10 Helmut Lütkepohl (1996), *Handbook of Matrices*, Chapter 10

Vector and Matrix Derivatives

In this chapter all vectors and matrices are assumed to be real unless otherwise stated. Differentiable always means continuously differentiable of sufficiently high order so that all expressions are well-defined.

10.1 Notation

Let $f(x)$ be a differentiable real valued function of the real $(m \times 1)$ vector $x = (x_1, \dots, x_m)'$.

$$\frac{\partial f}{\partial x} \equiv \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{bmatrix} \quad \text{or} \quad \frac{\partial f(x)}{\partial x} \equiv \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_m} \end{bmatrix} \quad (m \times 1)$$

and

$$\frac{\partial f}{\partial x'} \equiv \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}\right) \quad \text{or} \quad \frac{\partial f(x)}{\partial x'} \equiv \left(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_m}\right) \quad (1 \times m)$$

are vectors of first order partial derivatives. $\partial f(x)/\partial x$ is sometimes called the **gradient vector** of $f(x)$.

$$\left.\frac{\partial f}{\partial x}\right|_{x_0} = \left.\frac{\partial f}{\partial x}\right|_{x=x_0} = \frac{\partial f(x_0)}{\partial x} \equiv \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{x=x_0} \\ \vdots \\ \left.\frac{\partial f}{\partial x_m}\right|_{x=x_0} \end{bmatrix} \quad (m \times 1)$$

and

$$\left.\frac{\partial f}{\partial x'}\right|_{x_0} = \left.\frac{\partial f}{\partial x'}\right|_{x=x_0} = \frac{\partial f(x_0)}{\partial x'} \equiv \left[\frac{\partial f(x_0)}{\partial x}\right]' \quad (1 \times m)$$

are vectors of **first order partial derivatives evaluated at** the $(m \times 1)$ vector x_0.

$$\frac{\partial^2 f}{\partial x \partial x'} = \frac{\partial^2 f(x)}{\partial x \partial x'} \equiv \left[\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right] = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m} \end{bmatrix} \quad (m \times m)$$

is the **Hessian matrix** of second order partial derivatives of $f(x)$ and

$$\left.\frac{\partial^2 f}{\partial x \partial x'}\right|_{x=x_0} = \left.\frac{\partial^2 f}{\partial x \partial x'}\right|_{x_0} = \frac{\partial^2 f(x_0)}{\partial x \partial x'} \equiv \left[\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{x=x_0}\right] \quad (m \times m)$$

is the **Hessian matrix** of second order partial derivatives of $f(x)$ **evaluated at** $x = x_0$.

Let $f(X)$ be a differentiable real valued function of the real $(m \times n)$ matrix $X = [x_{ij}]$.

$$\frac{\partial f}{\partial X} = \frac{\partial f(X)}{\partial X} \equiv \left[\frac{\partial f(X)}{\partial x_{ij}}\right] = \begin{bmatrix} \frac{\partial f}{\partial x_{11}} & \cdots & \frac{\partial f}{\partial x_{1n}} \\ \vdots & & \vdots \\ \frac{\partial f}{\partial x_{m1}} & \cdots & \frac{\partial f}{\partial x_{mn}} \end{bmatrix} \quad (m \times n)$$

is the matrix of first order partial derivatives of $f(X)$ and

$$\left.\frac{\partial f}{\partial X}\right|_{X=X_0} = \left.\frac{\partial f}{\partial X}\right|_{X_0} = \frac{\partial f(X_0)}{\partial X} \equiv \left[\left.\frac{\partial f}{\partial x_{ij}}\right|_{X=X_0}\right] \quad (m \times n)$$

is the matrix of first order partial derivatives of $f(X)$ evaluated at the $(m \times n)$ matrix X_0.

$$\frac{\partial^2 f}{\partial \mathrm{vec}(X) \partial \mathrm{vec}(X)'} = \frac{\partial^2 f(X)}{\partial \mathrm{vec}(X) \partial \mathrm{vec}(X)'} \quad (mn \times mn)$$

is the **Hessian matrix** of second order partial derivatives of $f(X)$. Here it is important to note the order in which the partial derivatives are arranged. They have the same order as for the function $f(\mathrm{vec}(X))$. Accordingly,

$$\left.\frac{\partial^2 f}{\partial \mathrm{vec}(X) \partial \mathrm{vec}(X)'}\right|_{X=X_0} = \frac{\partial^2 f(X_0)}{\partial \mathrm{vec}(X) \partial \mathrm{vec}(X)'}$$

is the Hessian matrix evaluated at $X = X_0$.

Let $y(x) = [y_1(x), \ldots, y_n(x)]'$ be a real $(n \times 1)$ vector of differentiable functions of the real $(m \times 1)$ vector $x = (x_1, \ldots, x_m)'$, that is, y is a function mapping a subset of $\mathbb{R}^m$ on a subset of $\mathbb{R}^n$.

$$\frac{\partial y}{\partial x'} = \frac{\partial y(x)}{\partial x'} \equiv \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{bmatrix} \quad (n \times m)$$

and

$$\frac{\partial y'}{\partial x} = \frac{\partial y(x)'}{\partial x} \equiv \left(\frac{\partial y}{\partial x'}\right)' \quad (m \times n)$$

are matrices of first order partial derivatives of $y(x)$. $\partial y/\partial x'$ is sometimes called the **Jacobian matrix** of y and $\partial y'/\partial x$ is sometimes called the **gradient** of y.

$$\left.\frac{\partial y}{\partial x'}\right|_{x=x_0} = \left.\frac{\partial y}{\partial x'}\right|_{x_0} = \frac{\partial y(x_0)}{\partial x'} \equiv \begin{bmatrix} \left.\frac{\partial y_1}{\partial x_1}\right|_{x=x_0} & \cdots & \left.\frac{\partial y_1}{\partial x_m}\right|_{x=x_0} \\ \vdots & & \vdots \\ \left.\frac{\partial y_n}{\partial x_1}\right|_{x=x_0} & \cdots & \left.\frac{\partial y_n}{\partial x_m}\right|_{x=x_0} \end{bmatrix}$$

is the Jacobian matrix evaluated at $x = x_0$ and its transpose is the gradient of y evaluated at x_0. For $m = n$,

$$\det\left(\frac{\partial y(x)}{\partial x'}\right) = \det\left(\frac{\partial y}{\partial x'}\right)$$

is the **Jacobian** or **Jacobian determinant** of $y(x)$. The **Hessian matrix** of $y(x)$ is

$$\frac{\partial \operatorname{vec}(\partial y/\partial x')}{\partial x'}.$$

For a matrix valued function of a single variable, $A : S \subset \mathbb{R} \to \mathbb{R}^{m\times n}$, $x \mapsto A(x) = [a_{ij}(x)]$, the matrix of first order derivatives is

$$\frac{dA(x)}{dx} = \frac{dA}{dx} \equiv \begin{bmatrix} \frac{da_{11}}{dx} & \cdots & \frac{da_{1n}}{dx} \\ \vdots & & \vdots \\ \frac{da_{m1}}{dx} & \cdots & \frac{da_{mn}}{dx} \end{bmatrix}$$

and the corresponding matrix of first order derivatives evaluated at x_0 is

$$\frac{dA(x_0)}{dx} = \left.\frac{dA}{dx}\right|_{x=x_0} \equiv \left[\left.\frac{da_{ij}}{dx}\right|_{x=x_0}\right].$$

In the remainder of this chapter all functions are assumed to be differentiable.

10.2 Gradients and Hessian Matrices of Real Valued Functions with Vector Arguments

10.2.1 Gradients

(1) x $(m \times 1), c \in \mathbb{R}$ constant: $\quad \frac{\partial c}{\partial x} = O_{m\times 1}.$

(2) (Linearity)
x $(m \times 1)$, $f(x), g(x)$ real valued functions, $c_1, c_2 \in \mathbb{R}$:

$$\frac{\partial[c_1 f(x) + c_2 g(x)]}{\partial x} = c_1\frac{\partial f(x)}{\partial x} + c_2\frac{\partial g(x)}{\partial x}.$$

(3) (Product rule)
x $(m \times 1)$, $f(x), g(x)$ real valued functions:

$$\frac{\partial f(x)g(x)}{\partial x} = f(x)\frac{\partial g(x)}{\partial x} + g(x)\frac{\partial f(x)}{\partial x}.$$

(4) x $(m \times 1)$, $f(x), g(x), h(x)$ real valued functions:

$$\frac{\partial f(x)g(x)h(x)}{\partial x} = f(x)g(x)\frac{\partial h(x)}{\partial x} + f(x)h(x)\frac{\partial g(x)}{\partial x} + g(x)h(x)\frac{\partial f(x)}{\partial x}.$$

(5) (Ratio rule)
x $(m \times 1)$, $f(x), g(x) \neq 0$ real valued functions:

$$\frac{\partial[f(x)/g(x)]}{\partial x} = \frac{1}{g(x)^2}\left[g(x)\frac{\partial f(x)}{\partial x} - f(x)\frac{\partial g(x)}{\partial x}\right].$$

(6) (Chain rule)
x $(m \times 1)$, $y(x)$ $(n \times 1)$, $f(y)$ a real valued function:

$$\frac{\partial f(y(x))}{\partial x} = \frac{\partial y(x)'}{\partial x}\frac{\partial f(y)}{\partial y}.$$

(7) x, a $(m \times 1)$: $\quad \frac{\partial a'x}{\partial x} = a.$

(8) x $(m \times 1)$, $a, y(x)$ $(n \times 1)$: $\quad \frac{\partial a'y(x)}{\partial x} = \frac{\partial y(x)'}{\partial x}a.$

(9) x $(m \times 1)$, A $(m \times m)$: $\quad \frac{\partial x'Ax}{\partial x} = (A + A')x.$

(10) $x\ (m\times 1), a\ (n\times 1), A\ (n\times m), B\ (n\times n)$ symmetric:

$$\frac{(a-Ax)'B(a-Ax)}{\partial x} = -2A'B(a-Ax).$$

(11) $x\ (m\times 1), y(x)\ (n\times 1), A\ (n\times n)$:

$$\frac{\partial y(x)'Ay(x)}{\partial x} = \frac{\partial y(x)'}{\partial x}(A+A')y(x).$$

(12) $x\ (m\times 1), y(x)\ (n\times 1), z(x)\ (p\times 1), A\ (n\times p)$:

$$\frac{\partial y(x)'Az(x)}{\partial x} = \frac{\partial y(x)'}{\partial x}Az(x) + \frac{\partial z(x)'}{\partial x}A'y(x).$$

Note: (1) – (6) are standard rules of calculus for functions with vector arguments. For the chain rule see, e.g., Magnus & Neudecker (1988, Chapter 5, Sec. 12). (7) and (8) follow from (2). (9) is obtained from standard rules for derivatives by writing $x'Ax$ in summation notation and computing the derivative for each individual component of x. (10) and (11) follow from (9) and the chain rule (6). (12) is a consequence of the product and chain rules.

10.2.2 Hessian Matrices

(1) $x, a\ (m\times 1)$: $\dfrac{\partial^2 a'x}{\partial x\partial x'} = O_{m\times m}.$

(2) $x\ (m\times 1), A\ (m\times m)$: $\dfrac{\partial^2 x'Ax}{\partial x\partial x'} = A + A'.$

(3) $x\ (m\times 1), a\ (n\times 1), A\ (n\times m), B\ (n\times n)$ symmetric:

$$\frac{\partial^2(a-Ax)'B(a-Ax)}{\partial x\partial x'} = 2A'BA.$$

(4) $x\ (m\times 1), y(x)\ (n\times 1), A\ (n\times n)$:

$$\begin{aligned}\frac{\partial^2 y(x)'Ay(x)}{\partial x\partial x'} &= \frac{\partial y(x)'}{\partial x}(A+A')\frac{\partial y(x)}{\partial x'} \\ &\quad + [y(x)'(A+A')\otimes I_m]\frac{\partial}{\partial x'}\left[\text{vec}\left(\frac{\partial y(x)'}{\partial x}\right)\right].\end{aligned}$$

(5) $x\ (m\times 1), y(x)\ (n\times 1), z(x)\ (p\times 1), A\ (n\times p)$:

$$\begin{aligned}\frac{\partial^2 y(x)'Az(x)}{\partial x\partial x'} &= \frac{\partial y(x)'}{\partial x}A\frac{\partial z(x)}{\partial x'} + \frac{\partial z(x)'}{\partial x}A'\frac{\partial y(x)}{\partial x'} \\ &\quad + [z(x)'A'\otimes I_m]\frac{\partial}{\partial x'}\left[\text{vec}\left(\frac{\partial y(x)'}{\partial x}\right)\right] \\ &\quad + [y(x)'A\otimes I_m]\frac{\partial}{\partial x'}\left[\text{vec}\left(\frac{\partial z(x)'}{\partial x}\right)\right].\end{aligned}$$

Note: All results are simple consequences of the results in the previous subsection and the rules for vector valued functions with vector arguments given in the following sections.

10.3 Derivatives of Real Valued Functions with Matrix Arguments

10.3.1 General and Miscellaneous Rules

(1) $X\ (m\times n), c\in\mathbb{R}$ constant: $\dfrac{\partial c}{\partial X} = O_{m\times n}.$

(2) (Linearity)
$X\ (m\times n), f(X), g(X)$ real valued functions, $c_1, c_2\in\mathbb{R}$:

$$\frac{\partial[c_1f(X)+c_2g(X)]}{\partial X} = c_1\frac{\partial f(X)}{\partial X} + c_2\frac{\partial g(X)}{\partial X}.$$

(3) (Product rule)
$X\ (m\times n), f(X), g(X)$ real valued functions:

$$\frac{\partial f(X)g(X)}{\partial X} = f(X)\frac{\partial g(X)}{\partial X} + g(X)\frac{\partial f(X)}{\partial X}.$$

(4) $X\ (m\times n), f(X), g(X), h(X)$ real valued functions:

$$\begin{aligned}\frac{\partial f(X)g(X)h(X)}{\partial X} &= f(X)g(X)\frac{\partial h(X)}{\partial X} \\ &\quad + f(X)h(X)\frac{\partial g(X)}{\partial X} + g(X)h(X)\frac{\partial f(X)}{\partial X}.\end{aligned}$$

(5) (Ratio rule)
$X\ (m\times n), f(X), g(X)\neq 0$ real valued functions:

$$\frac{\partial[f(X)/g(X)]}{\partial X} = \frac{1}{g(X)^2}\left[g(X)\frac{\partial f(X)}{\partial X} - f(X)\frac{\partial g(X)}{\partial X}\right].$$

(6) (Chain rule)
$X\ (m\times n),\ y = f(X),\ g(y)$ real valued functions:

$$\frac{\partial g(f(X))}{\partial X} = \frac{\mathrm{d}g(y)}{\mathrm{d}y}\frac{\partial f(X)}{\partial X}.$$

(7) $X\ (m\times n), f(X)$ a real valued function:

$$\text{vec}\left(\frac{\partial f(X)}{\partial X}\right) = \frac{\partial f(X)}{\partial\,\text{vec}(X)}.$$

(8) $X\ (m \times n), a\ (m \times 1), b\ (n \times 1)$:

$$\frac{\partial a'Xb}{\partial X} = ab'.$$

(9) $X\ (m \times m), a, b\ (m \times 1)$:

$$\frac{\partial a'X^i b}{\partial X} = \sum_{j=0}^{i-1} (X^j)' ab' (X^{i-1-j})', \quad i = 1, 2, \ldots$$

(10) $X\ (m \times m)$ nonsingular, $a, b\ (m \times 1)$:

$$\frac{\partial a'X^{-1}b}{\partial X} = -(X^{-1})'ab'(X^{-1})'.$$

(11) $X\ (m \times n), a, b\ (n \times 1)$:

$$\frac{\partial a'X'Xb}{\partial X} = X(ba' + ab').$$

(12) $X\ (m \times n), a, b\ (m \times 1)$:

$$\frac{\partial a'XX'b}{\partial X} = (ba' + ab')X.$$

(13) $X\ (m \times m)$ symmetric with simple eigenvalue λ and corresponding eigenvector v satisfying $v'v = 1$:

$$\frac{\partial \lambda}{\partial X} = vv'.$$

Note: (1) – (7) are standard results from calculus. (9) and (10) are given in Magnus & Neudecker (1988, Chapter 9, Sec. 13) and (8) is a special case of (9). (11) and (12) follow from results given in Magnus & Neudecker (1988, Chapter 9, Sec. 9) by noting that $a'Ab = \mathrm{tr}(Aba')$. Result (13) is from Magnus & Neudecker (1988, Chapter 9, Sec. 11).

10.3.2 Derivatives of the Trace

First Order Derivatives

(1) $X\ (m \times m)$: $\dfrac{\partial \mathrm{tr}(X)}{\partial X} = \dfrac{\partial \mathrm{tr}(X')}{\partial X} = I_m.$

(2) $X\ (m \times n),\ A\ (n \times m)$: $\dfrac{\partial \mathrm{tr}(AX)}{\partial X} = \dfrac{\partial \mathrm{tr}(XA)}{\partial X} = A'.$

(3) $X\ (m \times n),\ A\ (m \times n)$: $\dfrac{\partial \mathrm{tr}(X'A)}{\partial X} = \dfrac{\partial \mathrm{tr}(AX')}{\partial X} = A.$

(4) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times p)$: $\dfrac{\partial \mathrm{tr}(AXB)}{\partial X} = A'B'.$

(5) $X\ (m \times n),\ A\ (p \times n),\ B\ (m \times p)$: $\dfrac{\partial \mathrm{tr}(AX'B)}{\partial X} = BA.$

(6) $X\ (m \times m)$: $\dfrac{\partial \mathrm{tr}(X^2)}{\partial X} = 2X',$

$$\frac{\partial \mathrm{tr}(X^i)}{\partial X} = i(X')^{i-1}, \quad i = 1, 2, \ldots$$

(7) $X, A, B\ (m \times m)$:

$$\frac{\partial \mathrm{tr}(AX^iB)}{\partial X} = \sum_{j=0}^{i-1} (X^j)'A'B'(X^{i-1-j})', \quad i = 1, 2, \ldots$$

(8) $X\ (m \times n)$: $\dfrac{\partial \mathrm{tr}(X'X)}{\partial X} = \dfrac{\partial \mathrm{tr}(XX')}{\partial X} = 2X.$

(9) $X\ (m \times n),\ A\ (m \times m)$: $\dfrac{\partial \mathrm{tr}(X'AX)}{\partial X} = (A + A')X.$

(10) $X\ (m \times n),\ A\ (m \times m)$ symmetric : $\dfrac{\partial \mathrm{tr}(X'AX)}{\partial X} = 2AX.$

(11) $X\ (m \times n),\ A\ (n \times n)$: $\dfrac{\partial \mathrm{tr}(XAX')}{\partial X} = X(A + A').$

(12) $X\ (m \times n),\ A\ (n \times n)$ symmetric : $\dfrac{\partial \mathrm{tr}(XAX')}{\partial X} = 2XA.$

(13) $X, A\ (m \times m)$: $\dfrac{\partial \mathrm{tr}(XAX)}{\partial X} = X'A' + A'X'.$

(14) $X\ (m \times n),\ A\ (p \times m)$: $\dfrac{\partial \mathrm{tr}(AXX'A')}{\partial X} = 2A'AX.$

(15) $X\ (m \times n),\ A\ (p \times n)$: $\dfrac{\partial \mathrm{tr}(AX'XA')}{\partial X} = 2XA'A.$

(16) $X\ (m \times n),\ A\ (p \times m),\ B\ (m \times p)$: $\dfrac{\partial \mathrm{tr}(AXX'B)}{\partial X} = (BA + A'B')X.$

(17) $X\ (m \times n),\ A\ (p \times n),\ B\ (n \times p)$: $\dfrac{\partial \mathrm{tr}(AX'XB)}{\partial X} = X(BA + A'B').$

(18) $X\ (m \times n),\ A\ (n \times n),\ B\ (m \times m)$: $\dfrac{\partial \mathrm{tr}(XAX'B)}{\partial X} = B'XA' + BXA.$

(19) $X\ (m \times n),\ A, B\ (n \times m)$: $\dfrac{\partial \mathrm{tr}(XAXB)}{\partial X} = B'X'A' + A'X'B'.$

(20) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times n),\ C\ (m \times p)$:

$$\frac{\partial \mathrm{tr}(AXBX'C)}{\partial X} = A'C'XB' + CAXB.$$

(21) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times m),\ C\ (n \times p)$:

$$\frac{\partial \mathrm{tr}(AXBXC)}{\partial X} = A'C'X'B' + B'X'A'C'.$$

(22) $X\ (m \times m)$ nonsingular : $\dfrac{\partial \mathrm{tr}(X^{-1})}{\partial X} = -(X^{-2})'.$

(23) $X\ (m \times m)$ nonsingular, $A, B\ (m \times m)$:

$$\frac{\partial \mathrm{tr}(AX^{-1}B)}{\partial X} = -(X^{-1}BAX^{-1})'.$$

(24) $X\ (m \times n),\ F(X)\ (p \times p)$: $\dfrac{\partial \mathrm{tr}[F(X)]}{\partial \mathrm{vec}(X)'} = \mathrm{vec}(I_p)'\dfrac{\partial \mathrm{vec}\, F(X)}{\partial \mathrm{vec}(X)'}.$

(25) $X\ (m \times n),\ F(X)\ (p \times q),\ G(X)\ (r \times s),\ A\ (q \times r),\ B\ (s \times p)$:

$$\frac{\partial \mathrm{tr}[F(X)AG(X)B]}{\partial \mathrm{vec}(X)'} =$$

$$\mathrm{vec}(I_p)'\left[[B'G(X)'A' \otimes I_p]\frac{\partial \mathrm{vec}\, F(X)}{\partial \mathrm{vec}(X)'} + [B' \otimes F(X)A]\frac{\partial \mathrm{vec}\, G(X)}{\partial \mathrm{vec}(X)'}\right].$$

Hessian Matrices

(1) $X\ (m \times m)$: $\dfrac{\partial^2 \mathrm{tr}(X)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = O_{m^2 \times m^2}.$

(2) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times p)$: $\dfrac{\partial^2 \mathrm{tr}(AXB)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = O_{mn \times mn}.$

(3) $X\ (m \times n),\ A\ (p \times n),\ B\ (m \times p)$: $\dfrac{\partial^2 \mathrm{tr}(AX'B)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = O_{mn \times mn}.$

(4) $X\ (m \times m)$: $\dfrac{\partial^2 \mathrm{tr}(X^2)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = 2K_{mm},$

$$\frac{\partial^2 \mathrm{tr}(X^i)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = iK_{mm}\left(\sum_{j=0}^{i-2}(X')^{i-2-j} \otimes X^j\right), \quad i = 2, 3, \ldots$$

(5) $X\ (m \times n)$: $\dfrac{\partial^2 \mathrm{tr}(X'X)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = 2I_{mn}.$

(6) $X\ (m \times n),\ A\ (m \times m)$: $\dfrac{\partial^2 \mathrm{tr}(X'AX)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = I_n \otimes (A + A').$

(7) $X\ (m \times n),\ A\ (m \times m)$ symmetric: $\dfrac{\partial^2 \mathrm{tr}(X'AX)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = 2(I_n \otimes A).$

(8) $X\ (m \times n),\ A\ (n \times n)$: $\dfrac{\partial^2 \mathrm{tr}(XAX')}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = (A' + A) \otimes I_m.$

(9) $X\ (m \times n),\ A\ (n \times n)$ symmetric: $\dfrac{\partial^2 \mathrm{tr}(XAX')}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = 2(A \otimes I_m).$

(10) $X, A\ (m \times m)$: $\dfrac{\partial^2 \mathrm{tr}(XAX)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = (A \otimes I_m + I_m \otimes A')K_{mm}.$

(11) $X\ (m \times n),\ A\ (p \times m)$: $\dfrac{\partial^2 \mathrm{tr}(AXX'A')}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = 2(I_n \otimes A'A).$

(12) $X\ (m \times n),\ A\ (p \times n)$: $\dfrac{\partial^2 \mathrm{tr}(AX'XA')}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = 2(A'A \otimes I_m).$

(13) $X\ (m \times n),\ A\ (p \times m),\ B\ (m \times p)$:

$$\frac{\partial^2 \mathrm{tr}(AXX'B)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = I_n \otimes (BA + A'B').$$

(14) $X\ (m \times n),\ A\ (p \times n),\ B\ (n \times p)$:

$$\frac{\partial^2 \mathrm{tr}(AX'XB)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = (BA + A'B') \otimes I_m.$$

(15) $X\ (m \times n),\ A\ (n \times n),\ B\ (m \times m)$:

$$\frac{\partial^2 \mathrm{tr}(XAX'B)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = A \otimes B' + A' \otimes B.$$

(16) $X\ (m \times n),\ A, B\ (n \times m)$:

$$\frac{\partial^2 \mathrm{tr}(XAXB)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = (A \otimes B' + B \otimes A')K_{mn}.$$

(17) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times n),\ C\ (m \times p)$:

$$\frac{\partial^2 \mathrm{tr}(AXBX'C)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = B \otimes A'C' + B' \otimes CA.$$

(18) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times m),\ C\ (n \times p)$:

$$\frac{\partial^2 \mathrm{tr}(AXBXC)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = (B \otimes A'C' + CA \otimes B')K_{mn}.$$

(19) X $(m \times m)$ nonsingular :

$$\frac{\partial^2 \mathrm{tr}(X^{-1})}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = K_{mm}(X'^{-2} \otimes X^{-1} + X'^{-1} \otimes X^{-2}).$$

(20) X $(m \times m)$ nonsingular, A, B $(m \times m)$:

$$\frac{\partial^2 \mathrm{tr}(AX^{-1}B)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'}$$
$$= K_{mm}(X'^{-1}A'B'X'^{-1} \otimes X^{-1} + X'^{-1} \otimes X^{-1}BAX^{-1}).$$

Note: The results of this subsection are partly given in Magnus & Neudecker (1988, Chapter 9, Sec. 9 and 13) and partly follow from their results by straightforward application of the rules for the trace and matrix derivatives, notably the chain rule and the product rules of Section 10.5.

10.3.3 Derivatives of Determinants

(1) X $(m \times m)$ nonsingular : $\dfrac{\partial \det(X)}{\partial X} = \det(X)(X')^{-1} = (X^{adj})'$.

(2) X $(m \times n)$, A $(p \times m)$, B $(n \times p)$, $\det(AXB) \neq 0$:

$$\frac{\partial \det(AXB)}{\partial X} = \det(AXB)A'(B'X'A')^{-1}B'.$$

(3) X $(m \times n)$, $\mathrm{rk}(X) = m$: $\dfrac{\partial \det(XX')}{\partial X} = 2\det(XX')(XX')^{-1}X$.

(4) X $(m \times n)$, $\mathrm{rk}(X) = n$: $\dfrac{\partial \det(X'X)}{\partial X} = 2\det(X'X)X(X'X)^{-1}$.

(5) X $(m \times m)$ nonsingular:

(a) $\dfrac{\partial \det(X^2)}{\partial X} = 2(\det X)^2(X')^{-1}$.

(b) $\dfrac{\partial \det(X^{-1})}{\partial X} = -(\det X)^{-1}(X')^{-1}$.

(c) $\dfrac{\partial \det(X^i)}{\partial X} = i(\det X)^i(X')^{-1}, \quad i = \pm 1, \pm 2, \ldots$

(6) X, A, B $(m \times m)$, $\det(AX^iB) \neq 0$:

$$\frac{\partial \det(AX^iB)}{\partial X} = \det(AX^iB)\sum_{j=1}^{i-1}[(X^{i-1-j})B(AX^iB)^{-1}AX^j]'.$$

(7) X $(m \times n)$, A $(p \times m)$, B $(n \times n)$, C $(m \times p)$, $\det(AXBX'C) \neq 0$:

$$\frac{\partial \det(AXBX'C)}{\partial X} =$$
$$\det(AXBX'C)\{C[AXBX'C]^{-1}AXB + A'[C'XB'X'A']^{-1}C'XB'\}.$$

(8) X $(m \times n)$, A $(p \times n)$, B $(m \times m)$, C $(n \times p)$, $\det(AX'BXC) \neq 0$:

$$\frac{\partial \det(AX'BXC)}{\partial X} =$$
$$\det(AX'BXC)\{BXC[AX'BXC]^{-1}A + B'XA'[C'X'B'XA']^{-1}C'\}.$$

(9) X $(m \times n)$, A $(p \times m)$, B $(n \times m)$, C $(n \times p)$, $\det(AXBXC) \neq 0$:

$$\frac{\partial \det(AXBXC)}{\partial X} =$$
$$\det(AXBXC)\{C[AXBXC]^{-1}AXB + BXC[AXBXC]^{-1}A\}'.$$

(10) X $(m \times m)$, $\det(X) > 0$: $\dfrac{\partial \ln \det(X)}{\partial X} = (X')^{-1}$.

(11) X $(m \times n)$, A $(m \times m)$ positive definite, $\det(X'AX) > 0$:

$$\frac{\partial \ln \det(X'AX)}{\partial X} = 2AX(X'AX)^{-1}.$$

(12) X $(m \times n)$, A $(n \times n)$ positive definite, $\det(XAX') > 0$:

$$\frac{\partial \ln \det(XAX')}{\partial X} = 2(XAX')^{-1}XA.$$

(13) X $(m \times n)$, A $(n \times p)$, B $(m \times m)$ positive definite, $\det(A'X'BXA) > 0$:

$$\frac{\partial \ln \det(A'X'BXA)}{\partial X} = 2BXA(A'X'BXA)^{-1}A'.$$

(14) X $(m \times n)$, A $(p \times m)$, B $(n \times n)$ positive definite, $\det(AXBX'A') > 0$:

$$\frac{\partial \ln \det(AXBX'A')}{\partial X} = 2A'(AXBX'A')^{-1}AXB.$$

(15) X $(m \times m)$, $\det(X) > 0$:

$$\frac{\partial^2 \ln \det(X)}{\partial \mathrm{vec}(X)\partial \mathrm{vec}(X)'} = -K_{mm}\left[(X')^{-1} \otimes X^{-1}\right].$$

Note: (1) – (5) and (7) – (9) are either given in or follow straightforwardly from Magnus & Neudecker (1988, Chapter 9, Sec. 10). Rules (10) – (15) follow from the chain rule and the derivative of the logarithm. In (15), the derivative of $\mathrm{vec}(X^{-1})$ from Sec. 10.6 is used in addition. (6) results from (2), a chain rule and a product rule.

10.4 Jacobian Matrices of Linear Functions

10.4.1 Linear Functions with General Matrix Arguments

(1) $X\ (m \times n)$: $$\frac{\partial \operatorname{vec}(X)}{\partial \operatorname{vec}(X)'} = I_{mn},$$

$$\frac{\partial \operatorname{vec}(X')}{\partial \operatorname{vec}(X)'} = K_{mn},$$

$$\frac{\partial \operatorname{vec}(X)}{\partial \operatorname{vec}(X')'} = K_{nm}.$$

(2) $X\ (m \times n), c \in \mathbb{R}$: $$\frac{\partial \operatorname{vec}(cX)}{\partial \operatorname{vec}(X)'} = cI_{mn}.$$

(3) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times q)$: $$\frac{\partial \operatorname{vec}(AXB)}{\partial \operatorname{vec}(X)'} = B' \otimes A.$$

(4) $X\ (m \times n),\ A\ (p \times n),\ B\ (m \times q)$: $$\frac{\partial \operatorname{vec}(AX'B)}{\partial \operatorname{vec}(X)'} = (B' \otimes A)K_{mn}.$$

(5) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times q),\ C\ (p \times q)$:

$$\frac{\partial \operatorname{vec}(AXB + C)}{\partial \operatorname{vec}(X)'} = B' \otimes A.$$

(6) $X\ (m \times n),\ A, C\ (p \times m),\ B, D\ (n \times q)$:

$$\frac{\partial \operatorname{vec}(AXB \pm CXD)}{\partial \operatorname{vec}(X)'} = B' \otimes A \pm D' \otimes C.$$

(7) $X\ (m \times n),\ A, C\ (p \times n),\ B, D\ (m \times q)$:

$$\frac{\partial \operatorname{vec}(AX'B \pm CX'D)}{\partial \operatorname{vec}(X)'} = (B' \otimes A \pm D' \otimes C)K_{mn}.$$

(8) $X\ (m \times n),\ A\ (p \times m),\ B\ (n \times q),\ C\ (p \times n),\ D\ (m \times q)$:

$$\begin{aligned}\frac{\partial \operatorname{vec}(AXB + CX'D)}{\partial \operatorname{vec}(X)'} &= \frac{\partial \operatorname{vec}(CX'D + AXB)}{\partial \operatorname{vec}(X)'} \\ &= B' \otimes A + (D' \otimes C)K_{mn}.\end{aligned}$$

(9) $X\ (m \times n),\ A_i\ (p \times m),\ B_i\ (n \times q),\ i = 1, \ldots, r$:

$$\frac{\partial \operatorname{vec}(\sum_{i=1}^r A_i X B_i)}{\partial \operatorname{vec}(X)'} = \sum_{i=1}^r B_i' \otimes A_i.$$

(10) $X\ (m \times n),\ A_i\ (p \times n),\ B_i\ (m \times q),\ i = 1, \ldots, r$:

$$\frac{\partial \operatorname{vec}(\sum_{i=1}^r A_i X' B_i)}{\partial \operatorname{vec}(X)'} = \left(\sum_{i=1}^r B_i' \otimes A_i\right) K_{mn}.$$

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(11) $X\ (m \times n),\ A\ (p \times q)$:

(a) $$\frac{\partial \operatorname{vec}(A) \otimes \operatorname{vec}(X)}{\partial \operatorname{vec}(X)'} = \operatorname{vec}(A) \otimes I_{mn}.$$

(b) $$\frac{\partial \operatorname{vec}(X) \otimes \operatorname{vec}(A)}{\partial \operatorname{vec}(X)'} = I_{mn} \otimes \operatorname{vec}(A).$$

(c) $$\frac{\partial \operatorname{vec}(A \otimes X)}{\partial \operatorname{vec}(X)'} = (I_q \otimes K_{np} \otimes I_m)[\operatorname{vec}(A) \otimes I_{mn}].$$

(d) $$\frac{\partial \operatorname{vec}(X \otimes A)}{\partial \operatorname{vec}(X)'} = (I_n \otimes K_{qm} \otimes I_p)[I_{mn} \otimes \operatorname{vec}(A)].$$

(12) $X\ (m \times n),\ A\ (p \times q),\ B\ (r \times m),\ C\ (n \times s)$:

$$\frac{\partial \operatorname{vec}(A \otimes BXC)}{\partial \operatorname{vec}(X)'} = (I_q \otimes K_{sp} \otimes I_r)[\operatorname{vec}(A) \otimes C' \otimes B],$$

$$\frac{\partial \operatorname{vec}(BXC \otimes A)}{\partial \operatorname{vec}(X)'} = (I_s \otimes K_{qr} \otimes I_p)[C' \otimes B \otimes \operatorname{vec}(A)].$$

(13) $X\ (m \times n),\ A\ (p \times q), B\ (r \times s)$:

$$\frac{\partial \operatorname{vec}(A \otimes X \otimes B)}{\partial \operatorname{vec}(X)'}$$
$$= (I_q \otimes K_{ns,p} \otimes I_{mr})[\operatorname{vec}(A) \otimes (I_n \otimes K_{sm} \otimes I_r)(I_{mn} \otimes \operatorname{vec}(B))].$$

(14) $X\ (m \times n),\ A\ (p \times q),\ B\ (r \times s),\ C\ (k \times m),\ D\ (n \times l)$:

$$\frac{\partial \operatorname{vec}(A \otimes CXD \otimes B)}{\partial \operatorname{vec}(X)'}$$
$$= (I_q \otimes K_{sl,p} \otimes I_{kr})[\operatorname{vec}(A) \otimes (I_l \otimes K_{sk} \otimes I_r)(D' \otimes C \otimes \operatorname{vec}(B))].$$

(15) $X\ (m \times n),\ A\ (p \times q),\ B\ (mp \times nq)$:

$$\frac{\partial \operatorname{vec}[(A \otimes X) \odot B]}{\partial \operatorname{vec}(X)'} = \operatorname{diag}(\operatorname{vec} B)(I_q \otimes K_{np} \otimes I_m)[\operatorname{vec}(A) \otimes I_{mn}],$$

$$\frac{\partial \operatorname{vec}[(X \otimes A) \odot B]}{\partial \operatorname{vec}(X)'} = \operatorname{diag}(\operatorname{vec} B)(I_n \otimes K_{qm} \otimes I_p)[I_{mn} \otimes \operatorname{vec}(A)].$$

(16) $X, A\ (m \times n)$:

$$\frac{\partial \text{vec}(A \odot X)}{\partial \text{vec}(X)'} = \frac{\partial \text{vec}(X \odot A)}{\partial \text{vec}(X)'} = \text{diag}(\text{vec}\, A).$$

(17) $X\ (m \times n),\ A\ (p \times q),\ B\ (p \times m),\ C\ (n \times q)$:

$$\frac{\partial \text{vec}(A \odot BXC)}{\partial \text{vec}(X)'} = \frac{\partial \text{vec}(BXC \odot A)}{\partial \text{vec}(X)'} = \text{diag}(\text{vec}\, A)(C' \otimes B).$$

(18) $X\ (m \times n),\ A\ (p \times q),\ B\ (p \times n),\ C\ (m \times q)$:

$$\frac{\partial \text{vec}(A \odot BX'C)}{\partial \text{vec}(X)'} = \frac{\partial \text{vec}(BX'C \odot A)}{\partial \text{vec}(X)'} = \text{diag}(\text{vec}\, A)(C' \otimes B)K_{mn}.$$

(19) $X, A, B\ (m \times n)$:

$$\frac{\partial \text{vec}(A \odot X \odot B)}{\partial \text{vec}(X)'} = \text{diag}[\text{vec}(A \odot B)].$$

(20) $X\ (m \times n),\ A, B\ (n \times m)$:

$$\frac{\partial \text{vec}(A \odot X' \odot B)}{\partial \text{vec}(X)'} = \text{diag}[\text{vec}(A \odot B)]K_{mn}.$$

(21) $X, A\ (m \times n),\ B\ (p \times q)$:

$$\frac{\partial \text{vec}[(A \odot X) \otimes B]}{\partial \text{vec}(X)'} = (I_n \otimes K_{qm} \otimes I_p)[\text{diag}(\text{vec}\, A) \otimes \text{vec}(B)],$$

$$\frac{\partial \text{vec}[B \otimes (A \odot X)]}{\partial \text{vec}(X)'} = (I_q \otimes K_{np} \otimes I_m)[\text{vec}(B) \otimes \text{diag}(\text{vec}\, A)].$$

Note: (1) – (10) follow from basic properties of the derivative and the rule $\text{vec}(AXB) = (B' \otimes A)\text{vec}(X)$ for the vec operator (see Section 7.2). (11) – (21) may be derived using $\text{vec}(A \otimes B) = (I \otimes K \otimes I)[\text{vec}(A) \otimes \text{vec}(B)]$ and $\text{vec}(A \odot B) = \text{diag}(\text{vec}\, A)\text{vec}(B)$ and other standard rules for Kronecker and Hadamard products (see Chapter 2 and Magnus & Neudecker (1988, Chapter 9, Sec. 14)).

10.4.2 Linear Functions with Symmetric Matrix Arguments

Reminder: D_m denotes a duplication matrix and D_m^+ its Moore–Penrose inverse (see Section 9.5). L_m is an elimination matrix (see Section 9.6).

(1) $X\ (m \times m)$ symmetric: $\dfrac{\partial \text{vech}(X)}{\partial \text{vech}(X)'} = I_{m(m+1)/2}$,

$$\frac{\partial \text{vec}(X)}{\partial \text{vech}(X)'} = D_m.$$

(2) $X\ (m \times m)$ symmetric, $c \in \mathbb{R}$: $\dfrac{\partial \text{vec}(cX)}{\partial \text{vech}(X)'} = cD_m$.

(3) $X\ (m \times m)$ symmetric, $A\ (m \times n)$:

$$\frac{\partial \text{vech}(A'XA)}{\partial \text{vech}(X)'} = L_n(A' \otimes A')D_m = D_n^+(A' \otimes A')D_m.$$

(4) $X\ (m \times m)$ symmetric, $A\ (n \times m), B\ (m \times p)$:

$$\frac{\partial \text{vec}(AXB)}{\partial \text{vech}(X)'} = (B' \otimes A)D_m.$$

(5) $X\ (m \times m)$ symmetric, $A, B\ (m \times m)$:

$$\frac{\partial \text{vech}(AXA' \pm BXB')}{\partial \text{vech}(X)'} = D_m^+(A \otimes A \pm B \otimes B)D_m.$$

(6) $X\ (m \times m)$ symmetric, $A, B\ (m \times m)$:

$$\frac{\partial \text{vech}(AXB' + BXA')}{\partial \text{vech}(X)'} = D_m^+(B \otimes A + A \otimes B)D_m.$$

(7) $X\ (m \times m)$ symmetric, $A, C\ (n \times m),\ B, D\ (m \times p)$:

$$\frac{\partial \text{vec}(AXB \pm CXD)}{\partial \text{vech}(X)'} = (B' \otimes A \pm D' \otimes C)D_m.$$

(8) $X\ (m \times m)$ symmetric, $A_i\ (n \times m), B_i\ (m \times p),\ i = 1, \ldots, r$:

$$\frac{\partial \text{vec}(\sum_{i=1}^r A_i X B_i)}{\partial \text{vech}(X)'} = \sum_{i=1}^r (B_i' \otimes A_i)D_m.$$

(9) $X\ (m \times m)$ symmetric, $A_i\ (m \times n),\ i = 1, \ldots, r$:

$$\frac{\partial \text{vech}(\sum_{i=1}^r A_i' X A_i)}{\partial \text{vech}(X)'} = \sum_{i=1}^r L_n(A_i' \otimes A_i')D_m = \sum_{i=1}^r D_n^+(A_i' \otimes A_i')D_m.$$

Note: These results follow from basic properties of the derivative and the vec and vech operators. Notably, the rule $\text{vec}(AXB) = (B' \otimes A)\text{vec}(X)$ is useful (see Chapter 7). For proofs see also Magnus (1988, Capter 8, Sec. 8.2).

10.4.3 Linear Functions with Triangular Matrix Arguments

Reminder: L_m denotes an elimination matrix (see Section 9.6).

(1) X $(m \times m)$ lower triangular: $\dfrac{\partial \operatorname{vech}(X)}{\partial \operatorname{vech}(X)'} = I_{m(m+1)/2}$,

$$\frac{\partial \operatorname{vec}(X)}{\partial \operatorname{vech}(X)'} = L_m'.$$

(2) X $(m \times m)$ lower triangular, $c \in \mathbb{R}$: $\dfrac{\partial \operatorname{vec}(cX)}{\partial \operatorname{vech}(X)'} = cL_m'$.

(3) X $(m \times m)$ lower triangular, A $(n \times m)$, B $(m \times p)$:

$$\frac{\partial \operatorname{vec}(AXB)}{\partial \operatorname{vech}(X)'} = (B' \otimes A)L_m'.$$

(4) X $(m \times m)$ lower triangular, A, C $(n \times m)$, B, D $(m \times p)$:

$$\frac{\partial \operatorname{vec}(AXB \pm CXD)}{\partial \operatorname{vech}(X)'} = (B' \otimes A \pm D' \otimes C)L_m'.$$

(5) X $(m \times m)$ lower triangular, A_i $(n \times m)$, B_i $(m \times p)$, $i = 1, \ldots, r$:

$$\frac{\partial \operatorname{vec}(\sum_{i=1}^r A_i X B_i)}{\partial \operatorname{vech}(X)'} = \sum_{i=1}^r (B_i' \otimes A_i)L_m'.$$

(6) X, A, B $(m \times m)$ lower triangular: $\dfrac{\partial \operatorname{vech}(AXB)}{\partial \operatorname{vech}(X)'} = L_m(B' \otimes A)L_m'$.

(7) X, A, B, C, D $(m \times m)$ lower triangular:

$$\frac{\partial \operatorname{vech}(AXB \pm CXD)}{\partial \operatorname{vech}(X)'} = L_m(B' \otimes A \pm D' \otimes C)L_m'.$$

(8) X, A_i, B_i $(m \times m)$ lower triangular, $i = 1, \ldots, r$:

$$\frac{\partial \operatorname{vech}(\sum_{i=1}^r A_i X B_i)}{\partial \operatorname{vech}(X)'} = L_m\left(\sum_{i=1}^r B_i' \otimes A_i\right)L_m'.$$

Note: The results of this subsection follow from basic properties of derivatives and the vec and vech operators. In particular, the rules $\operatorname{vec}(A) = L_m'\operatorname{vech}(A)$ for lower triangular matrices A (see Section 9.14) and $\operatorname{vec}(AXB) = (B' \otimes A)\operatorname{vec}(X)$ (see Section 7.2) are useful. For proofs see also Magnus (1988, Chapter 8, Sec. 8.3).

10.4.4 Linear Functions of Vector and Matrix Valued Functions with Vector Arguments

(1) x $(m \times 1)$, c $(n \times 1)$ constant: $\dfrac{\partial c}{\partial x'} = O_{n \times m}$.

(2) x $(m \times 1)$, $y(x)$, $z(x)$ $(n \times 1)$, $c_1, c_2 \in \mathbb{R}$:

$$\frac{\partial[c_1 y(x) \pm c_2 z(x)]}{\partial x'} = c_1\frac{\partial y(x)}{\partial x'} \pm c_2\frac{\partial z(x)}{\partial x'}.$$

(3) x $(m \times 1)$, $y_i(x)$ $(n \times 1)$, $c_i \in \mathbb{R}$, $i = 1, \ldots, r$:

$$\frac{\partial[\sum_{i=1}^r c_i y_i(x)]}{\partial x'} = \sum_{i=1}^r c_i\frac{\partial y_i(x)}{\partial x'}.$$

(4) x $(m \times 1)$, $y(x)$ $(n \times 1)$, $z(x)$ $(p \times 1)$, A $(q \times n)$, B $(q \times p)$:

$$\frac{\partial[Ay(x) \pm Bz(x)]}{\partial x'} = A\frac{\partial y(x)}{\partial x'} \pm B\frac{\partial z(x)}{\partial x'}.$$

(5) x $(m \times 1)$, $Y(x)$ $(n \times p)$, $c \in \mathbb{R}$: $\dfrac{\partial \operatorname{vec}(cY)}{\partial x'} = c\dfrac{\partial \operatorname{vec}(Y)}{\partial x'}$.

(6) x $(m \times 1)$, $Y(x)$ $(n \times p)$, A $(q \times n)$, B $(p \times r)$:

$$\frac{\partial \operatorname{vec}(AYB)}{\partial x'} = (B' \otimes A)\frac{\partial \operatorname{vec}(Y)}{\partial x'}.$$

(7) x $(m \times 1)$, $Y_i(x)$ $(n_i \times p_i)$, A_i $(q \times n_i)$, B_i $(p_i \times r)$, $i = 1, \ldots, s$:

$$\frac{\partial \operatorname{vec}(\sum_{i=1}^s A_i Y_i B_i)}{\partial x'} = \sum_{i=1}^s (B_i' \otimes A_i)\frac{\partial \operatorname{vec}(Y_i)}{\partial x'}.$$

(8) x $(m \times 1)$, $Y(x)$ $(n \times p)$, A $(q \times n)$, B $(p \times r)$, C $(q \times r)$:

$$\frac{\partial \operatorname{vec}(AYB + C)}{\partial x'} = (B' \otimes A)\frac{\partial \operatorname{vec}(Y)}{\partial x'}.$$

(9) x $(m \times 1)$, $Y(x)$ $(n \times p)$, A $(r \times s)$:

$$\frac{\partial \operatorname{vec}(A \otimes Y)}{\partial x'} = (I_s \otimes K_{pr} \otimes I_n)\left(\operatorname{vec}(A) \otimes \frac{\partial \operatorname{vec}(Y)}{\partial x'}\right),$$

$$\frac{\partial \operatorname{vec}(Y \otimes A)}{\partial x'} = (I_p \otimes K_{sn} \otimes I_r)\left(\frac{\partial \operatorname{vec}(Y)}{\partial x'} \otimes \operatorname{vec}(A)\right).$$

(10) $x\ (m\times 1)$, $Y(x)\ (n\times p)$, $A\ (r\times s)$, $B\ (k\times l)$:

$$\frac{\partial \mathrm{vec}(A\otimes Y\otimes B)}{\partial x'} = (I_s\otimes K_{p\cdot l,r}\otimes I_{nk}) \times\left[\mathrm{vec}(A)\otimes(I_p\otimes K_{ln}\otimes I_k)\left(\frac{\partial \mathrm{vec}(Y)}{\partial x'}\otimes \mathrm{vec}(B)\right)\right].$$

(11) $x\ (m\times 1)$, $Y(x)\ (n\times p)$, $A\ (r\times s)$, $B\ (k\times n)$, $C\ (p\times l)$:

$$\frac{\partial \mathrm{vec}(BYC\otimes A)}{\partial x'} = (I_l\otimes K_{sk}\otimes I_r)\left(\left[(C'\otimes B)\frac{\partial \mathrm{vec}(Y)}{\partial x'}\right]\otimes \mathrm{vec}(A)\right),$$

$$\frac{\partial \mathrm{vec}(A\otimes BYC)}{\partial x'} = (I_s\otimes K_{lr}\otimes I_k)\left(\mathrm{vec}(A)\otimes\left[(C'\otimes B)\frac{\partial \mathrm{vec}(Y)}{\partial x'}\right]\right).$$

(12) $x\ (m\times 1)$, $Y(x)\ (n\times p)$, $A\ (n\times p)$:

$$\frac{\partial \mathrm{vec}(Y\odot A)}{\partial x'} = \frac{\partial \mathrm{vec}(A\odot Y)}{\partial x'} = \mathrm{diag}(\mathrm{vec}\, A)\frac{\partial \mathrm{vec}(Y)}{\partial x'}.$$

Note: The results of this subsection follow from the linearity of the functions considered, the rules for the vec operator and basic matrix operations.

10.5 Product Rules

10.5.1 Matrix Products

(1) $X\ (m\times m)$:

(a) $$\frac{\partial \mathrm{vec}(X^2)}{\partial \mathrm{vec}(X)'} = X'\otimes I_m + I_m\otimes X.$$

(b) $$\frac{\partial \mathrm{vec}(X^i)}{\partial \mathrm{vec}(X)'} = \sum_{j=0}^{i-1}(X')^{i-1-j}\otimes X^j,\quad i=1,2,\ldots$$

(c) $$\frac{\partial \mathrm{vec}(X'^2)}{\partial \mathrm{vec}(X)'} = (X\otimes I_m + I_m\otimes X')K_{mm}.$$

(2) $X\ (m\times n)$:

(a) $$\frac{\partial \mathrm{vec}(X'X)}{\partial \mathrm{vec}(X)'} = (I_{n^2}+K_{nn})(I_n\otimes X').$$

(b) $$\frac{\partial \mathrm{vec}(XX')}{\partial \mathrm{vec}(X)'} = (I_{m^2}+K_{mm})(X\otimes I_m).$$

(3) $X\ (m\times n)$, $A\ (n\times m)$:

$$\frac{\partial \mathrm{vec}(XAX)}{\partial \mathrm{vec}(X)'} = X'A'\otimes I_m + I_n\otimes XA.$$

(4) $X\ (m\times n)$, $A\ (m\times m)$:

$$\frac{\partial \mathrm{vec}(X'AX)}{\partial \mathrm{vec}(X)'} = (X'A'\otimes I_n)K_{mn} + (I_n\otimes X'A).$$

(5) $X\ (m\times n)$, $A\ (m\times m)$ symmetric:

$$\frac{\partial \mathrm{vec}(X'AX)}{\partial \mathrm{vec}(X)'} = (I_{n^2}+K_{nn})(I_n\otimes X'A).$$

(6) $X\ (m\times n)$, $A\ (n\times n)$:

$$\frac{\partial \mathrm{vec}(XAX')}{\partial \mathrm{vec}(X)'} = (XA'\otimes I_m) + (I_m\otimes XA)K_{mn}.$$

(7) $X\ (m\times n)$, $A\ (n\times n)$ symmetric:

$$\frac{\partial \mathrm{vec}(XAX')}{\partial \mathrm{vec}(X)'} = (I_{m^2}+K_{mm})(XA\otimes I_m).$$

(8) $X\ (m\times n)$, $A\ (p\times m)$, $B\ (n\times m)$, $C\ (n\times q)$:

$$\frac{\partial \mathrm{vec}(AXBXC)}{\partial \mathrm{vec}(X)'} = C'X'B'\otimes A + C'\otimes AXB.$$

(9) $X\ (m\times n)$, $A\ (p\times n)$, $B\ (m\times m)$, $C\ (n\times q)$:

$$\frac{\partial \mathrm{vec}(AX'BXC)}{\partial \mathrm{vec}(X)'} = (C'X'B'\otimes A)K_{mn} + (C'\otimes AX'B).$$

(10) $X\ (m\times n)$, $A\ (p\times n)$, $B\ (m\times m)$ symmetric:

$$\frac{\partial \mathrm{vec}(AX'BXA')}{\partial \mathrm{vec}(X)'} = (I_{p^2}+K_{pp})(A\otimes AX'B).$$

(11) $X\ (m\times n)$, $A\ (p\times m)$, $B\ (n\times n)$, $C\ (m\times q)$:

$$\frac{\partial \mathrm{vec}(AXBX'C)}{\partial \mathrm{vec}(X)'} = (C'XB'\otimes A) + (C'\otimes AXB)K_{mn}.$$

(12) $X\ (m\times n)$, $A\ (p\times m)$, $B\ (n\times n)$ symmetric:

$$\frac{\partial \mathrm{vec}(AXBX'A')}{\partial \mathrm{vec}(X)'} = (I_{p^2}+K_{pp})(AXB\otimes A).$$

(13) $X\ (m\times m),\ A\ (n\times m),\ B\ (m\times p)$:

$$\frac{\partial \text{vec}(AX^iB)}{\partial \text{vec}(X)'}=\sum_{j=0}^{i-1}B'(X')^{i-1-j}\otimes AX^j,\quad i=1,2,\ldots$$

(14) $X\ (m\times n),\ A\ (p\times m),\ B\ (n\times p)$:

$$\frac{\partial \text{vec}(AXB)^i}{\partial \text{vec}(X)'}=\sum_{j=0}^{i-1}(B'X'A')^{i-1-j}B'\otimes(AXB)^jA,\quad i=1,2,\ldots$$

(15) $X, A\ (m\times m)$:

$$\frac{\partial \text{vec}(A+X)^2}{\partial \text{vec}(X)'}\ =\ (A'+X')\otimes I_m+I_m\otimes(A+X),$$

$$\frac{\partial \text{vec}(A+X)^i}{\partial \text{vec}(X)'}\ =\ \sum_{j=0}^{i-1}(A'+X')^{i-1-j}\otimes(A+X)^j,\quad i=1,2,\ldots$$

(16) $X, A\ (m\times n)$:

$$\frac{\partial \text{vec}[(A+X)'(A+X)]}{\partial \text{vec}(X)'}=(I_{n^2}+K_{nn})[I_n\otimes(A'+X')],$$

$$\frac{\partial \text{vec}[(A+X)(A+X)']}{\partial \text{vec}(X)'}=(I_{m^2}+K_{mm})[(A+X)\otimes I_m].$$

(17) $X\ (m\times n),\ A\ (p\times p),\ B\ (p\times m),\ C\ (n\times p)$:

$$\frac{\partial \text{vec}(A+BXC)^2}{\partial \text{vec}(X)'}=(A'+C'X'B')C'\otimes B+C'\otimes(A+BXC)B,$$

$$\frac{\partial \text{vec}(A+BXC)^i}{\partial \text{vec}(X)'}=\sum_{j=0}^{i-1}(A'+C'X'B')^{i-1-j}C'\otimes(A+BXC)^jB,$$

$i=1,2,\ldots$

(18) $X\ (m\times n),\ A\ (p\times q),\ B\ (p\times m),\ C\ (n\times q)$:

$$\frac{\partial \text{vec}[(A+BXC)'(A+BXC)]}{\partial \text{vec}(X)'}=(I_{q^2}+K_{qq})[C'\otimes(A'+C'X'B')B].$$

$$\frac{\partial \text{vec}[(A+BXC)(A+BXC)']}{\partial \text{vec}(X)'}=(I_{p^2}+K_{pp})[(A+BXC)C'\otimes B].$$

Note: (1) – (2) are given in Magnus & Neudecker (1988, Chapter 9, Sec. 13). The remaining results follow via the product and chain rules of differential calculus.

10.5.2 Kronecker and Hadamard Products

(1) $X\ (m\times n)$:

(a) $$\frac{\partial \text{vec}(X)\otimes\text{vec}(X)}{\partial \text{vec}(X)'}=I_{mn}\otimes\text{vec}(X)+\text{vec}(X)\otimes I_{mn}.$$

(b) $$\frac{\partial \text{vec}(X\otimes X)}{\partial \text{vec}(X)'}=(I_n\otimes K_{nm}\otimes I_m)[I_{mn}\otimes\text{vec}(X)+\text{vec}(X)\otimes I_{mn}].$$

(c) $$\frac{\partial \text{vec}(X\otimes X')}{\partial \text{vec}(X)'}$$
$$=(I_n\otimes K_{mm}\otimes I_n)[I_{mn}\otimes\text{vec}(X')+\text{vec}(X)\otimes K_{mn}].$$

(d) $$\frac{\partial \text{vec}(X'\otimes X)}{\partial \text{vec}(X)'}$$
$$=(I_m\otimes K_{nn}\otimes I_m)[K_{mn}\otimes\text{vec}(X)+\text{vec}(X')\otimes I_{mn}].$$

(e) $$\frac{\partial \text{vec}(X'\otimes X')}{\partial \text{vec}(X)'}$$
$$=(I_m\otimes K_{mn}\otimes I_n)[K_{mn}\otimes\text{vec}(X')+\text{vec}(X')\otimes K_{mn}].$$

(2) $X\ (m\times n),\ A\ (p\times m),\ B\ (n\times q),\ C\ (r\times m),\ D\ (n\times s)$:

$$\frac{\partial \text{vec}(AXB\otimes CXD)}{\partial \text{vec}(X)'}$$
$$=(I_q\otimes K_{sp}\otimes I_r)[B'\otimes A\otimes\text{vec}(CXD)+\text{vec}(AXB)\otimes D'\otimes C].$$

(3) $X\ (m\times n),\ A\ (p\times m),\ B\ (n\times q),\ C\ (r\times n),\ D\ (m\times s)$:

$$\frac{\partial \text{vec}(AXB\otimes CX'D)}{\partial \text{vec}(X)'}$$
$$=(I_q\otimes K_{sp}\otimes I_r)$$
$$\times[B'\otimes A\otimes\text{vec}(CX'D)+\text{vec}(AXB)\otimes(D'\otimes C)K_{mn}].$$

(4) $X\ (m\times n),\ A\ (p\times n),\ B\ (m\times q),\ C\ (r\times m),\ D\ (n\times s)$:

$$\frac{\partial \text{vec}(AX'B\otimes CXD)}{\partial \text{vec}(X)'}$$
$$=(I_q\otimes K_{sp}\otimes I_r)$$
$$\times[(B'\otimes A)K_{mn}\otimes\text{vec}(CXD)+\text{vec}(AX'B)\otimes D'\otimes C].$$

(5) $X\ (m\times n),\ A\ (p\times n),\ B\ (m\times q),\ C\ (r\times n),\ D\ (m\times s)$:

$$\frac{\partial \text{vec}(AX'B\otimes CX'D)}{\partial \text{vec}(X)'}$$
$$=(I_q\otimes K_{sp}\otimes I_r)[(B'\otimes A)K_{mn}\otimes\text{vec}(CX'D)+$$

$$+ \operatorname{vec}(AX'B) \otimes (D' \otimes C)K_{mn}].$$

(6) $X\ (m \times n)$:

(a) $$\frac{\partial \operatorname{vec}(X \odot X)}{\partial \operatorname{vec}(X)'} = 2 \operatorname{diag}(\operatorname{vec} X).$$

(b) $$\frac{\partial \operatorname{vec}(X \odot X \odot X)}{\partial \operatorname{vec}(X)'} = 3 \operatorname{diag}[\operatorname{vec}(X \odot X)].$$

(c) $$\frac{\partial \operatorname{vec}(X' \odot X')}{\partial \operatorname{vec}(X)'} = 2 \operatorname{diag}(\operatorname{vec} X')K_{mn}.$$

(7) $X\ (m \times m)$:

$$\frac{\partial \operatorname{vec}(X \odot X')}{\partial \operatorname{vec}(X)'} = \frac{\partial \operatorname{vec}(X' \odot X)}{\partial \operatorname{vec}(X)'} = \operatorname{diag}(\operatorname{vec} X)K_{mn} + \operatorname{diag}(\operatorname{vec} X').$$

(8) $X\ (m \times n),\ A, C\ (p \times m),\ B, D\ (n \times q)$:

$$\begin{aligned}&\frac{\partial \operatorname{vec}(AXB \odot CXD)}{\partial \operatorname{vec}(X)'}\\ &\quad = \operatorname{diag}[\operatorname{vec}(AXB)](D' \otimes C) + \operatorname{diag}[\operatorname{vec}(CXD)](B' \otimes A).\end{aligned}$$

(9) $X\ (m \times n),\ A\ (p \times n),\ B\ (m \times q),\ C\ (p \times m),\ D\ (n \times q)$:

$$\begin{aligned}\frac{\partial \operatorname{vec}(AX'B \odot CXD)}{\partial \operatorname{vec}(X)'} &= \frac{\partial \operatorname{vec}(CXD \odot AX'B)}{\partial \operatorname{vec}(X)'}\\ &= \operatorname{diag}[\operatorname{vec}(AX'B)](D' \otimes C)\\ &\quad + \operatorname{diag}[\operatorname{vec}(CXD)](B' \otimes A)K_{mn}.\end{aligned}$$

(10) $X\ (m \times n),\ A, C\ (p \times n),\ B, D\ (m \times q)$:

$$\begin{aligned}\frac{\partial \operatorname{vec}(AX'B \odot CX'D)}{\partial \operatorname{vec}(X)'} &= \{\operatorname{diag}[\operatorname{vec}(AX'B)](D' \otimes C)\\ &\quad + \operatorname{diag}[\operatorname{vec}(CX'D)](B' \otimes A)\}K_{mn}.\end{aligned}$$

Note: The results of this subsection follow from the basic product rule for differentiation and the rules for Kronecker and Hadamard products. See also Magnus & Neudecker (1988, Chapter 9, Sec. 14).

10.5.3 Functions with Symmetric Matrix Arguments

Reminder: D_m denotes a duplication matrix and D_m^+ its Moore–Penrose inverse (see Section 9.5). L_m is an elimination matrix (see Section 9.6).

(1) $X\ (m \times m)$ symmetric:

(a) $$\frac{\partial \operatorname{vec}(X^2)}{\partial \operatorname{vech}(X)'} = (X \otimes I_m + I_m \otimes X)D_m.$$

(b) $$\frac{\partial \operatorname{vec}(X^i)}{\partial \operatorname{vech}(X)'} = \sum_{j=0}^{i-1}(X^{i-1-j} \otimes X^j)D_m, \quad i = 1, 2, \ldots$$

(c) $$\frac{\partial \operatorname{vech}(X^i)}{\partial \operatorname{vech}(X)'} = D_m^+ \left(\sum_{j=0}^{i-1} X^{i-1-j} \otimes X^j\right) D_m, \quad i = 1, 2, \ldots$$

(2) $X, A\ (m \times m)$ symmetric:

$$\begin{aligned}\frac{\partial \operatorname{vech}(XAX)}{\partial \operatorname{vech}(X)'} &= D_m^+(XA \otimes I_m + I_m \otimes XA)D_m\\ &= L_m(XA \otimes I_m + I_m \otimes XA)D_m.\end{aligned}$$

(3) $X\ (m \times m)$ symmetric, $A\ (p \times m),\ B\ (m \times m),\ C\ (m \times q)$:

$$\frac{\partial \operatorname{vec}(AXBXC)}{\partial \operatorname{vech}(X)'} = (C'XB' \otimes A + C' \otimes AXB)D_m.$$

(4) $X\ (m \times m)$ symmetric, $A\ (p \times m),\ B\ (m \times q)$:

$$\frac{\partial \operatorname{vec}(AX^iB)}{\partial \operatorname{vech}(X)'} = \sum_{j=0}^{i-1}(B'X^{i-1-j} \otimes AX^j)D_m, \quad i = 1, 2, \ldots$$

(5) $X\ (m \times m)$ symmetric, $A\ (p \times m),\ B\ (m \times p)$:

$$\frac{\partial \operatorname{vec}(AXB)^i}{\partial \operatorname{vech}(X)'} = \left(\sum_{j=0}^{i-1}(B'XA')^{i-1-j}B' \otimes (AXB)^j A\right) D_m,$$

$i = 1, 2, \ldots$

(6) $X\ (m \times m)$ symmetric, $A\ (p \times m),\ B\ (m \times p),\ C\ (p \times p)$:

$$\begin{aligned}&\frac{\partial \operatorname{vec}(AXB + C)^i}{\partial \operatorname{vech}(X)'}\\ &\quad = \left(\sum_{j=0}^{i-1}(B'XA' + C')^{i-1-j}B' \otimes (AXB + C)^j A\right) D_m,\end{aligned}$$

$i = 1, 2, \ldots$

(7) $X\ (m \times m)$ symmetric, $A\ (p \times m),\ B\ (m \times q),\ C\ (p \times q)$:

$$\frac{\partial \operatorname{vec}[(AXB + C)'(AXB + C)]}{\partial \operatorname{vech}(X)'} = (I_{q^2} + K_{qq})[B' \otimes (B'XA' + C')A]D_m.$$

(8) X $(m \times m)$ symmetric, A $(p \times m)$, B $(m \times q)$, C $(p \times q)$:

$$\frac{\partial \mathrm{vec}[(AXB+C)(AXB+C)']}{\partial \mathrm{vech}(X)'} = (I_{p^2}+K_{pp})[(AXB+C)B' \otimes A]D_m.$$

(9) X $(m \times m)$ symmetric:

$$\frac{\partial \mathrm{vec}(X \otimes X)}{\partial \mathrm{vech}(X)'} = (I_m \otimes K_{mm} \otimes I_m)[I_{m^2} \otimes \mathrm{vec}(X) + \mathrm{vec}(X) \otimes I_{m^2}]D_m.$$

(10) X $(m \times m)$ symmetric, A $(p \times m)$, B $(m \times q)$, C $(r \times m)$, D $(m \times s)$:

$$\frac{\partial \mathrm{vec}(AXB \otimes CXD)}{\partial \mathrm{vech}(X)'}$$
$$= (I_q \otimes K_{sp} \otimes I_r)[B' \otimes A \otimes \mathrm{vec}(CXD) + \mathrm{vec}(AXB) \otimes D' \otimes C]D_m.$$

(11) X $(m \times m)$ symmetric:

(a) $$\frac{\partial \mathrm{vec}(X \odot X)}{\partial \mathrm{vech}(X)'} = 2\ \mathrm{diag}(\mathrm{vec}\, X)D_m.$$

(b) $$\frac{\partial \mathrm{vec}(X \odot X \odot X)}{\partial \mathrm{vech}(X)'} = 3\ \mathrm{diag}[\mathrm{vec}(X \odot X)]D_m.$$

(12) X $(m \times m)$ symmetric, A $(p \times m)$, B $(m \times q)$, C $(p \times m)$, D $(m \times q)$:

$$\frac{\partial \mathrm{vec}(AXB \odot CXD)}{\partial \mathrm{vech}(X)'} = \{\mathrm{diag}[\mathrm{vec}(AXB)](D' \otimes C) + \mathrm{diag}[\mathrm{vec}(CXD)](B' \otimes A)\}D_m.$$

Note: These results follow from those of the previous subsection and the chain rule for matrix differentiation. See also Magnus (1988, Chapter 8, Sec. 8.2).

10.5.4 Functions with Lower Triangular Matrix Arguments

Reminder: D_m^+ is the Moore-Penrose inverse of the duplication matrix D_m (see Section 9.5) and L_m denotes an elimination matrix (see Section 9.6).

(1) X $(m \times m)$ lower triangular:

$$\frac{\partial \mathrm{vech}(X^i)}{\partial \mathrm{vech}(X)'} = L_m \left(\sum_{j=0}^{i-1} (X')^{i-1-j} \otimes X^j \right) L_m', \quad i = 1, 2, \ldots$$

(2) X, A $(m \times m)$ lower triangular:

$$\frac{\partial \mathrm{vech}(XAX)}{\partial \mathrm{vech}(X)'} = L_m(X'A' \otimes I_m + I_m \otimes XA)L_m'.$$

(3) X $(m \times m)$ lower triangular:

$$\frac{\partial \mathrm{vech}(X'X)}{\partial \mathrm{vech}(X)'} = 2D_m^+(I_m \otimes X')L_m',$$

$$\frac{\partial \mathrm{vech}(XX')}{\partial \mathrm{vech}(X)'} = 2D_m^+(X \otimes I_m)L_m'.$$

Note: For proofs see Magnus (1988, Chapter 8, Sec. 8.3 - 8.4).

10.5.5 Products of Matrix Valued Functions with Vector Arguments

(1) x $(m \times 1)$, $Y(x)$ $(n \times n)$:

$$\frac{\partial \mathrm{vec}(Y^i)}{\partial x'} = \sum_{j=0}^{i-1} [(Y')^{i-1-j} \otimes Y^j] \frac{\partial \mathrm{vec}(Y)}{\partial x'}.$$

(2) x $(m \times 1)$, $Y(x)$ $(n \times p)$:

$$\frac{\partial \mathrm{vec}(Y'Y)}{\partial x'} = (I_{p^2} + K_{pp})(I_p \otimes Y')\frac{\partial \mathrm{vec}(Y)}{\partial x'},$$

$$\frac{\partial \mathrm{vec}(YY')}{\partial x'} = (I_{n^2} + K_{nn})(Y \otimes I_n)\frac{\partial \mathrm{vec}(Y)}{\partial x'}.$$

(3) x $(m \times 1)$, $Y(x)$ $(n \times p)$, $Z(x)$ $(p \times q)$:

$$\frac{\partial \mathrm{vec}(YZ)}{\partial x'} = (I_q \otimes Y)\frac{\partial \mathrm{vec}(Z)}{\partial x'} + (Z' \otimes I_n)\frac{\partial \mathrm{vec}(Y)}{\partial x'}.$$

(4) x $(m \times 1)$, $Y(x)$ $(n \times p)$, $Z(x)$ $(q \times r)$, A $(s \times n)$, B $(p \times q)$, C $(r \times k)$:

$$\frac{\partial \mathrm{vec}(AYBZC)}{\partial x'} = (C' \otimes AYB)\frac{\partial \mathrm{vec}(Z)}{\partial x'} + (C'Z'B' \otimes A)\frac{\partial \mathrm{vec}(Y)}{\partial x'}.$$

(5) x $(m \times 1)$, $Y(x)$ $(n \times p)$, A $(q \times n)$, B $(p \times q)$, C $(q \times q)$:

$$\frac{\partial \mathrm{vec}(AYB + C)^i}{\partial x'}$$
$$= \left(\sum_{j=0}^{i-1} (B'Y'A' + C')^{i-1-j} B' \otimes (AYB + C)^j A \right) \frac{\partial \mathrm{vec}(Y)}{\partial x'},$$

$i = 1, 2, \ldots$

(6) x $(m \times 1)$, $Y(x)$ $(n \times p)$, A $(q \times n)$, B $(p \times r)$, C $(q \times r)$:

(a) $$\frac{\partial \text{vec}[(AYB+C)'(AYB+C)]}{\partial x'} = (I_{r^2}+K_{rr})[B'\otimes(B'Y'A'+C')A]\frac{\partial \text{vec}(Y)}{\partial x'}.$$

(b) $$\frac{\partial \text{vec}[(AYB+C)(AYB+C)']}{\partial x'} = (I_{q^2}+K_{qq})[(AYB+C)B'\otimes A]\frac{\partial \text{vec}(Y)}{\partial x'}.$$

(7) x $(m\times 1)$, $Y(x)$ $(n\times p)$, $Z(x)$ $(q\times r)$:

(a) $$\frac{\partial[\text{vec}(Y)\otimes\text{vec}(Z)]}{\partial x'} = \frac{\partial \text{vec}(Y)}{\partial x'}\otimes\text{vec}(Z)+\text{vec}(Y)\otimes\frac{\partial \text{vec}(Z)}{\partial x'}.$$

(b) $$\frac{\partial \text{vec}(Y\otimes Z)}{\partial x'} = (I_p\otimes K_{rn}\otimes I_q)\left[\frac{\partial \text{vec}(Y)}{\partial x'}\otimes\text{vec}(Z)+\text{vec}(Y)\otimes\frac{\partial \text{vec}(Z)}{\partial x'}\right].$$

(8) x $(m\times 1)$, $Y(x)$ $(n\times p)$, $Z(x)$ $(q\times r)$, A $(s\times n)$, B $(p\times k)$, C $(l\times q)$, D $(r\times h)$:

$$\begin{aligned}\frac{\partial \text{vec}(AYB\otimes CZD)}{\partial x'} &= (I_k\otimes K_{hs}\otimes I_l)\\ &\quad\times\left[(B'\otimes A)\frac{\partial \text{vec}(Y)}{\partial x'}\otimes\text{vec}(CZD)\right.\\ &\quad\left.+\text{vec}(AYB)\otimes(D'\otimes C)\frac{\partial \text{vec}(Z)}{\partial x'}\right].\end{aligned}$$

(9) x $(m\times 1)$, $Y(x), Z(x)$ $(n\times p)$:

$$\begin{aligned}\frac{\partial \text{vec}(Y\odot Z)}{\partial x'} &= \frac{\partial \text{vec}(Z\odot Y)}{\partial x'}\\ &= \text{diag}(\text{vec}\,Z)\frac{\partial \text{vec}(Y)}{\partial x'}+\text{diag}(\text{vec}\,Y)\frac{\partial \text{vec}(Z)}{\partial x'}.\end{aligned}$$

(10) x $(m\times 1)$, $Y(x)$ $(n\times p)$, $Z(x)$ $(q\times r)$, A $(s\times n)$, B $(p\times k)$, C $(s\times q)$, D $(r\times k)$:

$$\begin{aligned}\frac{\partial \text{vec}(AYB\odot CZD)}{\partial x'} &= \frac{\partial \text{vec}(CZD\odot AYB)}{\partial x'}\\ &= \text{diag}[\text{vec}(CZD)](B'\otimes A)\frac{\partial \text{vec}(Y)}{\partial x'}\\ &\quad+\text{diag}[\text{vec}(AYB)](D'\otimes C)\frac{\partial \text{vec}(Z)}{\partial x'}.\end{aligned}$$

Note: The results of this subsection follow from the chain rule and the rules for basic matrix operations. See also the results of the previous subsections, in particular Sections 10.5.1 and 10.5.2.

10.6 Jacobian Matrices of Functions Involving Inverse Matrices

(1) X $(m\times m)$ nonsingular:

$$\frac{\partial \text{vec}(X^{-1})}{\partial \text{vec}(X)'} = -X'^{-1}\otimes X^{-1},$$

$$\frac{\partial \text{vec}(X'^{-1})}{\partial \text{vec}(X)'} = -(X^{-1}\otimes X'^{-1})K_{mn}.$$

(2) X $(m\times m)$ symmetric nonsingular:

$$\frac{\partial \text{vech}(X^{-1})}{\partial \text{vech}(X)'} = -D_m^+(X^{-1}\otimes X^{-1})D_m,$$

where D_m denotes a duplication matrix and D_m^+ its Moore–Penrose inverse (see Section 9.5).

Note: For proofs of these results see Magnus & Neudecker (1988, Chapter 9, Sec. 13) and Magnus (1988, Chapter 8, Sec. 8.2).

10.6.1 Matrix Products

(1) X $(m\times m)$ nonsingular, A $(n\times m)$, B $(m\times p)$:

$$\frac{\partial \text{vec}(AX^{-1}B)}{\partial \text{vec}(X)'} = -B'X'^{-1}\otimes AX^{-1}.$$

(2) X $(m\times m)$ nonsingular :

$$\frac{\partial \text{vec}[(X^{-1})^i]}{\partial \text{vec}(X)'} = -\sum_{j=0}^{i-1}(X')^{j-i}\otimes X^{-j-1}, \quad i=1,2,\ldots$$

(3) X $(m\times n)$, $\text{rk}(X)=n$:

$$\frac{\partial \text{vec}[(X'X)^{-1}]}{\partial \text{vec}(X)'} = -(I_{n^2}+K_{nn})[(X'X)^{-1}\otimes(X'X)^{-1}X'].$$

(4) X $(m\times n)$, $\text{rk}(X)=m$:

$$\frac{\partial \text{vec}[(XX')^{-1}]}{\partial \text{vec}(X)'} = -(I_{m^2}+K_{mm})[(XX')^{-1}X\otimes(XX')^{-1}].$$

(5) X $(m\times m)$ nonsingular, A $(n\times m)$, B $(m\times m)$, C $(m\times p)$:

(a) $\dfrac{\partial \text{vec}(AX^{-1}BXC)}{\partial \text{vec}(X)'} = C' \otimes AX^{-1}B - C'X'B'X'^{-1} \otimes AX^{-1}.$

(b) $\dfrac{\partial \text{vec}(AXBX^{-1}C)}{\partial \text{vec}(X)'} = C'X'^{-1}B' \otimes A - C'X'^{-1} \otimes AXBX^{-1}.$

(c) $\dfrac{\partial \text{vec}(AX^{-1}BX^{-1}C)}{\partial \text{vec}(X)'}$

$= -C'X'^{-1}B'X'^{-1} \otimes AX^{-1} - C'X'^{-1} \otimes AX^{-1}BX^{-1}.$

Note: The results of this subsection follow from the chain rule and the results of the previous sections.

10.6.2 Kronecker and Hadamard Products

(1) X $(m \times m)$ nonsingular, A $(p \times q)$:

$$\frac{\partial \text{vec}(A \otimes X^{-1})}{\partial \text{vec}(X)'} = -(I_q \otimes K_{mp} \otimes I_m)[\text{vec}(A) \otimes X'^{-1} \otimes X^{-1}],$$

$$\frac{\partial \text{vec}(X^{-1} \otimes A)}{\partial \text{vec}(X)'} = -(I_m \otimes K_{qm} \otimes I_p)[X'^{-1} \otimes X^{-1} \otimes \text{vec}(A)].$$

(2) X $(m \times m)$ nonsingular:

(a) $\dfrac{\partial \text{vec}(X^{-1}) \otimes \text{vec}(X)}{\partial \text{vec}(X)'} = \text{vec}(X^{-1}) \otimes I_{m^2} - X'^{-1} \otimes X^{-1} \otimes \text{vec}(X).$

(b) $\dfrac{\partial \text{vec}(X) \otimes \text{vec}(X^{-1})}{\partial \text{vec}(X)'} = I_{m^2} \otimes \text{vec}(X^{-1}) - \text{vec}(X) \otimes X'^{-1} \otimes X^{-1}.$

(c) $\dfrac{\partial \text{vec}(X^{-1}) \otimes \text{vec}(X^{-1})}{\partial \text{vec}(X)'}$

$= -[X'^{-1} \otimes X^{-1} \otimes \text{vec}(X^{-1}) + \text{vec}(X^{-1}) \otimes X'^{-1} \otimes X^{-1}].$

(d) $\dfrac{\partial \text{vec}(X^{-1} \otimes X)}{\partial \text{vec}(X)'}$

$= (I_m \otimes K_{mm} \otimes I_m)[\text{vec}(X^{-1}) \otimes I_{m^2} - X'^{-1} \otimes X^{-1} \otimes \text{vec}(X)].$

(e) $\dfrac{\partial \text{vec}(X \otimes X^{-1})}{\partial \text{vec}(X)'}$

$= (I_m \otimes K_{mm} \otimes I_m)$

$\times [I_{m^2} \otimes \text{vec}(X^{-1}) - \text{vec}(X) \otimes X'^{-1} \otimes X^{-1}].$

(f) $\dfrac{\partial \text{vec}(X^{-1} \otimes X^{-1})}{\partial \text{vec}(X)'}$

$= -(I_m \otimes K_{mm} \otimes I_m)$

$\times [X'^{-1} \otimes X^{-1} \otimes \text{vec}(X^{-1}) + \text{vec}(X^{-1}) \otimes X'^{-1} \otimes X^{-1}].$

(3) X $(m \times m)$ nonsingular, A $(m \times m)$:

$$\frac{\partial \text{vec}(A \bullet X^{-1})}{\partial \text{vec}(X)'} = \frac{\partial \text{vec}(X^{-1} \bullet A)}{\partial \text{vec}(X)'} = -\text{diag}(\text{vec}\, A)(X'^{-1} \otimes X^{-1}).$$

(4) X $(m \times m)$ nonsingular:

(a) $\dfrac{\partial \text{vec}(X^{-1} \odot X)}{\partial \text{vec}(X)'} = \dfrac{\partial \text{vec}(X \odot X^{-1})}{\partial \text{vec}(X)'}$

$= \text{diag}(\text{vec}\, X^{-1}) - \text{diag}(\text{vec}\, X)(X'^{-1} \otimes X^{-1}).$

(b) $\dfrac{\partial \text{vec}(X^{-1} \bullet X^{-1})}{\partial \text{vec}(X)'} = -2\ \text{diag}(\text{vec}\, X^{-1})(X'^{-1} \otimes X^{-1}).$

Note: The results of this subsection follow from the chain rule for derivatives and the results of the previous sections.

10.6.3 Matrix Valued Functions with Vector Arguments

(1) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular:

$$\frac{\partial \text{vec}(Y^{-1})}{\partial x'} = -(Y'^{-1} \otimes Y^{-1})\frac{\partial \text{vec}(Y)}{\partial x'}.$$

(2) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(n \times p)$:

$$\frac{\partial \text{vec}(Y^{-1}Z)}{\partial x'} = (I_p \otimes Y^{-1})\frac{\partial \text{vec}(Z)}{\partial x'} - (Z'Y'^{-1} \otimes Y^{-1})\frac{\partial \text{vec}(Y)}{\partial x'}.$$

(3) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(p \times n)$:

$$\frac{\partial \text{vec}(ZY^{-1})}{\partial x'} = (Y'^{-1} \otimes I_p)\frac{\partial \text{vec}(Z)}{\partial x'} - (Y'^{-1} \otimes ZY^{-1})\frac{\partial \text{vec}(Y)}{\partial x'}.$$

(4) x $(m \times 1)$, $Y(x)$, $Z(x)$ $(n \times n)$ nonsingular:

$\dfrac{\partial \text{vec}(Y^{-1}Z^{-1})}{\partial x'}$

$= -(Z'^{-1}Y'^{-1} \otimes Y^{-1})\dfrac{\partial \text{vec}(Y)}{\partial x'} - (Z'^{-1} \otimes Y^{-1}Z^{-1})\dfrac{\partial \text{vec}(Z)}{\partial x'}.$

(5) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(p \times p)$ nonsingular, $U(x)$ $(q \times n)$, $V(x)$ $(n \times p)$, $W(x)$ $(p \times r)$:

$$\begin{aligned}
\frac{\partial \operatorname{vec}(UY^{-1}VZ^{-1}W)}{\partial x'} &= (W'Z'^{-1}V'Y'^{-1} \otimes I_q)\frac{\partial \operatorname{vec}(U)}{\partial x'} \\
&\quad - (W'Z'^{-1}V'Y'^{-1} \otimes UY^{-1})\frac{\partial \operatorname{vec}(Y)}{\partial x'} \\
&\quad + (W'Z'^{-1} \otimes UY^{-1})\frac{\partial \operatorname{vec}(V)}{\partial x'} \\
&\quad - (W'Z'^{-1} \otimes UY^{-1}VZ^{-1})\frac{\partial \operatorname{vec}(Z)}{\partial x'} \\
&\quad + (I_r \otimes UY^{-1}VZ^{-1})\frac{\partial \operatorname{vec}(W)}{\partial x'}.
\end{aligned}$$

(6) x $(m \times 1)$, $Y(x)$ $(n \times q)$, $Z(x)$ $(p \times p)$ nonsingular:

$$\begin{aligned}
\frac{\partial[\operatorname{vec}(Y) \otimes \operatorname{vec}(Z^{-1})]}{\partial x'} &= \frac{\partial \operatorname{vec}(Y)}{\partial x'} \otimes \operatorname{vec}(Z^{-1}) \\
&\quad - \operatorname{vec}(Y) \otimes (Z'^{-1} \otimes Z^{-1})\frac{\partial \operatorname{vec}(Z)}{\partial x'}.
\end{aligned}$$

(7) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(p \times q)$:

$$\begin{aligned}
\frac{\partial[\operatorname{vec}(Y^{-1}) \otimes \operatorname{vec}(Z)]}{\partial x'} &= \operatorname{vec}(Y^{-1}) \otimes \frac{\partial \operatorname{vec}(Z)}{\partial x'} \\
&\quad - \left[(Y'^{-1} \otimes Y^{-1})\frac{\partial \operatorname{vec}(Y)}{\partial x'}\right] \otimes \operatorname{vec}(Z).
\end{aligned}$$

(8) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(p \times p)$ nonsingular:

$$\begin{aligned}
\frac{\partial[\operatorname{vec}(Y^{-1}) \otimes \operatorname{vec}(Z^{-1})]}{\partial x'} &= -\operatorname{vec}(Y^{-1}) \otimes \left[(Z'^{-1} \otimes Z^{-1})\frac{\partial \operatorname{vec}(Z)}{\partial x'}\right] \\
&\quad - \left[(Y'^{-1} \otimes Y^{-1})\frac{\partial \operatorname{vec}(Y)}{\partial x'}\right] \otimes \operatorname{vec}(Z^{-1}).
\end{aligned}$$

(9) x $(m \times 1)$, $Y(x)$ $(n \times q)$, $Z(x)$ $(p \times p)$ nonsingular:

$$\begin{aligned}
\frac{\partial \operatorname{vec}(Y \otimes Z^{-1})}{\partial x'} &= (I_q \otimes K_{pn} \otimes I_p) \\
&\quad \times \left[\frac{\partial \operatorname{vec}(Y)}{\partial x'} \otimes \operatorname{vec}(Z^{-1})\right. \\
&\quad \left. - \operatorname{vec}(Y) \otimes \left((Z'^{-1} \otimes Z^{-1})\frac{\partial \operatorname{vec}(Z)}{\partial x'}\right)\right].
\end{aligned}$$

(10) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(p \times q)$:

$$\begin{aligned}
\frac{\partial \operatorname{vec}(Y^{-1} \otimes Z)}{\partial x'} &= (I_n \otimes K_{qn} \otimes I_p) \\
&\quad \times \left[\operatorname{vec}(Y^{-1}) \otimes \frac{\partial \operatorname{vec}(Z)}{\partial x'}\right. \\
&\quad \left. - \left((Y'^{-1} \otimes Y^{-1})\frac{\partial \operatorname{vec}(Y)}{\partial x'}\right) \otimes \operatorname{vec}(Z)\right].
\end{aligned}$$

(11) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(p \times p)$ nonsingular:

$$\begin{aligned}
\frac{\partial \operatorname{vec}(Y^{-1} \otimes Z^{-1})}{\partial x'} &= -(I_n \otimes K_{pn} \otimes I_p) \\
&\quad \times \left[\operatorname{vec}(Y^{-1}) \otimes \left((Z'^{-1} \otimes Z^{-1})\frac{\partial \operatorname{vec}(Z)}{\partial x'}\right)\right. \\
&\quad \left. + \left((Y'^{-1} \otimes Y^{-1})\frac{\partial \operatorname{vec}(Y)}{\partial x'}\right) \otimes \operatorname{vec}(Z^{-1})\right].
\end{aligned}$$

(12) x $(m \times 1)$, $Y(x)$ $(n \times n)$, $Z(x)$ $(n \times n)$ nonsingular:

$$\begin{aligned}
\frac{\partial \operatorname{vec}(Y \odot Z^{-1})}{\partial x'} &= \frac{\partial \operatorname{vec}(Z^{-1} \odot Y)}{\partial x'} \\
&= \operatorname{diag}(\operatorname{vec} Z^{-1})\frac{\partial \operatorname{vec}(Y)}{\partial x'} \\
&\quad - \operatorname{diag}(\operatorname{vec} Y)(Z'^{-1} \otimes Z^{-1})\frac{\partial \operatorname{vec}(Z)}{\partial x'}.
\end{aligned}$$

(13) x $(m \times 1)$, $Y(x)$ $(n \times n)$ nonsingular, $Z(x)$ $(n \times n)$ nonsingular:

$$\begin{aligned}
\frac{\partial \operatorname{vec}(Y^{-1} \odot Z^{-1})}{\partial x'} &= -\left[\operatorname{diag}(\operatorname{vec} Z^{-1})(Y'^{-1} \otimes Y^{-1})\frac{\partial \operatorname{vec}(Y)}{\partial x'}\right. \\
&\quad \left. + \operatorname{diag}(\operatorname{vec} Y^{-1})(Z'^{-1} \otimes Z^{-1})\frac{\partial \operatorname{vec}(Z)}{\partial x'}\right].
\end{aligned}$$

Note: The results of this subsection follow from the chain rule for derivatives and the rules of the previous sections.

10.7 Chain Rules and Miscellaneous Jacobian Matrices

Reminder: D_m denotes a duplication matrix and D_m^+ its Moore-Penrose inverse (see Section 9.5). L_m is an elimination matrix (see Section 9.6).

(1) $x\ (m\times 1)$, $y(x)\ (n\times 1)$, $z(y)\ (p\times 1)$:

$$\frac{\partial z(y(x))}{\partial x'} = \frac{\partial z(y)}{\partial y'}\frac{\partial y(x)}{\partial x'}.$$

(2) $X\ (m\times n)$, $Y(X)\ (p\times q)$, $Z(Y)\ (r\times s)$:

$$\frac{\partial \mathrm{vec}\, Z(Y(X))}{\partial \mathrm{vec}(X)'} = \frac{\partial \mathrm{vec}\, Z(Y)}{\partial \mathrm{vec}(Y)'}\frac{\partial \mathrm{vec}\, Y(X)}{\partial \mathrm{vec}(X)'}.$$

(3) $X\ (m\times n)$, $\mathcal{S}$ open subset of $\mathbb{R}^{m\times n}$: $\mathrm{rk}(X)=r$ for all $X\in\mathcal{S}$ $\Rightarrow$ X^+ is a differentiable function of X on $\mathcal{S}$ and

$$\begin{aligned}\frac{\partial \mathrm{vec}(X^+)}{\partial \mathrm{vec}(X)'} = &-X^{+\prime}\otimes X^+ + [(I_m - XX^+)\otimes X^+X^{+\prime}\\ &+ X^{+\prime}X^+\otimes(I_n - X^+X)]K_{mn}.\end{aligned}$$

(4) $X\ (m\times n)$, $Y(X)\ (p\times q)$, $\mathcal{S}$ open subset of $\mathbb{R}^{m\times n}$: $\mathrm{rk}[Y(X)]=r$ for all $X\in\mathcal{S}$ $\Rightarrow$ $Y(X)^+$ is differentiable on $\mathcal{S}$ and

$$\begin{aligned}\frac{\partial \mathrm{vec}\, Y(X)^+}{\partial \mathrm{vec}(X)'} = &\Big(-Y^{+\prime}\otimes Y^+ + [(I_p - YY^+)\otimes Y^+Y^{+\prime}\\ &+ Y^{+\prime}Y^+\otimes(I_q - Y^+Y)]K_{pq}\Big)\frac{\partial \mathrm{vec}(Y)}{\partial \mathrm{vec}(X)'}.\end{aligned}$$

(5) $X\ (m\times m)$ nonsingular:

$$\frac{\partial \mathrm{vec}(X^{adj})}{\partial \mathrm{vec}(X)'} = \det(X)[(\mathrm{vec}\, X^{-1})(\mathrm{vec}\, X'^{-1})' - X'^{-1}\otimes X^{-1}].$$

(6) $X\ (m\times m)$ symmetric nonsingular:

$$\frac{\partial \mathrm{vech}(X^{adj})}{\partial \mathrm{vech}(X)'} = \det(X)D_m^+[(\mathrm{vec}\, X^{-1})(\mathrm{vec}\, X^{-1})' - X^{-1}\otimes X^{-1}]D_m.$$

(7) $X\ (m\times m)$ lower triangular, nonsingular:

$$\frac{\partial \mathrm{vech}(X^{adj})}{\partial \mathrm{vech}(X)'} = \det(X)L_m[(\mathrm{vec}\, X^{-1})(\mathrm{vec}\, X'^{-1})' - X'^{-1}\otimes X^{-1}]L_m'.$$

(8) $X\ (m\times n)$, $Y(X)\ (p\times p)$ nonsingular:

$$\frac{\partial \mathrm{vec}[Y(X)^{adj}]}{\partial \mathrm{vec}\,(X)'} = \det(Y)[(\mathrm{vec}\, Y^{-1})(\mathrm{vec}\, Y'^{-1})' - Y'^{-1}\otimes Y^{-1}]\frac{\partial \mathrm{vec}(Y)}{\partial \mathrm{vec}(X)'}.$$

(9) $X\ (m\times m)$ positive definite, $Y(X)=[y_{ij}]\ (m\times m)$ lower triangular with $y_{ii}\geq 0$, $i=1,\dots,m$, such that $X=YY'$, that is, Y is obtained by a Choleski decomposition of X :

$$\begin{aligned}\frac{\partial \mathrm{vech}(Y)}{\partial \mathrm{vech}(X)'} &= \{L_m[(I_m\otimes Y)K_{mm} + (Y\otimes I_m)]L_m'\}^{-1}\\ &= \{L_m(I_{m^2}+K_{mm})(Y\otimes I_m)L_m'\}^{-1}\\ &= \tfrac{1}{2}[D_m^+(Y\otimes I_m)L_m']^{-1}.\end{aligned}$$

Note: (1) and (2) are just chain rules. (3) is given in Magnus & Neudecker (1988, Chapter 8, Sec. 5). (4) follows from (3) and a chain rule. (5) is established in Magnus & Neudecker (1988, Chapter 8, Sec. 6). (6), (7) and (8) follow from (5) and a chain rule (see also Magnus (1988, Chapter 8, Sec. 8.2, 8.3)). (9) is, e.g., given in Lütkepohl (1991, Appendix A, Sec. A.13).

10.8 Jacobian Determinants

10.8.1 Linear Transformations

(1) $X\ (m\times n)$:

(a) $\det\left(\dfrac{\partial \mathrm{vec}(X)}{\partial \mathrm{vec}(X)'}\right) = 1.$

(b) $\det\left(\dfrac{\partial \mathrm{vec}(X')}{\partial \mathrm{vec}(X)'}\right) = (-1)^{mn(m-1)(n-1)/4}.$

(c) $\det\left(\dfrac{\partial \mathrm{vec}(X)}{\partial \mathrm{vec}(X')'}\right) = (-1)^{mn(m-1)(n-1)/4}.$

(2) $X\ (m\times m)$: $\det\left(\dfrac{\partial \mathrm{vech}(X)}{\partial \mathrm{vech}(X)'}\right) = 1.$

(3) $X\ (m\times n)$, $c\in\mathbb{R}$: $\det\left(\dfrac{\partial \mathrm{vec}(cX)}{\partial \mathrm{vec}(X)'}\right) = c^{mn}.$

(4) $X\ (m\times m)$, $c\in\mathbb{R}$: $\det\left(\dfrac{\partial \mathrm{vech}(cX)}{\partial \mathrm{vech}(X)'}\right) = c^{m(m+1)/2}.$

(5) $X\ (m\times n)$, $A\ (m\times m)$, $B\ (n\times n)$:

$$\det\left(\frac{\partial \mathrm{vec}(AXB)}{\partial \mathrm{vec}(X)'}\right) = (\det A)^n(\det B)^m.$$

(6) $X\ (m\times n)$, $A\ (n\times n)$, $B\ (m\times m)$:

$$\det\left(\frac{\partial \mathrm{vec}(AX'B)}{\partial \mathrm{vec}(X)'}\right) = (-1)^{mn(m-1)(n-1)/4}(\det A)^m(\det B)^n.$$

(7) X, $A = [a_{ij}]$, $B = [b_{ij}]$ $(m \times m)$ lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(AXB)}{\partial \operatorname{vech}(X)'}\right) = \prod_{i=1}^{m} a_{ii}^{i} b_{ii}^{m-i+1}.$$

(8) X $(m \times m)$ symmetric, A $(m \times m)$:

$$\det\left(\frac{\partial \operatorname{vech}(A'XA)}{\partial \operatorname{vech}(X)'}\right) = (\det A)^{m+1}.$$

(9) X $(m \times m)$ symmetric, A, B $(m \times m)$, $\det(A) \neq 0$, $\lambda_1, \ldots, \lambda_m$ eigenvalues of BA^{-1} :

$$\det\left(\frac{\partial \operatorname{vech}(AXA' \pm BXB')}{\partial \operatorname{vech}(X)'}\right) = (\det A)^{m+1} \prod_{i \geq j} (1 \pm \lambda_i \lambda_j).$$

(10) X $(m \times m)$ symmetric, $A = [a_{ij}]$, $B = [b_{ij}]$ $(m \times m)$ lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(AXA' \pm BXB')}{\partial \operatorname{vech}(X)'}\right) = \prod_{i \geq j} (a_{ii}a_{jj} \pm b_{ii}b_{jj}).$$

(11) X $(m \times m)$ symmetric, A, B $(m \times m)$, $\det(B) \neq 0$, $\lambda_1, \ldots, \lambda_m$ eigenvalues of AB^{-1} :

$$\det\left(\frac{\partial \operatorname{vech}(AXB' + BXA')}{\partial \operatorname{vech}(X)'}\right) = 2^m \det(A)(\det B)^m \prod_{i>j} (\lambda_i + \lambda_j).$$

(12) $X, A = [a_{ij}], B = [b_{ij}], C = [c_{ij}], D = [d_{ij}]$ $(m \times m)$ lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(AXB + CXD)}{\partial \operatorname{vech}(X)'}\right) = \prod_{i \geq j} (a_{ii}b_{jj} + c_{ii}d_{jj}).$$

(13) X $(m \times m)$ lower triangular, A, B $(m \times m)$ nonsingular:

$$\det\left(\frac{\partial \operatorname{vech}(B'XA + A'X'B)}{\partial \operatorname{vech}(X)'}\right) = 2^m \det(A)(\det B)^m \prod_{i=1}^{m-1} \det(C_{(i)}),$$

where

$$C_{(i)} = \begin{bmatrix} c_{11} & \cdots & c_{1i} \\ \vdots & \ddots & \vdots \\ c_{i1} & \cdots & c_{ii} \end{bmatrix}, \quad i = 1, \ldots, m-1,$$

are the principal submatrices of $C = [c_{ij}] = AB^{-1}$.

(14) $X, A = [a_{ij}]$ $(m \times n)$, B $(m \times m)$, C $(n \times n)$:

$$\det\left(\frac{\partial \operatorname{vec}(A \odot BXC)}{\partial \operatorname{vec}(X)'}\right) = (\det B)^n (\det C)^m \prod_{i=1}^{m} \prod_{j=1}^{n} a_{ij}.$$

(15) X $(m \times n)$, $A = [a_{ij}]$ $(n \times m)$, B $(n \times n)$, C $(m \times m)$:

$$\det\left(\frac{\partial \operatorname{vec}(A \odot BX'C)}{\partial \operatorname{vec}(X)'}\right)$$
$$= (-1)^{mn(m-1)(n-1)/4} (\det B)^m (\det C)^n \prod_{i=1}^{n} \prod_{j=1}^{m} a_{ij}.$$

(16) X, $A = [a_{ij}]$, $B = [b_{ij}]$ $(m \times n)$:

$$\det\left(\frac{\partial \operatorname{vec}(A \odot X \odot B)}{\partial \operatorname{vec}(X)'}\right) = \prod_{i=1}^{m} \prod_{j=1}^{n} a_{ij} b_{ij}.$$

(17) X $(m \times n)$, $A = [a_{ij}]$, $B = [b_{ij}]$ $(n \times m)$:

$$\det\left(\frac{\partial \operatorname{vec}(A \odot X' \odot B)}{\partial \operatorname{vec}(X)'}\right) = (-1)^{mn(m-1)(n-1)/4} \prod_{i=1}^{n} \prod_{j=1}^{m} a_{ij} b_{ij}.$$

Note: The results of this subsection follow from the rules of the previous sections of this chapter and the rules for determinants (see also Magnus (1988, Chapter 8)).

10.8.2 Nonlinear Transformations

(1) (Chain rule)
x, $y(x)$, $z(y)$ $(m \times 1)$:

$$\det\left(\frac{\partial z(y(x))}{\partial x'}\right) = \det\left(\frac{\partial z(y)}{\partial y'}\right) \det\left(\frac{\partial y(x)}{\partial x'}\right).$$

(2) (Chain rule)
$X, Y(X), Z(Y)$ $(m \times n)$:

$$\det\left(\frac{\partial \operatorname{vec} Z(Y(X))}{\partial \operatorname{vec}(X)'}\right) = \det\left(\frac{\partial \operatorname{vec} Z(Y)}{\partial \operatorname{vec}(Y)'}\right) \det\left(\frac{\partial \operatorname{vec} Y(X)}{\partial \operatorname{vec}(X)'}\right).$$

(3) X $(m \times m)$ symmetric with eigenvalues $\lambda_1, \ldots, \lambda_m$:

$$\det\left(\frac{\partial \operatorname{vech}(X^i)}{\partial \operatorname{vech}(X)'}\right) = i^m (\det X)^{i-1} \prod_{k>l} \mu_{kl},$$

where

$$\mu_{kl} = \begin{cases} (\lambda_k^i - \lambda_l^i)/(\lambda_k - \lambda_l) & \text{if} \quad \lambda_k \neq \lambda_l \\ i\lambda_k^{i-1} & \text{if} \quad \lambda_k = \lambda_l \end{cases}.$$

(4) $X = [x_{kl}]$ $(m \times m)$ lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(X^i)}{\partial \operatorname{vech}(X)'}\right) = i^m (\det X)^{i-1} \prod_{k>l} \mu_{kl},$$

where

$$\mu_{kl} = \begin{cases} (x_{kk}^i - x_{ll}^i)/(x_{kk} - x_{ll}) & \text{if} \quad x_{kk} \neq x_{ll} \\ i x_{kk}^{i-1} & \text{if} \quad x_{kk} = x_{ll} \end{cases}.$$

(5) X, A $(m \times m)$ symmetric, $\lambda_1, \dots, \lambda_m$ eigenvalues of XA :

$$\det\left(\frac{\partial \operatorname{vech}(XAX)}{\partial \operatorname{vech}(X)'}\right) = 2^m \det(A)\det(X) \prod_{i>j} (\lambda_i + \lambda_j).$$

(6) $X = [x_{ij}]$, $A = [a_{ij}]$ $(m \times m)$ lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(XAX)}{\partial \operatorname{vech}(X)'}\right) = 2^m \det(A)\det(X) \prod_{i>j} (a_{ii}x_{ii} + a_{jj}x_{jj}).$$

(7) $X = [x_{ij}]$ $(m \times m)$ lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(X'X)}{\partial \operatorname{vech}(X)'}\right) = 2^m \prod_{i=1}^{m} x_{ii}^i,$$

$$\det\left(\frac{\partial \operatorname{vech}(XX')}{\partial \operatorname{vech}(X)'}\right) = 2^m \prod_{i=1}^{m} x_{ii}^{m-i+1}.$$

(8) X $(m \times m)$ nonsingular:

$$\det\left(\frac{\partial \operatorname{vec}(X^{-1})}{\partial \operatorname{vec}(X)'}\right) = (-1)^m (\det X)^{-2m}.$$

(9) X $(m \times m)$ symmetric nonsingular:

$$\det\left(\frac{\partial \operatorname{vech}(X^{-1})}{\partial \operatorname{vech}(X)'}\right) = (-1)^{m(m+1)/2} (\det X)^{-(m+1)}.$$

(10) X $(m \times m)$ nonsingular, lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(X^{-1})}{\partial \operatorname{vech}(X)'}\right) = (-1)^{m(m+1)/2} (\det X)^{-(m+1)}.$$

(11) X $(m \times m)$ symmetric nonsingular:

$$\det\left(\frac{\partial \operatorname{vech}(X^{adj})}{\partial \operatorname{vech}(X)'}\right) = (-1)^{m(m+1)/2} (1-m) (\det X)^{(m+1)(m-2)/2}.$$

(12) X $(m \times m)$ nonsingular, lower triangular:

$$\det\left(\frac{\partial \operatorname{vech}(X^{adj})}{\partial \operatorname{vech}(X)'}\right) = (-1)^{m(m+1)/2} (1-m) (\det X)^{(m+1)(m-2)/2}.$$

Note: Most results of this subsection are given in Magnus (1988, Chapter 8). They follow from rules of the previous sections and the results for determinants.

10.9 Matrix Valued Functions of a Scalar Variable

(1) $x \in \mathbb{R}$, $A(x) = A$ $(m \times n)$ constant: $\dfrac{\mathrm{d}A}{\mathrm{d}x} = O_{m \times n}$.

(2) (Linearity)
$x \in \mathbb{R}$, $A(x), B(x)$ $(m \times n), c_1, c_2 \in \mathbb{R}$:

$$\frac{\mathrm{d}[c_1 A(x) + c_2 B(x)]}{\mathrm{d}x} = c_1 \frac{\mathrm{d}A(x)}{\mathrm{d}x} + c_2 \frac{\mathrm{d}B(x)}{\mathrm{d}x}.$$

(3) (Product rule)
$x \in \mathbb{R}$, $A(x)$ $(m \times n)$, $B(x)$ $(n \times p)$:

$$\frac{\mathrm{d}A(x)B(x)}{\mathrm{d}x} = A(x)\frac{\mathrm{d}B(x)}{\mathrm{d}x} + \frac{\mathrm{d}A(x)}{\mathrm{d}x}B(x).$$

(4) $x \in \mathbb{R}$, $A(x)$ $(m \times n)$, $B(x)$ $(n \times p)$, $C(x)$ $(p \times q)$:

$$\frac{\mathrm{d}A(x)B(x)C(x)}{\mathrm{d}x} = A(x)B(x)\frac{\mathrm{d}C(x)}{\mathrm{d}x} + A(x)\frac{\mathrm{d}B(x)}{\mathrm{d}x}C(x) + \frac{\mathrm{d}A(x)}{\mathrm{d}x}B(x)C(x).$$

(5) $x \in \mathbb{R}$, $A(x)$ $(m \times m)$ nonsingular:

$$\frac{\mathrm{d}A(x)^{-1}}{\mathrm{d}x} = -A(x)^{-1}\frac{\mathrm{d}A(x)}{\mathrm{d}x}A(x)^{-1}.$$

(6) (Ratio rule)
$x \in \mathbb{R}$, $A(x)$ $(m \times m)$ nonsingular, $B(x)$ $(n \times m)$:

$$\frac{\mathrm{d}B(x)A(x)^{-1}}{\mathrm{d}x} = \frac{\mathrm{d}B(x)}{\mathrm{d}x}A(x)^{-1} - B(x)A(x)^{-1}\frac{\mathrm{d}A(x)}{\mathrm{d}x}A(x)^{-1}.$$

(7) (Generalized inverse rule)
$x \in \mathbb{R}$, $A(x)$ $(m \times n)$, $A(x)^-$ some generalized inverse of $A(x)$:

$$A(x)\frac{\mathrm{d}A(x)^-}{\mathrm{d}x}A(x) = -A(x)A(x)^- \frac{\mathrm{d}A(x)}{\mathrm{d}x} A(x)^- A(x).$$

Note: The first four rules follow from the corresponding rules for real valued functions by considering typical elements of the matrices involved. Rule (5) is obtained by applying the product rule to $AA^{-1} = I_m$, (6) follows from (5) and the product rule and the result in (7) follows by applying the product rule to $AA^-A = A$ and multiplying by AA^- from the left.