Bayesian variable selection

Feng Li
feng.li@cufe.edu.cn

School of Statistics and Mathematics
Central University of Finance and Economics

Revised on October 13, 2017

Today we are going to learn...

1. **Bayesian Variable Selection**
2. **Priors for Variable Selection Indicators**

Bayesian variable selection

- Linear regression:

$$y = \beta_0 + \beta_1 x_1 + ... + \beta_p x_p + \varepsilon.$$

- Which variables have **non-zero** coefficient? Example of hypotheses:

$$\begin{aligned} H_0 &: \beta_0 = \beta_1 = ... = \beta_p = 0 \\ H_1 &: \beta_1 = 0 \\ H_2 &: \beta_1 = \beta_2 = 0 \end{aligned}$$

- Introduce **variable selection indicators** $\mathcal{I} = (I_1, ..., I_p)$.
- Example: $\mathcal{I} = (1, 1, 0)$ means that $\beta_1 \neq 0$ and $\beta_2 \neq 0$, but $\beta_3 = 0$, so x_3 drops out of the model.

Bayesian variable selection, cont.

- Model inference, just crank the Bayesian machine:

$$p(\mathcal{I}|\mathbf{y}, \mathbf{X}) \propto p(\mathbf{y}|\mathbf{X}, \mathcal{I}) \cdot p(\mathcal{I})$$

- The prior $p(\mathcal{I})$ is typically taken to be $I_1, ..., I_p|\theta \overset{iid}{\sim} Bernoulli(\theta)$.
- θ is the **prior inclusion probability**.

Bayesian variable selection, cont.

- Model inference, just crank the Bayesian machine:

$$p(\mathcal{I}|\mathbf{y}, \mathbf{X}) \propto p(\mathbf{y}|\mathbf{X}, \mathcal{I}) \cdot p(\mathcal{I})$$

- The prior $p(\mathcal{I})$ is typically taken to be $I_1, ..., I_p|\theta \overset{iid}{\sim} Bernoulli(\theta)$.
- θ is the **prior inclusion probability**.

- Challenge: Computing the **marginal likelihood** for each model ($\mathcal{I}$)

$$p(\mathbf{y}|\mathbf{X}, \mathcal{I}) = \int p(\mathbf{y}|\mathbf{X}, \mathcal{I}, \beta) p(\beta|\mathbf{X}, \mathcal{I}) d\beta$$

Bayesian variable selection, cont.

- Let $\beta_{\mathcal{I}}$ denote the **non-zero** coefficients under $\mathcal{I}$.
- Prior:

$$\beta_{\mathcal{I}}|\sigma^2 \sim N\left(0, \sigma^2\Omega_{\mathcal{I},0}^{-1}\right)$$
$$\sigma^2 \sim Inv - \chi^2\left(\nu_0, \sigma_0^2\right)$$

- **Marginal likelihood**

$$p(\mathbf{y}|\mathbf{X}, \mathcal{I}) \propto \left|\mathbf{X}_{\mathcal{I}}'\mathbf{X}_{\mathcal{I}} + \Omega_{\mathcal{I},0}^{-1}\right|^{-1/2} \left|\Omega_{\mathcal{I},0}\right|^{1/2} \left(\nu_0\sigma_0^2 + RSS_{\mathcal{I}}\right)^{-(\nu_0+n-1)/2}$$

  where $\mathbf{X}_{\mathcal{I}}$ is the covariate matrix for the subset given by $\mathcal{I}$.
- $RSS_{\mathcal{I}}$ is (almost) the residual sum of squares under model implied by $\mathcal{I}$

$$RSS_{\mathcal{I}} = \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}_{\mathcal{I}}\left(\mathbf{X}_{\mathcal{I}}'\mathbf{X}_{\mathcal{I}} + \Omega_{\mathcal{I},0}\right)^{-1}\mathbf{X}_{\mathcal{I}}'\mathbf{y}$$

Bayesian variable selection via Gibbs sampling

- But there are 2^p model combinations to go through! Ouch!
- ... but most will have essentially zero posterior probability. Phew!

Bayesian variable selection via Gibbs sampling

- But there are 2^p model combinations to go through! Ouch!
- ... but most will have essentially zero posterior probability. Phew!
- **Simulate** from the joint posterior distribution:

$$p(\beta, \sigma^2, \mathcal{I}|\mathbf{y},\mathbf{X}) = p(\beta, \sigma^2|\mathcal{I}, \mathbf{y},\mathbf{X})p(\mathcal{I}|\mathbf{y}, \mathbf{X}).$$

- Simulate from $p(\mathcal{I}|\mathbf{y})$ using **Gibbs sampling**:
  - Draw $I_1|\mathcal{I}_{-1}, \mathbf{y}, \mathbf{X}$
  - Draw $I_2|\mathcal{I}_{-2},\mathbf{y}, \mathbf{X}$
  - ...
  - Draw $I_p|\mathcal{I}_{-p}, \mathbf{y}, \mathbf{X}$

Bayesian variable selection via Gibbs sampling

- But there are 2^p model combinations to go through! Ouch!
- ... but most will have essentially zero posterior probability. Phew!

- **Simulate** from the joint posterior distribution:

$$p(\beta, \sigma^2, \mathcal{I}|\mathbf{y},\mathbf{X}) = p(\beta, \sigma^2|\mathcal{I}, \mathbf{y},\mathbf{X})p(\mathcal{I}|\mathbf{y}, \mathbf{X}).$$

- Simulate from $p(\mathcal{I}|\mathbf{y})$ using **Gibbs sampling**:
  - Draw $I_1|\mathcal{I}_{-1}, \mathbf{y}, \mathbf{X}$
  - Draw $I_2|\mathcal{I}_{-2}, \mathbf{y}, \mathbf{X}$
  - ...
  - Draw $I_p|\mathcal{I}_{-p}, \mathbf{y}, \mathbf{X}$

- Only need to compute $Pr(I_i = 0|\mathcal{I}_{-i}, \mathbf{y}, \mathbf{X})$ and $Pr(I_i = 1|\mathcal{I}_{-i}, \mathbf{y}, \mathbf{X})$.
- Automatic model averaging, all in one simulation run.
- If needed, simulate from $p(\beta, \sigma^2|\mathcal{I}, \mathbf{y},\mathbf{X})$ for each draw of $\mathcal{I}$.

PSEUDO CODE FOR BAYESIAN VARIABLE SELECTION

0 Initialize $\mathcal{I}^{(0)} = (I_1^{(0)}, I_2^{(0)} \dots, I_p^{(0)})$

1 Simulate σ^2 and β from [Note: $\nu_n, \sigma_n^2, \mu_n, \Omega_n$ all depend on $\mathcal{I}^{(0)}$]

- $\sigma^2 | \mathcal{I}^{(0)}, \mathbf{y}, \mathbf{X} \sim Inv - \chi^2 \left(\nu_n, \sigma_n^2\right)$
- $\beta | \sigma^2, \mathcal{I}^{(0)}, \mathbf{y}, \mathbf{X} \sim N\left[\mu_n, \sigma^2 \Omega_n^{-1}\right]$

2.1 Simulate $I_1 | \mathcal{I}_{-1}, \mathbf{y}, \mathbf{X}$ by [define $\mathcal{I}_{prop}^{(0)} = (1 - I_1^{(0)}, I_2^{(0)} \dots, I_p^{(0)})$]

- compute marginal likelihoods: $p(\mathbf{y}|\mathbf{X}, \mathcal{I}^{(0)})$ and $p(\mathbf{y}|\mathbf{X}, \mathcal{I}_{prop}^{(0)})$
- Simulate $I_1^{(1)} \sim Bernoulli(\kappa)$ where

$$\kappa = \frac{p(\mathbf{y}|\mathbf{X}, \mathcal{I}^{(0)}) \cdot p(\mathcal{I}^{(0)})}{p(\mathbf{y}|\mathbf{X}, \mathcal{I}^{(0)}) \cdot p(\mathcal{I}^{(0)}) + p(\mathbf{y}|\mathbf{X}, \mathcal{I}_{prop}^{(0)}) \cdot p(\mathcal{I}_{prop}^{(0)})}$$

2.2 Simulate $I_2 | \mathcal{I}_{-2}, \mathbf{y}, \mathbf{X}$ as in Step 2.1, but $\mathcal{I}^{(0)} = (I_1^{(1)}, I_2^{(0)}, \dots, I_p^{(0)})$

⋮

2.P Simulate $I_p | \mathcal{I}_{-p}, \mathbf{y}, \mathbf{X}$ as in Step 2.1, but $\mathcal{I}^{(0)} = (I_1^{(1)}, I_2^{(1)}, \dots, I_p^{(0)})$

3 Repeat Steps 1-2 many times.

Simple general Bayesian variable selection

- The previous algorithm only works when we can integrate out all the model parameters to obtain

$$p(\mathcal{I}|\mathbf{y},\mathbf{X}) = \int p(\beta,\sigma^2,\mathcal{I}|\mathbf{y},\mathbf{X})d\beta d\sigma$$

- **MH** - **propose** β and $\mathcal{I}$ jointly from the proposal distribution

$$q(\beta_p|\beta_c,\mathcal{I}_p)q(\mathcal{I}_p|\mathcal{I}_c)$$

- Main difficulty: how to propose the non-zero elements in β_p?
- Simple approach:
  - Approximate posterior with all variables in the model: $\beta|\mathbf{y},\mathbf{X} \overset{approx}{\sim} N\left[\hat{\beta}, J_{\mathbf{y}}^{-1}(\hat{\beta})\right]$
  - Propose β_p from $N\left[\hat{\beta}, J_{\mathbf{y}}^{-1}(\hat{\beta})\right]$, conditional on the zero restrictions implied by $\mathcal{I}_p$. Formulas are available.

Variable selection in more complex models

Posterior summary of the one-component split-t model.[a]

Parameters	Mean	Stdev	Post.Incl.
Location μ			
Const	0.084	0.019	–
Scale ϕ			
Const	0.402	0.035	–
LastDay	−0.190	0.120	0.036
LastWeek	**−0.738**	**0.193**	**0.985**
LastMonth	**−0.444**	**0.086**	**0.999**
CloseAbs95	0.194	0.233	0.035
CloseSqr95	0.107	0.226	0.023
MaxMin95	**1.124**	**0.086**	**1.000**
CloseAbs80	0.097	0.153	0.013
CloseSqr80	0.143	0.143	0.021
MaxMin80	−0.022	0.200	0.017
Degrees of freedom ν			
Const	2.482	0.238	–
LastDay	0.504	0.997	0.112
LastWeek	**−2.158**	**0.926**	**0.638**
LastMonth	0.307	0.833	0.089
CloseAbs95	0.718	1.437	0.229
CloseSqr95	1.350	1.280	0.279
MaxMin95	1.130	1.488	0.222
CloseAbs80	0.035	1.205	0.101
CloseSqr80	0.363	1.211	0.112
MaxMin80	−1.672	1.172	0.254
Skewness λ			
Const	−0.104	0.033	–
LastDay	−0.159	0.140	0.027
LastWeek	−0.341	0.170	0.135
LastMonth	−0.076	0.112	0.016
CloseAbs95	−0.021	0.096	0.008
CloseSqr95	−0.003	0.108	0.006
MaxMin95	0.016	0.075	0.008
CloseAbs80	0.060	0.115	0.009
CloseSqr80	0.059	0.111	0.010
MaxMin80	0.093	0.096	0.013

Model averaging

- Let γ be a quanitity with an interpretation which stays the same across the two models.
- Example: Prediction $\gamma = (y_{T+1}, ..., y_{T+h})'$.
- The marginal posterior distribution of γ reads

$$p(\gamma|\mathbf{y}) = p(M_1|\mathbf{y})p_1(\gamma|\mathbf{y}) + p(M_2|\mathbf{y})p_2(\gamma|\mathbf{y}),$$

  where $p_k(\gamma|\mathbf{y})$ is the marginal posterior of γ conditional on model k.
- Predictive distribution includes three sources of uncertainty:
  - **Future errors**/disturbances (e.g. the ε's in a regression)
  - **Parameter uncertainty** (the predictive distribution has the parameters integrated out by their posteriors)
  - **Model uncertainty** (by model averaging)

Variable-selection priors

- The standard modern practice in Bayesian variable-selection problems is to treat variable inclusions as exchangeable Bernoulli trials with common success probability p.
- This implies that the **prior probability of a model** is given by

$$p(M_\gamma|p) = p^{k_\gamma}(1-p)^{m-k_\gamma}$$

  with k_γ representing the number of included variables in the model.
- This indicates that as m grows with the true k remaining fixed, the posterior distribution of p will concentrate near 0. That means **using a fixed p will yield a null model when m is big** (no variable will be selected).
- Selecting $p = 1/2$ does not provide multiplicity correction. Treating p as an unknown parameter to be estimated from the data will, however, yield an automatic multiple-testing penalty.

Fully Bayesian variable-selection priors

- Assume that p has a Beta distribution, $p \sim \mathrm{Beta}(a, b)$, giving

$$p(M_\gamma) = \frac{\mathrm{Beta}(a + k_\gamma, b + m - k_\gamma)}{\mathrm{Beta}(a, b)}$$

- For the default choice of $a = b = 1$, implying a uniform prior on p

$$p(M_\gamma) = \frac{1}{m+1}\binom{m}{k_\gamma}^{-1}$$