pdf